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Orthonormal Jordan bases in finite dimensional Hilbert spaces

1. Introduction and terminology

The aim of this paper is twofold: first, we give a necessary and sufficient condition for a linear operator in a finite dimensional complex or real Hilbert space to have a *Jordan form in an orthonormal basis.* Recall that Veselic [V] established such a condition for the special case of a nilpotent operator by an elegant reasoning. Second, since this characterization is not always simple to check, we give necessary conditions with the help of the (self-adjoint) self-commutator operators.

At first some *notation and terminology.* In a finite dimensional Hilbert space H with scalar product $<,>$ the symbol $L(H)$ will denote the space of linear operators in H, $|\cdot|$ the norm of a vector or of a linear operator, and $T|S$ the restriction of the operator T to the linear subspace S. $M_n(\mathbf{C})$ denotes the space of $n\times n$ matrices over $\mathbf{C}$, and $\mathbf{N}_0$ the set of all nonnegative integers. We shall use the notation of the *commutator* $[A,B]:=AB-BA$ for an ordered pair of *operators* and also for an ordered pair of *matrices.* $\oplus$ will denote direct sum of spaces or operators, and we shall explicitly say if it is orthogonal.

We recall in a very simple case the definition of a *multiset.* Let A be a finite set, and $m:A\to\mathbf{N}_0$ be a function. The pair (A,m) is called a *multiset* with *set of elements* (equivalently: basic set) A and *multiplicity function* $m.$ The basic set can be written as $A=\{a_1,a_2,\dots,a_k\}$, and the multiset as

$$(A,m)=\{[a_1,m(1)],\dots,[a_k,m(k)]\}.$$

As a natural example we can consider the basic set $\sigma(T)$ of the spectrum of a self-adjoint operator T in a finite dimensional Hilbert space with the generally used concept $m(T)$ of multiplicity function there. The corresponding multiset $s(T):=(\sigma(T),m(T))$ can also be called the *spectral list* of T, and yields much more information than the basic set $\sigma(T)$.

It will be convenient to agree that (for any n-dimensional space H, ordered bases e,f, and linear operator $T\in L(H)$) we denote the $e-f$ basis representation (matrix) of T by

$${}_f[T]_e:=[[Te_1]_f,\dots[Te_n]_f]],$$

where $[Te_k]_f$ is the column of the coordinates of the vector Te_k in the basis f (cf., e.g., [HJ, pp. 39-40]). Further, the shorter notation $[T]_b$ will denote the same as ${}_b[T]_b$.

Let $<,>$ be a fixed scalar product (i.e. a bilinear form on a real, or a sesquilinear form conjugate-linear in the *second* variable on a complex H). Note that if e is an ordered basis, and E is the Gram matrix of e with respect to the scalar product $<,>$, defined entrywise by

$$e_{jk}:=<e_k,e_j>,$$

then (see, e.g., [D, pp. 163-164]), we have

$${}_e[T^*]_e=E^{-1}({}_e[T]_e)^hE.$$

Here and in what follows * will denote the adjoint of an *operator*, and ^{h} denote the conjugate transpose of a *matrix.* Introducing the notation

$$M:={}_e[T]_e,$$

we obtain for the *self-commutator operator* $[T,T^*]$ of T

$${}_e[TT^*-T^*T]_e=ME^{-1}M^hE-E^{-1}M^hEM.$$

If the ordered basis e is *orthonormal*, then the corresponding Gram matrix E is the identity, and the preceding equality simplifies to

$${}_e[TT^*-T^*T]_e=MM^h-M^hM,$$

where $M^h={}_e[T]_e{}^h={}_e[T^*]_e$. If the ordered basis e is, in addition, a Jordan basis for the operator T, then M is a *Jordan matrix* for T.

The symbol $s(\cdot)$ will denote, as above, *the multiset or spectral list of the spectrum* of the self-adjoint operator or matrix $\cdot$, i.e. a finite list of numbers with possible repetitions. For any ordered basis e we have then

$$s([T,T^*]) = s({}_e[T,T^*]_e) = s(ME^{-1}M^hE - E^{-1}M^hEM).$$

If the basis e is *orthonormal*, then $s([T,T^*]) = s([M,M^h])$. In words: the spectral list of the *operator* $[T,T^*]$ is then equal to the spectral list of the *matrix* $[M,M^h]$.

