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Multiplicities, generalized Jacobi matrices and symmetric operators

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Abstract. The global multiplicity of *bounded* linear operators in Banach spaces has been studied for a number of classes of operators. We introduce a definition of multiplicity of a *general unbounded* operator, and compare it with a known version (essentially reducing it to bounded cases) for certain symmetric operators. We study the connection of this concept with generalized (regular or irregular, block) Jacobi matrices. We establish the multiplicities of pure maximal symmetric operators, and show how this reveals the structure of the elementary symmetric operators and their simplest matrix representations: a problem unsolved in a classical paper by v.Neumann.

Keywords: global multiplicity, multicyclicity, (pure) maximal symmetric operator, elementary symmetric operator, deficiency index, infinite generalized (regular or irregular, Hermitian symmetric) Jacobi matrix, matrix representation of a symmetric operator.

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1. Introduction

The global multiplicity (or, equivalently, multicyclicity) of *bounded* linear operators in Banach spaces has been studied by a large number of writers. An excellent introduction to the subject and some of the basic results are contained in the paper and book by N.K.Nikolski [Ni1],[Ni2] together with a review of the references until that time. In [Ni2] a method is suggested to connect the (undefined) multiplicity of *some unbounded* operators (semigroup generators) to the multiplicity of corresponding bounded operators (resolvents, Cayley transforms). In this paper we shall introduce a definition of multiplicity of a *general unbounded* operator, compare it with the known version above, and show its usefulness in concrete problems: in connection with generalized Jacobi matrices and with symmetric operators.

It is well known that in the structure problem of maximal symmetric operators in a (separable) Hilbert space a basic role is played by the *elementary symmetric operators*, which were introduced by J.v.Neumann [NAE] in a number of unitarily equivalent forms. The (perhaps) simplest way to do this is to fix an orthonormal basis $\{e_k : k \in \mathbf{N}\}$, and define the Cayley transform V of the elementary symmetric operator S by

$$Ve_k := e_{k+1} \quad (k \in \mathbf{N}).$$

V is then an (everywhere defined, bounded) isometry with deficiency indices (0,1), which are the same for its (inverse) Cayley transform S. It is also clear that V has global multiplicity 1 (e_1 is a cyclic vector for V), but the similar question for S seems to be not so simple. Apparently, J.v.Neumann himself sought a simple matrix representation in [NAE], [NT], but he did not consider the problem of defining the multiplicity of S. Though his matrices are constructed ingeniously, even the second one is far from having the simplest structure (cf. [NAE, pp.126-128]). We shall show how successfully the introduced concept of multiplicity of an unbounded operator can be used for a solution to this problem.

Infinite Jacobi matrices and their generalized variants have been playing a useful role for a long time in handling various questions in analysis and operator theory. Our basic references will be the papers by Hamburger [H], M.G. Krein [KD] and the monographs by Stone [St], Akhieser and Glasman [AG] and by Akhieser [A]. We shall give an answer to a question posed in the last reference. In addition, we shall clear the relation between the generalized types $J[p]$ and J_p of Jacobi matrices as well as between regular and irregular variants.

It is well known that general linear operators, closed linear operators or even closed symmetric linear operators may have surprising pathological properties. E.g., Berberian gave an example (see, e.g., [Gol, p.53]) of a linear operator T such that its domain $D(T)$ is all of the Hilbert space $X = l^2$, and the domain $D(T^*)$ of the adjoint operator is the set $\{0\}$. A densely defined closed linear operator may have ([Nag, p.226]) a quotient operator (with respect to an invariant closed subspace) that is not closable. Or: a closed densely defined symmetric operator T can have a square such that $D(T^2) = \{0\}$ [Nai]. On the other hand, Schmüdgen [Sch] proved that if at least one of the deficiency numbers of a densely defined closed symmetric operator T is finite, then the linear manifold $D(T^\infty)$ (see below) is dense in the Hilbert space X.

Similar phenomena may be a warning that we must exercise some care when extending the definition and basic properties of the global multiplicity (or multicyclicity) function to the case of not necessarily bounded linear operators in linear normed (Banach, Hilbert) spaces. The basic reference for the properties of linear operators under such general conditions will be the monograph by S.Goldberg [Gol].

2. Terminology and notation

Let T be a linear operator with domain $D(T)$ and range $rg(T)$ in a complex linear space X. For any subset $M \subset X$ let $L(M)$ denote the linear hull of M, and let

$$TM := \{Tm : \, m \in M \cap D(T)\} \equiv T(M \cap D(T)).$$

We say that M is *T-invariant* if $TM \subset M$. If E is a T-invariant subspace of X, we define (as usual, cf. [Vas]) the quotient space X/E consisting of all cosets $\hat{x}$ of the form $x+E$ ($x \in X$), and the *quotient operator* T/E by

$$D(T/E) := \{\hat{x} : \, \exists x \in \hat{x} \cap D(T)\}, \quad (T/E)\hat{x} := Tx + E.$$

It is easy to see that these definitions are independent of the choice of x in $\hat{x} \cap D(T)$. Let $\mathbf{N}$ denote the set of all positive, $\mathbf{N}_0$ the set of all nonnegative integers, and let

$$D(T^\infty) := \cap_{n \in \mathbf{N}} D(T^n).$$

If the space X is normed and separable, we shall denote by $sp[S]$ the *closed linear hull* of the set $S \subset X$. In the family of all linear subspaces C in $D(T^\infty)$ we *define* the family of *T-cyclic subspaces* by

$$Cyc(T) := \{C \subset D(T^\infty) : \, sp[T^n C : \, n \in \mathbf{N}_0] = X\}.$$

The (linear) dimension of a space will be understood to be a nonnegative integer or $+\infty$, and the *global multiplicity (or, equivalently, multicyclicity)* of T will be defined by

$$\mu(T) := \min\{\dim C : \, C \in Cyc(T)\} \in \mathbf{N} \cup \{\infty\},$$

if the family $Cyc(T)$ is nonvoid or, equivalently, the subspace $D(T^\infty)$ is dense in the space X. In the *opposite case we define*

$$\mu(T) := \infty.$$

Note that in the case studied by Schmüdgen [Sch] and quoted in the Introduction above, further in the case of the generator operator T of a strongly continuous semigroup of class (C_0), as well as for many other operators, the subspace $D(T^\infty)$ is dense in the space X. Further, if T is a linear operator with any linear extension $\hat{T}$, then $Cyc(T) \subset Cyc(\hat{T})$, hence $\mu(\hat{T}) \le \mu(T)$.

For any *bounded* linear operator W in a Banach space X our definition *agrees* with the classical one:

$$\mu(W) := \min\{\dim C : \, sp[W^n C : n \in \mathbf{N}_0] = X\}.$$

We recall now that for *certain* unbounded operators concepts close to the global multiplicity of T were studied, e.g., in [Ni2]. We shall show that, in general, *the two approaches do not yield the same cardinal number.*

Let $\{T(t) : \, t \ge 0\}$ be a strongly continuous semigroup in the Banach space X with growth bound

$$\alpha(T) := \lim_{t \to \infty} t^{-1} \log |T(t)|$$

(here $|\cdot|$ denotes the operator norm), and generator operator A. It is well known that $\sup Re\sigma(A) \le \alpha(T)$ (here σ denotes the spectrum of the operator). For the following definitions of multiplicities we cite [Ni2, pp.240-241]:

$$\mu[T(\cdot)] := \min\{\dim C : \, sp[T(t)C : t \ge 0] = X\}.$$

Further, if $\alpha(T) < 1$, then the following Cayley transform $C^{-}(A)$ of A is bounded:

$$C^{-}(A) := (I + A)(I - A)^{-1} \equiv 2(I - A)^{-1} - I.$$

It is shown in [Ni2] that under the stated conditions

$$\mu[T(\cdot)] = \mu[C^{-}(A)] = \mu[R(z, A)]$$

for every $z > \alpha(T)$ [here $R(z, A) := (z - A)^{-1}$].

Recall that if the basic space X is Hilbert, then similar questions concerning semigroups of contractions, their generators, co-generators and unitary dilations were studied by Sz.-Nagy and Foias [SzNF1], [SzNF2] and by Sz.-Nagy [SzN]. It follows from their work, but also in a direct way that if S is the elementary symmetric operator from the Introduction with deficiency indices $(0, 1)$ (with the isometry V as its Cayley transform) then, since S is maximal symmetric (cf. also [Dav, Theorem 6.5]), the operator iS is the generator of a strongly continuous semigroup T of isometries. Hence $\alpha(T) = 0$, and we clearly have

$$\mu[T(\cdot)] = \mu(V) = \mu[R(z, iS)] = 1.$$

To the contrary, we shall prove that

$$\mu(iS) = \mu(S) = 2 \neq \mu[R(z, iS)].$$

It is well known that for a densely defined closed symmetric operator S in a separable Hilbert space X and for a given orthonormal basis $E \subset D(S) \subset X$ the infinite (Hermitian symmetric) matrix A of the operator in the given basis is defined as $a_{ik} := (Se_k, e_i) \quad (i, k \in \mathbf{N})$. In the converse direction: given the pair (A, E), the closure S_0 of the elementary linear operator (the latter is defined on the linear hull of the basis vectors) is a densely defined closed symmetric operator, which may have closed symmetric extensions S. We shall accept the following

Definition. Under the conditions above we shall call S_0 the operator *determined* by the matrix A, and call any such S an operator *generated* by the matrix A.

Let $p \in \mathbf{N}$. In order to fix terminology, we shall say (cf. [KD]) that the *infinite Hermitian symmetric matrix (over* $\mathbf{C}$*)* $A = (a_{mn}) \quad (m, n \in \mathbf{N})$ is a *regular* $J[p]$ *(block or, equivalently, generalized Jacobi) matrix* iff A is partitioned as $A = (A_{ik}) \quad (i, k \in \mathbf{N})$, where the A_{ik} blocks are $p \times p$ matrices over $\mathbf{C}$ such that $A_{ik} = 0$ for $|i - k| > 1$, and all the matrices $A_{k,k+1}$ are nonsingular. In particular, we shall say that A is a *regular* J_p *matrix* (cf., e.g., [Ci]) iff, in addition, all the blocks $A_{k,k+1}$ are lower triangular (hence $A_{k+1,k}$ upper triangular). We do not qualify, or say explicitly that A is *irregular*, if the regularity conditions above do not necessarily hold for every k.

For a matrix A as above, we define the value Tf of a linear operator T in $l^2(\mathbf{N})$ as the matrix product Af for any vector $f \in l^2(\mathbf{N})$ of *finite type* (i.e. with only a finite number of nonzero components with respect to the canonical basis in $l^2(\mathbf{N})$). It is well known that T has a (linear, densely defined, symmetric) closure T_p, which we shall call the *operator determined by the Jacobi block matrix* $A \in J[p]$ (See also [Wei, Theorem 6.20].) The nonnegative integers

$$d(z) := \dim \ker(T_p^* - zI)$$

are identical for z in the open upper (and also in the lower) half-planes: we shall call them, correspondingly, the deficiency *numbers* d_1 and d_2, call the ordered pair (d_1, d_2) the deficiency *index* (or sometimes indices) of the operator T_p, and write $def[T_p] = (d_1, d_2)$. It is known that *if the matrix* $A \in J[p]$ *is regular, then*

$$0 \leq d_1,\, d_2 \leq p \qquad \text{(Krein [KD])}, \qquad d_1 = p \Longleftrightarrow d_2 = p \qquad \text{(Kogan [Kog])}.$$

In the converse direction: Dyukarev [D] has recently proved that for any pair of deficiency numbers satisfying

$$0 \leq d_1, d_2 \leq p - 1$$

there is a *regular* generalized Jacobi matrix of the class $J[p]$ such that the determined operator has these deficiency numbers.

3. Basic properties of the multiplicity function

We shall need the following lemmas, which are "unbounded extensions" of some basic results in [Ni2, p.242].

Lemma 1. *Let T be a linear operator in the normed linear space X, and let the (not necessarily closed) subspace E be T-invariant. Then*

$$\mu(T/E) \le \mu(T).$$

Proof. It is immediate to check that

$$D[(T/E)^\infty] = \cap_{n\in\mathbf{N}} D[(T/E)^n] \supseteq D(T^\infty)/E.$$

Hence if $D(T^\infty)$ is dense in X, then $D[(T/E)^\infty]$ is dense in X/E. This proves the stated inequality if one of these density conditions fails. Otherwise let $q : X \to X/E$ denote the canonical quotient mapping. If a subspace $C \subset D(T^\infty)$ is cyclic for T, then the subspace $qC \subset D[(T/E)^\infty] \subset X/E$ is cyclic for T/E, and $\dim qC \le \dim C$. This proves the inequality in this case. $\triangle$

Lemma 2. *Let X be a normed linear space, and the linear operator T be densely defined in X. Then the dual operator T^* exists, and*

$$\mu(T) \ge \sup_{\lambda\in\mathbf{C}} \dim[\ker(T^* - \lambda I)].$$

Proof. We may assume that $\mu(T) < \infty$. Let $E := \overline{TX}$ (the bar denotes closure). By the preceding Lemma, then $\mu(T/E) \le \mu(T) < \infty$. For every $\hat{x}$ in the quotient subspace $\widehat{D(T)} \equiv D(T) + E$ we have $(T/E)\hat{x} = \hat{0}$. Hence $\widehat{D(T)} \subset D[(T/E)^\infty]$, and we obtain that the number $\mu(T/E)$ is equal to the dimension of some dense subspace (in $D[(T/E)^\infty]$) of X/E, i.e., to $\dim(X/E)$. By [Gol, Theorem I.6.4], there is a linear isometry from $(X/\overline{TX})^*$ onto $[\overline{TX}]^\perp$, where $H^\perp$ denotes the annihilator (in the dual space) of the set H. Hence

$$\dim(X/\overline{TX}) = \dim[(X/\overline{TX})^*] = \dim[(\overline{TX})^\perp].$$

By [Gol, Theorem II.3.7], the last annihilator is the kernel of the dual operator T^*, so the preceding Lemma implies

$$\mu(T) \ge \mu(T/E) = \dim(X/E) = \dim[\ker(T^*)].$$

For every $\lambda \in \mathbf{C}$ we clearly have

$$\mu(T - \lambda I) = \mu(T).$$

Hence we obtain the statement of the lemma. $\triangle$

Lemma 3. *Let the Banach space X be the direct sum of the closed subspaces X_1 and X_2, and the linear operator T be the direct sum of the corresponding operators T_1 and T_2. (Equivalently: let X_k reduce T, or: let $P_kT \subset TP_k$ $(k = 1, 2)$ for the corresponding projections). Then*

$$\max[\mu(T_1), \mu(T_2)] \le \mu(T) \le \mu(T_1) + \mu(T_2).$$

Proof. $D(T^\infty)$ is dense in X if and only if both subspaces $D(T_k^\infty)$ are dense in X_k $(k = 1, 2)$. This proves both inequalities if one of the density conditions fails.