We shall say that $T \in L(H)$ *has an orthonormal Jordan basis* if H has an orthonormal basis in which the matrix of T is a (complex or real) Jordan matrix. Hence, if there exists an *orthonormal Jordan basis* e for T, the preceding equality holds with $M :=_e [T]_e$, where M is a Jordan matrix.

We know that every operator $T \in L(H)$ is the (uniquely determined) sum of a simple structure (in a complex space: diagonal) operator S plus a nilpotent operator N commuting with S. We shall call this *the Jordan-Dunford decomposition of* T (valid in both complex and real spaces).

Recall that $J \in L(H)$ is a *partial isometry* iff the operator J^*J is an orthogonal projection. J is called a *power partial isometry (p.p.i.)* iff each nonnegative integer power of J is a partial isometry.

2. The case of a complex Hilbert space

Let $T \in L(H)$, where H is a Hilbert space over $\mathbf{C}$. The problem of when an *orthonormal* basis b exists such that ${}_b[T]_b$ is a Jordan matrix is clearly equivalent to the problem of when a matrix representation ${}_e[T]_e$ of T in an orthonormal basis e is unitarily similar to a Jordan matrix. The problem was solved by Veselic [V] for the case when T is a nilpotent operator (using ideas, among others, of Ptak [Pt] and of Halmos and Wallen [HW]). A solution to the general question is the following:

Theorem 1. *Consider the Jordan-Dunford decomposition $T = S + N$ of the operator T, where S is the diagonal ($\equiv$ scalar-type spectral) and N is the nilpotent part of T (which commutes with S). There is an orthonormal basis b such that $[T]_b$ is a Jordan matrix if and only if N is a power partial isometry, and the operator S is normal.*

Proof. If $[T]_b$ is a Jordan matrix, then $[N]_b$ is clearly a Jordan matrix for the operator N. If b is, in addition, orthonormal, then it is an orthonormal basis for N. Veselic [V, Theorem 2] shows that then N is a power partial isometry. $[S]_b$ is a diagonal matrix, and b is orthonormal, hence the operator S is normal.

In the converse direction we shall need several important results of Veselic's paper [V], which we cite here as Facts. Note that, by [V], they are valid both in complex and in real Hilbert spaces.

Fact 1. Let X be a finite dimensional Hilbert space and $N \in L(X)$ satisfy for some $p \in \mathbf{N}, p > 1$

$$N^p = 0, \quad |N| = |N^{p-1}| = 1.$$

(Note that this implies $|N^2| = \cdots |N^{p-2}| = 1$.) Let

$$r := \dim span\{x \in X : |N^{p-1}x| = |x|\}.$$

Then $r > 0$, and there is an N-reducing orthogonal decomposition $X = X_1 \oplus X_0$ such that X_1 has an orthonormal basis

$$\{e_{ij}; \qquad i = 1, \dots, p, \quad j = 1, \dots, r\}$$

such that for every j we have

$$Ne_{ij} = e_{i-1,j} \quad (i > 1), \qquad Ne_{1j} = 0.$$

Fact 2. If $N \in L(X)$ is a power partial isometry satisfying $N^p = 0$, $N^{p-1} \neq 0$ for some $p \in \mathbf{N}, p > 1$, then $|N| = |N^{p-1}| = 1$. Further, the restriction $N|Y$ to *any* N-reducing subspace Y is again a p.p.i.

To start the proof of the converse direction, let $N \in L(H)$ satisfy the assumptions of Fact 2, and let the normal $S \in L(H)$ commute with N. The spectral decomposition of S is the finite sum

$$S = \sum_h z_h P_h \quad (z_h \in \mathbf{C}, \quad P_h P_k = \delta_{hk} P_h),$$

where the idempotents $P_h = P(z_h)$ are orthogonal projections. It is known that then each P_h commutes with N (cf. [K, I.4]). With the notation $N_h := N|P_hH$ it follows that $N = \oplus_h N|P_hH = \oplus_h N_h$. By Fact 2, the restrictions N_h are all p.p.i.-s (with possibly different pair p, r from the original one: it is also possible that $N_h = 0$, that is $p = 1$). Whatever the case, start with the operator $M := N_1$, and proceed as follows.