Otherwise recall that under the given conditions the quotient T/X_1 is similar to the restriction $T|X_2$: the linear isomorphism of X/X_1 onto X_2 defined by $x + X_1 \mapsto P_2x$ maps $D(T/X_1)$ onto $X_2 \cap D(T)$, and intertwines the operators. This implies the left-hand side inequality.

Let $C_j \in Cyc(T_j)$ be such that $\dim(C_j) = \mu(T_j)$ $(j = 1, 2)$. Then $C := C_1 \oplus C_2$ is in $D(T^\infty)$, and $\dim(C)$=$\dim(C_1)$+$\dim(C_2)$. Further, $T^kC \supset T_j^kC_j$ $(j = 1, 2;\ k \in \mathbf{N}_0)$. Hence $sp[T^kC : k \in \mathbf{N}_0] \supset X_1 \oplus X_2 = X$, and the right-hand side inequality follows. $\triangle$

4. Closed symmetric operators and Jacobi matrices

From now on let the basic space X be a *separable Hilbert space*, and assume that each considered symmetric operator S is *closed and $D(S^\infty)$ is dense in X.*

Remark. It is well known that each *closed symmetric operator* S is the orthogonal direct sum of a maximal selfadjoint part Q and of a *pure (equivalently, simple or prime or completely non-selfadjoint)* closed symmetric part R, i.e. of a restriction R having no selfadjoint part (see, e.g., [AG, Sect. 103] or [KrU, pp.8-9]). The latter source proves also that the direct summand subspace X_Q in which the operator Q acts is

$$X_Q = \cap_{Im\, z \neq 0}[(S - z)D(S)].$$

We shall call the (uniquely determined) orthogonal direct sum $S = Q \oplus R$ *the canonical decomposition of the closed symmetric operator* S.

The basic idea of the following considerations is contained in [Hal, Problem 167].

Theorem 1. *Assume that the multiplicity $\mu(S)$ of the closed symmetric operator S is $m \in \mathbf{N}$. Then there is an orthonormal basis sequence $E \subset D(S^\infty)$ with respect to which the matrix A of the operator S is a generalized Jacobi matrix of class J_m with the property that if in any row $r \geq m+1$ we have $a_{r1} = a_{r2} = \ldots = a_{ru} = 0$, then for every $j \in \mathbf{N}$ we also have*

$$a_{r+j,1} = a_{r+j,2} = \ldots = a_{r+j,u+j} = 0,$$

(i.e. the "whole subdiagonals vanish"). Consequently, there is an $u(r) \in \mathbf{N}_0$ among the column indices for which

$$a_{r1} = \ldots = a_{r,u(r)} = 0, \quad a_{r,u(r)+1} \neq 0.$$

Further, for every $r \geq m+1$ we have $u(r) < r-1$.

Proof. Let $M \in Cyc(S), \dim(M) = m$, and let $\{e_1, \ldots, e_m\} \subset D(S^\infty)$ be an orthonormal basis sequence of M. We shall extend this sequence inductively to an orthonormal basis sequence $E \subset D(S^\infty) \subset X$ having the stated properties. In each inductive step we shall extend the preceding (finite) sequence by either 0 or 1 new vector according to the following rule. Let $d(k)$ denote the cardinality of the finite orthonormal vector sequence $\{e_1, e_2, \ldots, e_{d(k)}\} \subset D(S^\infty)$ constructed after step k. By our assumption,

$$m = d(0) \leq d(1) \leq d(2) \leq \ldots \leq d(k), \qquad 0 \leq d(k) - d(k-1) \leq 1 \qquad (k \in \mathbf{N}).$$

Introduce the notation

$$M(d(k)) := sp[e_1, \ldots, e_{d(k)}],$$

so that $M(d(0)) = M \subset D(S^\infty)$. In the inductive **step** $k \in \mathbf{N}$ we distinguish two cases:

Case 1: if $Se_k \in M(d(k-1))$. Then we define $d(k) := d(k-1)$. Hence we add no new vectors to the sequence $\{e_1, e_2, \ldots, e_{d(k-1)}\}$.

Case 2: if $Se_k \notin M(d(k-1))$. Then we define $d(k) := d(k-1)+1$, and define one new vector with the help of the orthogonal projection $P_{M(d(k-1))}$ onto the subspace $M(d(k-1)) \subset X$:

$$e_{d(k)} := f_{d(k)}/||f_{d(k)}||, \qquad \text{where} \quad f_{d(k)} := [I - P_{M(d(k-1))}]Se_k.$$

Note that if $M(d(k-1)) \subset D(S^\infty)$, then the preceding line guarantees that $M(d(k)) \subset D(S^\infty)$. Further, we always have $Se_k \in M(d(k))$.

We show now that in each step k the vector e_k is already defined, i.e., the inequality

$$k \leq d(k-1) \qquad (k \in \mathbf{N})$$

holds. This will be proved by induction. For $k = 1$ we clearly have $k = 1 \leq m = d(0) = d(k-1)$. Assume now for $k > 1$ that $1 < k \leq d(k-1)$, which is equivalent to

$$e_k \in M(d(k-1)) \equiv sp[e_1, e_2, \ldots, e_{d(k-1)}].$$

Consider the vector Se_k, and assume first that we have Case 2 from above: $Se_k \notin M(d(k-1))$, hence $d(k) = d(k-1)+1$. It follows that $k < d(k)$, thus $k+1 \leq d(k)$, and this case is settled. Assume now that we have Case 1 from above: $Se_k \in M(d(k-1)) = M(d(k))$. If the assumed inequality was sharp: $k < d(k-1)$, then evidently $k+1 \leq d(k)$, and we are done. Assume now the less trivial other case: $k = d(k-1) = d(k)$. We clearly have that

$$1 \leq j \leq k \qquad \text{implies} \qquad d(0) \leq d(1) \leq d(j) \leq d(k).$$

Hence

$$sp[e_1,\dots,e_j,\dots,e_k]=M(d(k)).$$

Since we have $Se_j\in M(d(j))\subset M(d(k))$, we obtain $SM(d(k))\subset M(d(k))$. For every $n\in\mathbf{N}_0$ we have then

$$S^nM=S^nM(d(0))\subset S^nM(d(k))\subset M(d(k)),$$

which contradicts the assumption $M\in Cyc(S)$. Hence this case cannot occur, and the induction proof is complete.

Now let $r\geq m+1$, and consider the step $k\equiv k(r)$ in which the vector e_r has been constructed. Then we have $d(k-1)=r-1$ and $d(k)=r$. For the matrix A (in the constructed basis) it means

$$a_{r1}=a_{r2}=\ldots=a_{r,k-1}=0 \qquad \text{and} \qquad a_{rk}\neq 0.$$

The construction clearly shows that for any pair (k,r) satisfying $k\geq 1$, $r\geq m+1$ the relation $d(k-1)\leq r-1$ is equivalent to

$$a_{r1}=a_{r2}=\ldots=a_{r,k-1}=0.$$

For any $j\geq 1$ we have $d(r+j-1)-d(r-1)\leq j$. It follows that

$$a_{r1}=a_{r2}=\ldots=a_{ru}=0\Longrightarrow a_{r+j,1}=a_{r+j,2}=\ldots=a_{r+j,u+j}=0.$$

This is exactly the stated "vanishing of the whole subdiagonal". Further, since $M\in Cyc(S)$, the process produces an orthonormal *basis* in $D(S^\infty)$ for X.

The possibility $u(r)=0$ means that $a_{r1}\neq 0$. If for some $r\geq m+1$ the number $u(r)\in\mathbf{N}$ did not exist, it would mean that the rth row of the matrix A, hence, by symmetry, the rth column would contain only entries 0. The formally less strict statement $u(r)\geq r$ would lead (in view of the vanishing subdiagonals) to the same conclusion. In both cases we would be in contradiction to the assumption $M\in Cyc(S)$.

If for some $r\geq m+1$ we have $u(r)=r-1$, then A is the direct sum of the matrices of a selfadjoint operator in the $(r-1)$-dimensional space generated by the basis vectors $e_1,e_2,\dots,e_{r-1}$ plus of a symmetric operator in the subspace generated by the basis vectors $e_r,e_{r+1},\dots$, having a diagonal matrix (a direct summand in A). Further, we would have $S^kM\subset sp[e_1,\dots,e_{r-1}]$ for every $k\in\mathbf{N}_0$, which contradicts the assumption $M\in Cyc(S)$.

In any case, the infinite matrix A of the operator S in the orthonormal basis E constructed according to the indicated process is such as stated in the Theorem. △

Remark. The closed symmetric operator S_0 *determined* by the matrix A and the orthonormal basis E in the sense of v.Neumann [NAE], [NT] (see also [AG]), is not necessarily equal to S. Rather we have $S_0\subset S$, the latter is a finite dimensional extension of S_0, and S is *generated* by the pair (A,E) in the terminology of the Definition in Section 2.

Definition. A generalized Jacobi matrix J_m of the type above will be called an (in general: *irregular*) Jacobi matrix *with canonical diagonals.*

Corollary. Assume the situation and notation described in the preceding Theorem. *In the special case, if* the sequence of vectors

$$\{e_1,\dots,e_m,Se_1,\dots,Se_m,S^2e_1,\dots,S^2e_m,\dots\}\subset D(S^\infty)$$

is linearly independent, then the generalized Jacobi matrix J_m with canonical diagonals constructed in the proof of the Theorem will be *regular* , i.e. every entry $a_{m+j,j}$ ($j\in\mathbf{N}$) will be nonzero.

Proof. It follows from the construction process. △

Theorem 2. *Under the conditions and with the notation of Theorem 1 there is a positive integer $d\leq m$ such that the dth diagonal of the matrix A (of the operator S) below the main diagonal has from a certain row on only nonzero entries, and no integer greater than d has this property. It follows that the matrix A is the sum of a regular generalized Jacobi matrix $A(d)\in J_d$ plus of a direct sum of a Hermitian symmetric matrix M in a finite dimensional space and of an infinite zero matrix 0:*

$$A=A(d)+[M\oplus 0].$$

Hence the deficiency indices of the corresponding determined symmetric operators:

$$S_0 = S_0(d) + [S(M) \oplus 0]$$

satisfy $def(S_0) = def[S_0(d)]$, and both deficiency numbers are not greater than d. The (original) closed symmetric operator S is an extension of S_0, hence its deficiency numbers are not greater than d.

Proof. The sequence $\{u(r) : r \geq m+1\} \subset \mathbf{N}_0$ from Theorem 1 clearly has the following properties: it is strictly increasing, $u(m+1) \geq 0$, $\quad u(r) < r-1$ for each $r \geq m+1$. Hence the sequence can have "jumps" greater than 1 at most a finite number of times. Assume that the last such jump occurs for $r = r_0 + 1$, so that $r = u(r) + d + 1$ for each $r > r_0$. Then

$$a_{r,r-d} \equiv a_{r,u(r)+1} \neq 0 \qquad \text{if} \quad r > r_0,$$

i.e. the dth diagonal below the main one in A contains only nonzero entries below row r_0. Define now the matrix $A(d)$ to have the same entries as A in the main and in the by-diagonals $1, 2, \ldots, d$ below and above the main diagonal, except the new definitions:

$$a_{r,r-d} = a_{r-d,r} := 1 \qquad \text{if} \quad d < r \leq r_0,$$

and to have entries 0 everywhere else. Note that $A(d)$ is then a *regular* J_d matrix. Define further

$$B := A - A(d).$$

Then B has nonzero entries at most in rows and columns $1, 2, \ldots, r_0$, and is clearly Hermitian symmetric. Defining M to be the leading principal minor of order r_0 of B, we obtain the stated decomposition of the matrix A. The symmetric operator determined by $M \oplus 0$ is defined on the whole space, hence is selfadjoint. This implies the stated decomposition of the determined symmetric operator S_0. Since $S(M) \oplus 0$ is a bounded selfadjoint operator, [AG, 100°] shows that $def(S_0) = def[S_0(d)]$. Since $S_0(d)$ is determined by a *regular* J_d matrix, the penultimate statement follows from [KD]. The last sentence is then evident. $\triangle$

Now we want to clarify the relationship between the operators determined by certain block Jacobi matrices of the type $J[p]$ and J_p. Introduce the following

Notation: Let E, F be basis sequences in the finite dimensional subspaces $sp[E], sp[F]$ in the Hilbert space X, and let $T : sp[E] \to sp[F]$ be a bounded linear operator. We shall denote the matrix of T with respect to the bases E, F by $[T; E, F]$. Entrywise we have then

$$[T; E, F]_{ik} := (Te_k, f_i).$$

Theorem 3. *Let $p \in \mathbf{N}$. Assume that the closed symmetric operator S is determined by the infinite block Jacobi matrix $K \in J[p]$ with respect to the orthonormal basis $\{e_1, e_2, \ldots\}$. Then there is an orthonormal basis $\{f_1, f_2, \ldots\}$ such that S is determined by a generalized infinite Jacobi matrix $J \in J_p$ with respect to this new basis. It means that J has nonzero entries at most in the main diagonal and in the by-diagonals $1, 2, \ldots, p$ above and below the main diagonal. If every by-diagonal block in the matrix K has nonzero determinant, then we can achieve that both pth by-diagonals in J contain only nonzero complex entries.*