As mentioned above, it can be that $M = N|P_1H = 0$, though $P_1 \neq 0$. Then we have $T|P_1H = S|P_1H = z_1I|P_1H$, hence for this part of H and T we certainly have an orthonormal basis of the required sort. If $M \neq 0$, then apply Fact 1 to this $M = N|P_1H$ (with the corresponding $p > 1, r > 0$). Then there is an M-reducing orthogonal decomposition $M = M_1 \oplus M_0$ such that the subspace of M_1 has an orthonormal basis of the required sort. Continue then with the orthogonal summand part M_0 as above. After a finite number number of steps we shall have obtained an orthonormal basis for the subspace P_1H which is a Jordan basis for $N|P_1H$. In P_1H we have $S|P_1H = z_1I|P_1H$, thus every orthonormal basis in P_1H is an orthonormal Jordan basis for $S|P_1H$. The restriction $T|P_1H$ has the form $(z_1I + N)|P_1H$, hence it has an orthonormal Jordan basis.

In the next step let $M := N|P_2H$. Proceeding as above, we obtain an orthonormal basis for the subspace P_2H which is a Jordan basis for $N|P_2H$. Here the restriction $T|P_2H$ has the form $(z_2I + N)|P_2H$, hence it has an orthonormal Jordan basis. Since the subspaces P_jH and P_kH are orthogonal for $j \neq k$, the union of the bases obtained is an orthonormal basis. (More precisely, the basis vectors in the subspaces at first must and trivially can be extended to vectors in the whole space $H = \oplus_h P_hH$.) Proceeding similarly for all the subspaces $N|P_hH$ we obtain in a finite number of steps an orthonormal Jordan basis for the operator $T = S + N$. $\triangle$

Remark. Let $s(A)$ denote the spectral list of a self-adjoint operator A, and apply the similar notation $s(Q)$ for a self-adjoint matrix Q. For any $M \in M_n(\mathbf{C})$ it is clear then that $s([M, M^h])$ is a list of real numbers (with possible repetitions) $r_1, r_2, \ldots, r_n$ satisfying $r_1 + r_2 + \ldots + r_n = 0$. We recall ([BPW]) that any such list is the spectral list of some $M \in M_n(\mathbf{C})$.

Indeed, in [BPW, Example 4.8] it was shown that if we have such a list then, reordering it to satisfy $r_1 \geq r_2 \geq \ldots \geq r_n$, and defining

$$q_j := r_1 + \ldots + r_j \quad (j = 1, \ldots, n),$$

we obtain $q_j \geq 0$ for every $j = 1, \ldots, n$. Define p_j as the unique nonnegative square root of q_j $(j = 1, \ldots, n)$, and also $M \in M_n(\mathbf{C})$ as the matrix with exclusively zero entries except for the entries

$$m_{k,k+1} := p_k \quad (k = 1, 2, \ldots, n-1).$$

Then we obtain the diagonal matrix

$$[M, M^h] = diag(q_1, q_2 - q_1, \ldots, q_n - q_{n-1}) = diag(r_1, r_2, \ldots, r_n).$$

This means that any multiset of real numbers $r_1, r_2, \ldots, r_n$ as above can be realized as the spectral list of the self-commutator matrix $[M, M^h]$ for a suitable nilpotent $M \in M_n(\mathbf{C})$. Equivalently, *any multiset of real numbers* $r_1, r_2, \ldots, r_n$ as above can be realized *as the spectral list* of the self-commutator operator $TT^* - T^*T$ (of a nilpotent operator T) or, equivalently, of the self-commutator matrix $[T, T^*]_b \equiv [TT^* - T^*T]_b$ with respect to an *orthonormal basis* b.

Lemma 1. *Let H be a (possibly infinite dimensional) complex or real Hilbert space and*

$$T = S + Q, \qquad (T, S, Q \in L(H)),$$

where S is normal and Q commutes with S. Then $[T, T^*] = [Q, Q^*]$.

Proof. We have

$$[T, T^*] = [S + Q, S^* + Q^*] = SS^* + QS^* + SQ^* + QQ^* - S^*S - Q^*S - S^*Q - Q^*Q.$$

Since S is normal, $[S, S^*] = 0$.

Assume first that H is complex. Since Q commutes with the normal S, Fuglede's theorem (see, e.g., [Put, pp. 9-10]) implies that Q commutes with S^*, hence Q^* commutes with S. Thus six terms above have the sum 0, and we obtain the statement.