Proof. Let $f_1 := e_1, f_2 := e_2, \ldots, f_p := e_p$, and consider the block $B \equiv B_{21}$ (the block entry (2,1) in the matrix K). Consider the following orthonormal sequences in the Hilbert space X:

$$E_1 := \{e_1, \ldots, e_p\},\ E_2 := \{e_{p+1}, \ldots, e_{2p}\},\ G := \{g_1, \ldots, g_p\},$$

where the sequence G is an orthonormal basis in the subspace spanned by E_2, and let P_k denote the orthogonal projection of X onto the subspace spanned by E_k. We shall determine G so that the matrix $[S_{21}; E_1, G]$ of the operator $S_{21} := P_2(S|P_1X)$ (mapping the subspace generated by E_1 to the subspace generated by G) be upper triangular (with respect to the indicated orthonormal basis sequences). It is clear that, with the notation introduced above,

$$[S_{21}; E_1, G] = [I; E_2, G][S_{21}; E_1, E_2] = [I; E_2, G]B,$$

where I denotes the identity operator in the subspace generated by E_2 (or, equivalently, by G).

By [Gant, IX.7], there is a $p \times p$ unitary matrix V such that the matrix $C := VB$ is upper triangular. V (and then C) are determined up to a multiplying diagonal matrix with diagonal entries of moduli 1 if $det(B) \neq 0$, and then $|det(C)| = |det(B)| \neq 0$, hence the diagonal entries of C are nonzero. In any case, fix such a V, (hence the corresponding C), and determine G so that $[I; E_2, G] = V$. Equivalently, we require that

$$V^* = V^{-1} = [I; G, E_2] = [g_1, g_2, \ldots, g_p],$$

where the last $p \times p$ matrix consists of the components of the column vectors $g_1, g_2, \ldots, g_p$ of the basis G (with respect to the basis E_2). With this basis G the matrix

$$[S_{21}; E_1, G] = [I; E_2, G]B = [g_1, g_2, \ldots, g_p]^* B$$

is then upper triangular, and if $det(B) \neq 0$, then the diagonal entries of the left-hand side matrix are nonzero. Define now

$$f_{p+k} := g_k \quad (k = 1, 2, \ldots, p).$$

In the next step of the process we consider the matrix of the operator

$$S_{32} : \{sp[F_2] \equiv sp[G] \equiv sp[E_2]\} \to sp[E_3], \qquad S_{32} := P_3(S|P_2 X)$$

where the orthonormal sequences F_2 and E_3 are defined by

$$F_2 := \{f_{p+1}, \ldots, f_{2p}\}, \quad E_3 := \{e_{2p+1}, \ldots, e_{3p}\},$$

and proceed exactly as before to achieve that the matrix $[S_{32}; F_2, G_2]$ be upper triangular with respect to a suitable orthonormal basis G_2. Continuing the process inductively, we obtain a sequence of orthonormal bases (for the subspaces)

$$F := \{F_1, F_2, \ldots\} = \{f_1, \ldots, f_p, f_{p+1}, \ldots, f_{2p}, \ldots\}.$$

We take the sequence F as the new basis in the statement of the theorem. It is easy to check that both orthonormal bases: F and the original $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{2p}, \ldots\}$ are *bases of the matrix representation for the same closed operator* S in the sense of [AG, 53°]: the vectors of both bases are in $D(S^\infty)$, and S is the closure of the linear operator defined on finite linear combinations of the basis vectors (*whichever basis we may take*). The proof is complete. △

Corollary. Let $p \in \mathbf{N}$. If the *regular block* infinite Jacobi matrix of class $J[p]$ (with respect to some orthonormal basis E) determines the closed symmetric operator S, then there is a *regular generalized* Jacobi matrix of class J_p (with respect to another orthonormal basis F) determining S, and we have

$$\mu(S) \leq p.$$

Proof. The subspace $C := sp[f_1, \ldots, f_p]$ is clearly in $Cyc(S)$. △

Question. Can $\mu(S) < p$ happen? Let $S := S_1 \oplus S_2$ be the direct sum of the selfadjoint operators S_j of multiplication by the variable $t \in [j-1, j]$ in the Hilbert spaces $H_j := L^2([j-1, j])$, $(j = 1, 2)$. There are orthonormal basis sequences e^1, e^2 in the spaces such that S_j is determined by a regular Jacobi matrix (with respect to the basis e^j, cf. [St, Theorem 7.13]). Then S is determined by a regular generalized Jacobi matrix of class J_2 with respect to the amalgamated basis $\{e_1^1, e_1^2, e_2^1, e_2^2, \ldots\} \subset H_1 \oplus H_2$ (cf. [AG, 86°]). However, the multiplicity of S is clearly 1.

Remark. Let $p \in \mathbf{N}$. The question naturally arises, whether each *irregular* generalized Jacobi matrix of class J_p [or, equivalently, whether each *irregular* block Jacobi matrix of class $J[p]$] has the property that the multiplicity of the determined symmetric operator S is finite (possibly $\mu(S) \leq p$). The answer is, as the following simple argument shows, no.

For every $C \in Cyc(S)$ in the Hilbert space X we have

$$X = sp[S^n C : n \in \mathbf{N}_0] \subset sp[C \cup rg(S)].$$

Consider a generalized Jacobi matrix of class J_p that has an infinity of zero columns (and, by symmetry, an infinity of the corresponding zero rows). The corresponding elementary linear operator T (defined on vectors of finite type) has its range in the orthogonal complement of the subspace generated by the basis vectors corresponding to zero columns (or, equivalently, rows). S is the closure of T, hence the range of S is in the closure of $rg(T)$, consequently in the orthogonal complement above. It follows that no finite dimensional subspace C can satisfy the necessary condition above of belonging to the family $Cyc(S)$. △

5. On a question of Hamburger and Akhieser

We start from an infinite Hermitian symmetric Jacobi matrix J of the class J_1 defined by

$$J := \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & \dots \\ b_1 & a_2 & b_2 & 0 & 0 & \dots \\ 0 & b_2 & a_3 & b_3 & 0 & \dots \\ \dots & \dots & & & & \end{pmatrix},$$

where a_j is real, $b_j > 0$ $(j = 1, 2, \dots)$, and denote the corresponding orthonormal basis in the Hilbert space H by $\{e_j : j = 1, 2, \dots\}$, cf., e.g., [A, p. 10].

The operator $T \equiv T(J)$ is defined on the basis vectors by

$$Te_k := b_{k-1}e_{k-1} + a_k e_k + b_k e_{k+1} \qquad (k = 1, 2, \dots;\ b_0 := 0).$$

If $g = \sum x_k e_k$ is a *finite* linear combination, then extend the definition of T by linearity. We obtain for any pair of vectors f, g of such a finite type

$$(Tf, g) = (f, Tg).$$

Since the set F of such vectors is dense in H, the operator T with domain $D(T) := F$ is symmetric. Hence T is closable, and we denote its closure $\overline{T}$ by S. It is known that any vector $h = \sum_{k=1}^{\infty} x_k e_k$ is in the domain of the adjoint operator $S^* = \overline{T}^*$ if and only if

$$\sum_{k=1}^{\infty} |x_k|^2 < \infty, \qquad \sum_{k=1}^{\infty} |b_{k-1}x_{k-1} + a_k x_k + b_k x_{k+1}|^2 < \infty,$$

and then

$$S^*h = \sum_{k=1}^{\infty} [b_{k-1}x_{k-1} + a_k x_k + b_k x_{k+1}]e_k.$$

Further, the deficiency index $def(S)$ of the (closed symmetric) operator S is either (0,0) or (1,1). The former is the case exactly when the matrix J has the type D (the *limit point* case), the latter is exactly when J has the type C (the *limit circle* case, see, e.g., [A, Ch.I,3]). Clearly, S is *the operator determined by the infinite Jacobi matrix J and the orthonormal basis $\{e_n\}$*.

The vector e_1 is in $D(S^\infty)$, and the linear hull $L(S^k e_1 : k = 0, 1, 2, \dots)$ is equal to $L(e_k : k = 1, 2, \dots)$, hence is dense in H. It follows that $\mu(S) = 1$. For the case $def(S) = (0, 0)$ M.H.Stone [St, Theorem 7.13] proved the following converse:

Theorem (Stone). If an (in general, unbounded) selfadjoint operator S in a separable Hilbert space H has simple spectrum, then it is determined by some infinite Jacobi matrix J (with respect to some orthonormal basis) of type D.

A proof can also be found in [A, Ch.IV,2]. Recall that a selfadjoint operator S has *simple spectrum* if and only if $\mu(S) = 1$, i.e., there is a vector $v \in H$ such that v is in $D(S^\infty)$, and $sp[S^k v : k = 0, 1, 2, \dots] = H$. In this case it is customary to say that *the operator S is cyclic with cyclic vector v*.

H.L.Hamburger in [H] raised and answered the following question: when is a closed symmetric prime (or, equivalently, pure) operator of deficiency index (1,1) determined by an infinite Jacobi matrix J (necessarily of type C)? [Note that [A, Theorem 4.2.4] shows that if a closed symmetric operator S is determined by a Jacobi matrix of class J_1 of type C, then S is pure.] Hamburger's answer in [H] contains another known necessary condition, and is in (duly complicated) analytic terms. It may be interesting that an answer in general Hilbert space terms can be given (without explicitly assuming primeness), and can be formulated in a similar way to Stone's above result.

Theorem 4. *If a closed symmetric operator S (in a separable Hilbert space H) has $def(S) = (1,1)$ and $\mu(S) = 1$, then it is determined by some regular Jacobi matrix J (with respect to some orthonormal basis) of type C and such that a_j is real, $b_j > 0$ $(j = 1, 2, \ldots)$.*

Proof. Let v be a cyclic vector for S, i.e. $L(S^k v : \ k = 0, 1, 2, \ldots)$ be dense in H. Apply the orthonormalizing process to the sequence $\{S^k v : \ k = 0, 1, 2, \ldots\}$, obtaining the sequence $\{e_1 := v/|v|, e_k : k = 2, 3, \ldots\}$. Since v is a cyclic vector, this latter sequence is an orthonormal *basis* for H, and is contained in $D(S^\infty)$. Consider the subspaces

$$E_N := L(e_1, e_2, \ldots, e_N) = L(S^k v : \ k = 0, 1, 2, \ldots, N-1), \qquad (N \in \mathbf{N}).$$

We have $SE_N \subseteq E_{N+1}$, but $S^N v \notin L(S^k v : \ k = 0, 1, 2, \ldots, N-1) = E_N$ for every $N = 1, 2, \ldots$ (otherwise the Hilbert space H would be finite dimensional). Hence we have the *strict inclusions*

$$SE_N \subset E_{N+1} \quad (N = 1, 2, \ldots). \qquad\qquad (*)$$

Since S is symmetric, this implies for $j > k+1$

$$a_{kj} = (Se_j, e_k) = (e_j, Se_k) = 0.$$

Using $(*)$ again, for $j < k-1$ we obtain

$$a_{kj} = (Se_j, e_k) = 0.$$

Hence for every $k = 1, 2, \ldots$

$$Se_k = a_{k-1,k} e_{k-1} + a_{k,k} e_k + a_{k,k+1} e_{k+1} \qquad (e_0 := 0).$$

Further, we have

$$a_{k-1,k} = (Se_k, e_{k-1}) = (e_k, Se_{k-1}) = \overline{(Se_{k-1}, e_k)} = \overline{a_{k,k-1}} \neq 0, \qquad a_{k,k} = (Se_k, e_k) \in \mathbf{R}.$$

Introducing the shortened (usual) notation

$$a_k := a_{k,k}, \qquad b_k := a_{k,k+1} \qquad (k = 1, 2, \ldots),$$

we obtain the entries of an infinite Jacobi matrix A. Through the method outlined by Hamburger [H, p.502], we can slightly change the orthonormal basis, and obtain (with respect to this) the entries of the infinite Jacobi matrix J, where each b_j is positive.

Consider now the symmetric linear operator $T \equiv T(J)$ defined at the beginning of this Section with domain $D(T) = F$, the linear manifold of all finite linear combinations of the basis vectors e_k. Since

$$Te_k = b_{k-1} e_{k-1} + a_k e_k + b_k e_{k+1} = Se_k \qquad (k = 1, 2, \ldots; \ b_0 := 0),$$

T is equal to S restricted to F, hence the closure $\overline{T}$ has the closed symmetric extension S. Since $\overline{T}$ is the closed symmetric operator determined by the matrix J, it has the deficiency index either $def(\overline{T}) = (0,0)$ or else $def(\overline{T}) = (1,1)$, according as the matrix J is of type D or else of type C (cf. [A, Ch. IV,1-2]). In the first case $\overline{T}$ is selfadjoint having the closed symmetric extension S of deficiency index $(1,1)$, which is impossible. In the second case $\overline{T}$ has the deficiency index $(1,1)$, and has the closed symmetric extension S of deficiency index $(1,1)$. Hence $S = \overline{T}$ is the closed symmetric operator determined by the infinite Jacobi matrix J, which is in this case necessarily of type C. $\triangle$

6. The multiplicities of pure maximal symmetric operators

In this section we shall denote by $S(m,n)$ a densely defined closed symmetric operator with deficiency index $(m,n) \in \mathbf{N} \times \mathbf{N}$.