If H is a real Hilbert space, then consider its complexification H^c, and denote the complexification of any operator $A \in L(H)$ by A^c (see, e.g., [B], [G] or [KM]). By assumption, $SQ = QS$. Hence $S^cQ^c = (SQ)^c = (QS)^c = Q^cS^c$. Since the complexification S^c is a normal operator in H^c, Fuglede's theorem applies, and yields that Q^c commutes with S^{c*}. It follows that $(QS^*)^c = Q^cS^{*c} = Q^cS^{c*} = S^{c*}Q^c = S^{*c}Q^c = (S^*Q)^c$, hence $QS^* = S^*Q$. It means that Fuglede's theorem is valid also in a real Hilbert space, and the statement of the Lemma follows as above. $\triangle$

Theorem 2. *Let H be a finite dimensional complex Hilbert space and $T \in L(H)$. If there is an (ordered) orthonormal basis b in which the matrix $[T]_b$ has a Jordan form, then*

$$\sigma(TT^* - T^*T) \subset \{0, 1, -1\},$$

and the multiplicities of -1 and of 1 are equal. This multiplicity m is the number of the Jordan blocks of T of order larger than 1. In the converse direction: if the multiset Σ with basic set $\{0, 1, -1\}$ has the multiplicity property above, which is equivalent to

$$m(0) + 2m = dim(H) = n,$$

then there is $T \in L(H)$ and an (ordered) orthonormal basis b in which the matrix $[T]_b$ has a Jordan form, and $s(TT^* - T^*T) = \Sigma$.

Proof. Let b be an (ordered) orthonormal basis in H with the stated property, i.e.,

$$[T]_b = J(z_1; k_1) \oplus \cdots \oplus J(z_p; k_p),$$

where we have displayed the eigenvalues and the dimensions of the (upper type) Jordan blocks:

$$J(z;k) := \begin{pmatrix} z & 1 & 0 & 0 & \cdots & 0 \\ 0 & z & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & z \end{pmatrix}.$$

Let $T = S + N$ be the Jordan-Dunford decomposition of T. From the Lemma we know that $[T, T^*] = [N, N^*]$, hence

$$\sigma([TT^* - T^*T]_b) = \sigma([NN^* - N^*N]_b).$$

Since b is orthonormal, the matrix $[N^*]_b$ is the conjugate transpose of $[N]_b$, i.e.,

$$[N^*]_b = L(0; k_1) \oplus \cdots \oplus L(0; k_p),$$

where we have indicated the eigenvalues and the dimensions of the (lower type) Jordan blocks L:

$$L(z;k) := \begin{pmatrix} z & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & z & 0 & 0 & \cdots & 0 & 0 \\ \cdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & z \end{pmatrix}.$$

Taking the corresponding pair of blocks, we see that

$$J(0;k_r)L(0;k_r) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Similarly,

$$L(0;k_r)J(0;k_r) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hence

$$J(0;k_r)L(0;k_r) - L(0;k_r)J(0;k_r) = diag_{k_r}(1,0,\cdots,0,-1).$$

Here $diag_{k_r}(1,0,\cdots,0,-1)$ denotes the diagonal matrix of order k_r with the indicated diagonal. Note that if $k_r = 1$, then the diagonal consists of a single 0, whereas if $k_r = 2$, then the diagonal is $(1,-1)$. It follows that

$$[TT^* - T^*T]_b = diag_{k_1}(1,0,\cdots,0,-1) \oplus \cdots \oplus diag_{k_p}(1,0,\cdots,0,-1).$$

This proves the stated spectral containment relation with the slight addition that 0 is not in the spectrum exactly when all Jordan blocks of T have the dimension $k_r \geq 2$, and we have $\sigma(TT^* - T^*T) = \{0\}$ exactly when all Jordan blocks of T have the dimension $k_r = 1$, i.e., the operator T is normal. Note that this last statement holds also in an infinite dimensional space.

In the converse direction: Assume that in the n-element multiset Σ with basic set $S := \{0,1,-1\}$ the numbers 1 and -1 have the same multiplicity m. Consider all the n-element sequences built from all the elements of the spectral list Σ with the property that the ordering between the 1's and -1's be $\{1,-1,1,-1,\cdots,-1\}$ irrespective of the places of zeros (if any). An *interval of length* $q \geq 2$ be a subsequence of the form $\{1,0,0,\cdots,0,-1\}$ containing exactly $q-2$ copies of 0. It follows that in any sequence as above there are m intervals of lengths q_j $(j = 1,2,\ldots,m)$ and $n - q_1 - \ldots - q_m$ zeros outside or between these intervals: the latter types of zeros will be regarded as *intervals of length 1.* Thus for any sequence as above we have determined its *sequence of lengths* $\{k_1, k_2, \ldots, k_p\}$ consisting (in a fixed ordering) of the q_j $(j = 1,2,\ldots,m)$ and $n - q_1 - \ldots - q_m$ copies of 1. Then $k_1 + k_2 + \ldots + k_p = n$.

Consider any complex numbers $z_1, z_2, \ldots, z_p$, and the Jordan matrix J of the direct sum of (say, upper) Jordan cells, $J := \oplus_{j=1}^{p} J(z_j;k_j)$. For any orthonormal basis $b = \{b_1,\ldots,b_n\}$ of $H = \mathbf{C}^n$ let $T \in L(H)$ be the linear operator having $[T]_b := J$. The proof of the direct part shows that the multiset $s(TT^* - T^*T)$ is identical with Σ. It is clear that the choice of the operator T is highly nonunique. $\triangle$

To see that a stricter converse of the above theorem is not valid, consider the following

Example. *There is a 2-dimensional complex Hilbert space H and $T \in L(H)$ such that the self-commutator $TT^* - T^*T$ satisfies*

$$s(TT^* - T^*T) = \{1,-1\},$$

(hence the multiplicities of -1 and 1 are clearly 1), and there is no orthonormal basis b in which the matrix $[T]_b$ has a Jordan form.

To construct an example, let $<,>$ denote the standard scalar product in $\mathbf{C}^2$, and $e_1 := (1,1)^h$, $e_2 := (1,0)^h$. Then the corresponding Gram matrix and its inverse are

$$E = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad E^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Let $m, n \in \mathbf{R}$ satisfy $2(m-n)^4 = 1$, and define the operator T by

$${}_e[T]_e = M := diag(m,n).$$

Then the matrix

$${}_e[TT^* - T^*T]_e = ME^{-1}M^hE - E^{-1}M^hEM$$

has characteristic polynomial $x^2 - 2(m-n)^4$, hence its spectral list is $\{1,-1\}$.

Now if J is a Jordan basis for T, then ${}_J[T]_J$ is the direct sum of the 1-dimensional Jordan "blocks" m and n, hence

$${}_J[T]_J = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}.$$

The basis vectors must be parallel to the vectors e_1 and e_2, respectively. We have seen that $< e_1, e_2 > \neq 0$. This shows that there is no orthonormal (in the scalar product $<,>$) Jordan basis for the operator T, though we have

$$s(TT^* - T^*T) = \sigma(ME^{-1}M^hE - E^{-1}M^hEM) = \{1,-1\}.$$

3. The case of a real Hilbert space

Let H be a finite dimensional real Hilbert space and $A \in L(H)$. Recall that A is called *of simple structure, nilpotent, normal* exactly when its complexification $A^c \in L(H^c)$ has the corresponding property in the complexified space H^c (see, e.g., [G, Section 9.13]). For example, A is of simple structure iff A^c has a basis consisting of eigenvectors, and A is normal iff A commutes with its (real) adjoint A^t. It is known that A is of simple structure iff there is a basis b in H such that

$$[A]_b = diag(z_1, \cdots, z_p) \oplus J(c_1, d_1) \oplus \cdots \oplus J(c_q, d_q) \qquad (p + 2q = n),$$

where $J(c,d) := \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ $(c, 0 \neq d \in \mathbf{R})$, and n is the dimension of H. A is a normal operator iff there is an *orthonormal* basis $b \subset H$ such that $[A]_b$ has the form above. It is also known that every operator $A \in L(H)$ has a unique decomposition into a sum $S+N$, where the operator $S \in L(H)$ is of simple structure, $N \in L(H)$ is nilpotent, $SN = NS$, and they are called the corresponding *parts* of the operator A. A *real Jordan canonical form* of A is called a matrix representation of A of the form

$$J(z_1; k_1) \oplus \cdots \oplus J(z_p; k_p) \oplus J(c_1, d_1; g_1) \oplus \cdots \oplus J(c_q, d_q; g_q), \quad [k_1 + \cdots + k_p + 2(g_1 + \cdots + g_q) = n].$$