Theorem 5. *If the deficiency index of the closed symmetric operator $S \equiv S(m,n)$ is (m,n), then $\mu(S) \geq \max(m,n)$. There exists a closed symmetric operator $S(m,n)$ such that*

$$\mu[S(m,n)] \leq \max(m,n) + 1.$$

If a closed symmetric operator is pure, then we denote it by S_p, and we have

$$\mu[S_p(0,n)] = n+1, \qquad \mu[S_p(m,0)] = m+1, \qquad \text{hence} \quad \mu[S_p(m,0) \oplus S_p(0,n)] \leq m+n+2.$$

Proof. Assume that the deficiency index of the symmetric operator S is (m,n), and let $M \equiv M(m,n) := \max(m,n)$. By Lemma 2, we have then

$$\mu(S) \geq \sup_{\lambda \in \mathbf{C}} \dim[\ker(S^* - \lambda I)] = M(m,n).$$

By Dyukarev's result ([D, Theorem 1]), there exists an infinite *regular* Hermitian symmetric block Jacobi matrix of class $J[M+1]$ such that the determined (closed symmetric linear) operator T_{M+1} has the deficiency index (m,n). By Lemma 2 and by Corollary to Theorem 3, we have then

$$M(m,n) \leq \mu(T_{M+1}) \leq M(m,n) + 1,$$

which proves the second sentence in the Theorem. Note that the symmetric operator T_{M+1} is not necessarily pure.

Consider now the case when $m = 0, n \geq 1$, and the *pure* symmetric operator $S_p \equiv S_p(0,n)$ has the indicated deficiency index. Then a corresponding closed symmetric operator (via Dyukarev's cited result again) T_{n+1} has the same deficiency index $(0,n)$, and has the canonical decomposition

$$T_{n+1} = Q \oplus R_p(0,n),$$

where Q is selfadjoint, and $R_p(0,n)$ is a *pure* closed symmetric operator with the given deficiency index. Hence we obtain

$$n \leq \mu[R_p(0,n)] \leq \mu[T_{n+1}] \leq n+1.$$

Consider the possibility $\mu[R_p(0,n)] = n$. Then, by Theorem 1, there is an orthonormal basis with respect to which the matrix A of the operator $R_p(0,n)$ is a generalized Jacobi matrix of type J_n. By Theorem 2 (applying also the notation there), there is a positive integer $d \leq n$ such that the matrix A is the sum of a *regular* generalized Jacobi matrix $A(d) \in J_d$ plus of a direct sum:

$$A = A(d) + [M \oplus 0].$$

The deficiency numbers of the symmetric operator $S_0(d)$ determined by $A(d)$ are, by [KD], not greater than d. The deficiency numbers of the symmetric operator S_0 determined by A itself satisfy $def[S_0] = def[S_0(d)]$, hence they are not greater than d. The operator $R_p(0,n)$ is *generated by* A, hence is an extension of the operator S_0 (which is *determined by* A). It follows that both deficiency numbers of $R_p(0,n)$ are not greater than d. This implies $n \leq d$, hence $d = n$.

We have then the following inequalities for the deficiency indices (understood componentwise):

$$(0,n) = def[R_p(0,n)] \leq def[S_0] = def[S_0(d)] \leq (d,d) = (n,n).$$

The operator $S_0(d) \equiv S_0(n)$ is determined by the *regular* generalized Jacobi matrix $A(n) \in J_n$. By the already cited result of Kogan [Kog], one of its deficiency numbers can be equal to n if and only if both are. Hence

$$def[S_0] = def[S_0(n)] = (n,n).$$

However, the symmetric operator $R_p(0,n)$ is an extension of the symmetric operator S_0, hence for their deficiency indices we should have

$$(0,n) = (n-k, n-k) \qquad \text{for some} \qquad k \in \mathbf{N}_0,$$

a contradiction. By a result of v.Neumann, $S_p(0,n)$ and $R_p(0,n)$ are unitarily equivalent (see, e.g., [AG, 104°]). Hence we have obtained

$$\mu[S_p(0,n)] = \mu[R_p(0,n)] = n+1.$$

In a completely similar way we obtain that the multiplicity of the pure symmetric operator with deficiency index $(m,0)$, where $m \geq 1$, is $m+1$. Hence, if the symmetric operator $S \equiv S(m,n)$ is the orthogonal sum

$$S(m,n) = S_p(m,0) \oplus S_p(0,n),$$

then we have

$$\mu[S(m,n)] \leq m+n+2.$$

$\triangle$

Corollary. Each closed symmetric operator S satisfying $def(S) = (1,1)$ and $\mu(S) = 1$ is *not the direct sum* of elementary symmetric operators:

$$S \neq S_p(1,0) \oplus S_p(0,1).$$

Further, the n-fold orthogonal sum $S^{(n)} := S \oplus \ldots \oplus S$ satisfies

$$\mu[S^{(n)}] = n.$$

Hence, for every $n \in \mathbf{N}$ there is $S(n,n)$ such that $\mu[S(n,n)] = n$.

Proof. Assuming that S is the (negated) direct sum above, we should have

$$1 = \mu(S) \geq \mu[S_p(1,0)] = 2,$$

a contradiction. Further, the deficiency index of $S^{(n)}$ is clearly (n,n). Hence, by Lemma 3,

$$n = \max(n,n) \leq \mu[S^{(n)}] \leq n\mu(S) = n.$$

$\triangle$

Now we shall determine the matrix of the simplest structure of the elementary symmetric operator $S \equiv S_p(0,1)$ (which is necessarily pure, cf. [AG, 104°, Theorem 1]) defined by v.Neumann in [NAE]. We shall cite his representations in the form of the

Lemma *(v.Neumann [NAE, p. 130]). The closed symmetric operator S is unitarily equivalent to each one of the following operators T:*

1° $\qquad D(T) := \{x \equiv \{x_k\} \in l^2(\mathbf{N}) : |x_1|^2 + |x_1+x_2|^2 + |x_1+x_2+x_3|^2 + \ldots < \infty\},$

$$Tx := i\{x_1, 2x_1 + x_2, 2x_1 + 2x_2 + x_3, \ldots\}.$$

(Interestingly, for any $x \in D(T)$ we have $\sum_{k=1}^{\infty} x_k = 0$.)

2° Let $k \in \mathbf{Z}$. In $L^2(0,1)$ let $X \equiv H_k$ denote one of the closed linear subspaces generated by all the functions $\{x_n(t) := e^{2n\pi it} : \ n \in \mathbf{Z}, \ n \geq k\}$. Let

$$D(T) := \{f \in X : \ -cot(\pi t)f(t) \in X\}, \quad (Tf)(t) := -cot(\pi t)f(t).$$

3° $\qquad D(T) := \{f \in H^2(\mathbf{D}) : \ i\frac{z+1}{z-1}f(z) \in H^2(\mathbf{D})\}, \quad (Tf)(z) := i\frac{z+1}{z-1}f(z).$

Here $H^2(\mathbf{D})$ denotes the indicated Hardy space of the disc.

4° $\qquad D(T) := \{f \in L^2(0,\infty) : f \text{ is locally absolutely continuous}, \ f' \in L^2(0,\infty), \ f(0) = 0\}, \qquad Tf := if'.$

Theorem 6. *The multiplicity of the closed symmetric operator S is $\mu(S) = 2$. In the representation 4° the functions f, $g \in D(T^\infty)$ defined by*

$$f(t) := e^{-t-\frac{1}{t}}, \qquad g(t) := tf(t) = te^{-t-\frac{1}{t}}$$

satisfy

$$sp[\{T^k f : k \in \mathbf{N}_0\} \cup \{T^k g : k \in \mathbf{N}_0\}] = X = L^2(0,\infty).$$

Proof. We have obtained above $\mu(S) = \mu[S_p(0,1)] = 2$.