where $J(z_r; k_r)$ $(z_r \in \mathbf{R}, k_r \in \mathbf{N})$ are classical Jordan blocks as in Section 2, and

$$J(c,d;g) := \begin{pmatrix} K & I_2 & 0 & \ldots & 0 \\ 0 & K & I_2 & \ldots & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots & I_2 \\ 0 & 0 & 0 & \ldots & K \end{pmatrix}, \qquad K := J(c,d) = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \quad (c, 0 \neq d \in \mathbf{R}),$$

where $J(c,d;g)$ has g *block* rows and columns of 2×2 submatrices, and I_2 is the identity matrix of order 2. $J(c,d;g)$ is called a *real Jordan block* corresponding to the conjugate complex eigenvalues $c + id, c - id$ of multiplicity g. Note that if the last parameter in the notation of a Jordan block is 1, we shall sometimes omit it.

Theorem 3. *Consider the decomposition $T = S + N$ of the operator $T \in L(H)$, where S is the simple structure part and N is the nilpotent part of T (which commutes with S). There is an orthonormal basis $b \subset H$ such that $[T]_b$ is a real Jordan canonical form (matrix) if and only if N is a power partial isometry, and the operator S is normal.*

Proof. If $[T]_b$ is a real Jordan canonical matrix as above, then

$$[N]_b = J(0; k_1) \oplus \cdots \oplus J(0; k_p) \oplus J(0, 0; g_1) \oplus \cdots \oplus J(0, 0; g_q), \quad [k_1 + \cdots + k_p + 2(g_1 + \cdots + g_q) = n].$$

Note that in this formula we allow the second parameter in $J(c,d;g)$ to be 0. Define the basis β by reordering b as follows. For the blocks of the type $J(0;k)$ retain the ordering. For simplicity in notation we consider every block of the type $J(0,0;g)$ separately, and assume that the part of b in the subspace $span(b)$ is $\{b_1, \ldots, b_h\}$, where $h = 2g$. Define the corresponding part of the basis β by

$$\{\beta_1, \ldots, \beta_h\} := \{b_1, b_3, \ldots, b_{h-1}, b_2, b_4, \ldots, b_h\}.$$

The matrix of the operator N restricted to the subspace $span(b) = span(\beta)$ in the reordered basis β will then become $J(0; g) \oplus J(0; g)$. The basis vectors in these bases b and β can trivially be extended from the subspace $span(b) = span(\beta)$ to all of H by defining them to be the vector 0 on the orthogonally complementing subspace. We shall retain the notation for the extended vectors. The matrix of the operator N in the reordered basis β will be

$$[N]_\beta = J(0; k_1) \oplus \cdots \oplus J(0; k_p) \oplus J(0; g_1) \oplus J(0; g_1) \oplus \cdots \oplus J(0; g_q) \oplus J(0; g_q), \quad [k_1 + \cdots + k_p + 2(g_1 + \cdots + g_q) = n],$$

which is clearly a Jordan canonical form for N. If b is, in addition, orthonormal, then β is an orthonormal basis for N. Veselic [V, Theorems 1,2] are valid also in a real space, and show that then N is a power partial isometry. $[S]_\beta$ is a real matrix of simple structure, and β is orthonormal, hence the operator S is normal.

In the converse direction, let the nilpotent $N \in L(H)$ satisfy the assumptions of Facts 1,2 from the proof of Theorem 1, and let the normal $S \in L(H)$ commute with N. The complexification S^c of the operator S is normal in the complexified Hilbert space H^c, hence the spectral theorem yields that

$$S^c = \sum_{k=1}^{K} z_k Q(z_k) \quad (z_k \in \mathbf{C}), \qquad Q(z_j)Q(z_k) = \delta_{jk} Q(z_j), \qquad \sum_{k=1}^{K} Q(z_k) = I^c,$$

where the (orthogonal) spectral projections $Q(z_k)$ are in $L(H^c)$.

Since the Jordan-Dunford decomposition of the complexification T^c is $T^c = S^c + N^c$, for every eigenvalue z_k the projection $Q(z_k)$ commutes with the nilpotent operator N^c. Hence $N^c Q(z_k)H^c \subset Q(z_k)H^c$, and the latter subspace orthogonally reduces N^c. From general spectral theory we know that $Q(z_k)H^c = ker(S^c - z_k I)$.