Consider now for S the representation 4°, and the functions f, g defined above. For their successive derivatives it can be proved by induction that

$$f^{(k)}(t) = \frac{P_{2k}(t)f(t)}{t^{2k}}, \quad g^{(k)}(t) = \frac{Q_{2k}(t)f(t)}{t^{2k-1}} \qquad (k \in \mathbf{N}),$$

where P, Q are polynomials of the indicated degree and such that

$$P_{2k}(0) = Q_{2k}(0) = 1,$$

i.e. their lowest degree terms are 1. It follows that f, $g \in D(T^\infty)$.

Assume now that there is $h \in L^2(0,\infty)$ such that for every $n = 0, 1, 2, \ldots$ we have

$$\int_0^\infty f^{(n)}(t)h(t)dt = 0 = \int_0^\infty g^{(n)}(t)h(t)dt.$$

With the shortening

$$[f, n] := \int_0^\infty f^{(n)}(t)h(t)dt$$

we shall write the equation above as

$$[f, n] = 0 = [g, n] \qquad (n = 0, 1, 2, \ldots).$$

We state that this implies

$$\int_0^\infty t^j f(t)h(t)dt = 0 \qquad (j = 1, 0, -1, -2, -3, \ldots). \qquad (*)$$

Indeed, $[g, 0] = 0$ is (*) for $j = 1$, and $[f, 0] = 0$ is (*) for $j = 0$. Knowing these, $[g, 1] = 0$ implies (*) for $j = -1$, and then $[f, 1] = 0$ implies (*) for $j = -2$. Continuing in this way, we obtain successively (*) for $j = -3, -4, -5, \ldots$.

Applying now the substitution $t = x^{-1}$, (*) implies

$$\int_0^\infty x^k e^{-x-\frac{1}{x}} h(\frac{1}{x})dx = 0 \qquad (k = -3, -2, -1, 0, 1, 2, \ldots). \qquad (**)$$

For any b satisfying $0 < b < 1$ we have

$$\int_0^\infty e^{bx} f(x)|h(\frac{1}{x})|dx = \int_0^\infty |h(t)|e^{(b-1)\frac{1}{t}} e^{-t} t^{-2} dt < \infty,$$

since the integrand on the right-hand side is the product of two functions from $L^2(0,\infty)$. This fact together with (**) implies

$$\int_0^\infty x^k y(x)dx = 0 \qquad (k = -3, -2, -1, 0, 1, 2, \ldots),$$

where the function

$$y(x) := f(x)h(\frac{1}{x})$$

has been seen to have the property that for $0 < b < 1$ the function $x \mapsto e^{bx}y(x) \in L^1(0,\infty)$. A more or less standard reasoning concerning Fourier transforms in the complex domain (i.e. the Paley-Wiener circle of ideas) shows that then (cf., e.g., [KF, 8.4.2, 8.4.3]) $y = 0$ a.e. Hence $h = 0 \in L^2(0,\infty)$, which proves our claim. $\triangle$

We prove now the following

Theorem 7. *Apply the notation of the preceding proof, and consider the functions $f, g \in X := L^2(0,\infty)$ from there. Then the following sequence of functions:*

$$\{g, f, Tg, Tf, T^2g, T^2f, \ldots\} \qquad (+)$$

forms a linearly independent set. Hence the orthonormalized sequence of the above sequence is a basis with respect to which the matrix of the operator T is a regular J_2 matrix: it has nonzero entries only in the main and in the two neighboring diagonals in both directions, and the two indicated extreme diagonals contain exclusively nonzero entries.

Proof. In this proof we shall call an expression of the type

$$\sum_{k=-N}^{N} a_k t^k \quad (N \in \mathbf{N}, a_k \in \mathbf{C})$$

a *two-sided polynomial* or simply a polynomial in t over $\mathbf{C}$. We define the *positive degree* of a polynomial as the largest $k \geq 0$ such that $a_k \neq 0$, and the *negative degree* as the smallest $-j \leq 0$ such that $a_{-j} \neq 0$.

We have seen in the preceding proof that each derivative of g, f is a product of f and of a (two-sided) polynomial of t. It is immediate that a set of such functions is linearly independent in X if and only if the corresponding set of polynomials (after division by f) is linearly independent in $C(0,\infty)$. Denote the corresponding sequence of polynomials (in the order of (+)) by $\{p_1, p_2, p_3, \ldots\}$. We obtain that

$$p_1(t) = t,\ p_2(t) = 1,\ p_3(t) = -t + 1 + t^{-1},\ p_4(t) = -1 + t^{-2}.$$

In general, the negative degree of p_{2k+1} is $-2k+1$ for $k \in \mathbf{N}$, and its positive degree is always 1. The negative degree of p_{2k} is $-2k+2$ for $k \in \mathbf{N}$, and its positive degree is always 0. It follows that any "starting section" $\{p_1, p_2, \ldots, p_n\}$ of the sequence of these polynomials is a linearly independent set, which proves that the whole set (+) is.

Further, it is clear that

$$T(\{p_1, p_2, \ldots, p_n\}) \subset sp[p_1, p_2, \ldots, p_{n+2}] \qquad (n \in \mathbf{N}).$$

Since the orthonormalization process does not change the linear span of the starting sections, an application of the Corollary to Theorem 1 proves now the last sentence of the Theorem. $\triangle$

7. The completely indeterminate case revisited

It is a more or less generally accepted expression that for a (possibly block, Hermitian symmetric infinite) Jacobi matrix $A \in J[p]$ or, equivalently, for the determined minimal closed operator S in the Hilbert space l^2_p (sequences of vectors from $\mathbf{C}^p$ with square-summable sequences of norms) we have *the completely indeterminate case* iff the deficiency index of the operator S is (p, p). Kostyuchenko and Mirzoev claimed in [KM1], [KM2] that this is the case if and only if all solution vectors u of the equation $A \cdot u = 0$ belong to the space l^2_p. The following simple example shows that this claim is false.

Example. Let $p = 1$ and consider the regular Jacobi matrix $A \in J_1$ with zero main diagonal, and two identical by-diagonals with the entries $(1, 2, 2, 3, 3, 4, 4, \ldots)$ (in this order). Each solution vector u of the equation $A \cdot u = 0$ is a multiple of the vector

$$u = (1, 0, -1/2, 0, 1/3, 0, -1/4, 0, 1/5, 0, \ldots) \in l^2 \equiv l^2_p.$$

On the other hand, the reciprocals of the entries in both by-diagonals form a clearly divergent series. It is well known that this property of any Jacobi matrix with real main and positive by-diagonals implies that the determined (minimal closed) operator S in l^2 is self-adjoint (see, e.g., [A,Chap.1]). Hence $def(S) = (0,0) \neq (p,p)$, i.e. we do *not* have the completely indeterminate case. $\triangle$

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