The proof continues as follows. For each $z_k \in \mathbf{R}$ we shall construct one orthonormal real Jordan basis for the restriction $T^c|Q(z_k)H^c$ *from vectors in* H. Then for each $z_k \in \mathbf{C} \setminus \mathbf{R}$ we shall construct one orthonormal real Jordan basis for the restriction $T^c|[Q(z_k) + Q(\overline{z_k})]H^c$ *from vectors in* H. Since

$$H^c = \oplus_{k=1}^{K} Q(z_k)H^c, \qquad T^c = \oplus_{k=1}^{K} T^c|Q(z_k)H^c,$$

a suitable "orthogonal union" of these bases will yield an orthonormal real Jordan basis for the operator T^c *from vectors in* H. In view of the last fact, this basis will be an orthonormal real Jordan basis for the operator $T \in L(H)$ itself.

Let $z_k \in \mathbf{R}$. Then $T^c|Q(z_k)H^c = [S^c + N^c]|Q(z_k)H^c = [z_k I + N^c]|Q(z_k)H^c$. Since N^c is also a p.p.i., we see exactly as in the proof of Theorem 1 that $T^c|Q(z_k)H^c$ has an orthonormal Jordan basis *from vectors in* H.

If $z_k \in \mathbf{C} \setminus \mathbf{R}$, then the spectral projections $Q(z_k)$ and $Q(\overline{z_k})$ have the same ranks (dimensions). The restrictions $N^c|Q(z_k)H^c$ and $N^c|Q(\overline{z_k})H^c$ are, by Fact 2, p.p.i.-s, hence there are orthonormal Jordan bases for both of them. In fact, if β is such a basis for $N^c|Q(z_k)H^c$, then $\overline{\beta}$ (consisting of the conjugates of the vectors of β) is such a basis for $N^c|Q(\overline{z_k})H^c$, and they are (lying in orthogonal subspaces) orthogonal to each other. Further, if $\beta = \{b_1, \ldots, b_n\}$, $\overline{\beta} = \{\overline{b_1}, \ldots, \overline{b_n}\}$, and for $j = 2, \ldots, n$ we have $N^c b_j = \epsilon_j b_{j-1}$, where ϵ_j is equal to 0 or 1 then, since $NH \subset H$, we obtain $N^c \overline{b_j} = \overline{N^c b_j} = \epsilon_j \overline{b_{j-1}}$.

Consider now the ordered basis $e := \{b_1, \ldots, b_n, \overline{b_1}, \ldots, \overline{b_n}\}$ for the subspace $(Q(z_k) + Q(\overline{z_k}))H^c$. For simplicity, we shall write z instead of z_k, and obtain that

$$[T^c|(Q(z) + Q(\overline{z}))H^c]_e = \begin{pmatrix} z & \epsilon_2 & 0 & 0 & \ldots & 0 \\ 0 & z & \epsilon_3 & 0 & \ldots & 0 \\ \ldots & & & & & \\ 0 & 0 & 0 & 0 & \ldots & \epsilon_n \\ 0 & 0 & 0 & 0 & \ldots & z \end{pmatrix} \oplus \begin{pmatrix} \overline{z} & \epsilon_2 & 0 & 0 & \ldots & 0 \\ 0 & \overline{z} & \epsilon_3 & 0 & \ldots & 0 \\ \ldots & & & & & \\ 0 & 0 & 0 & 0 & \ldots & \epsilon_n \\ 0 & 0 & 0 & 0 & \ldots & \overline{z} \end{pmatrix}.$$

Reorder now the basis e to obtain the (orthonormal) basis $f := \{b_1, \overline{b_1}, \ldots, b_n, \overline{b_n}\}$. Then we obtain the following $2n \times 2n$ matrix $[T^c|(Q(z)+Q(\overline{z}))H^c]_f$: the main diagonal is $\{z, \overline{z}, z, \overline{z}, \ldots, z, \overline{z}\}$, the first bydiagonal upwards consists of $2n - 1$ zeros, the second bydiagonal upwards is $\{\epsilon_2, \epsilon_2, \epsilon_3, \epsilon_3, \ldots, \epsilon_n, \epsilon_n\}$, and all other entries are 0.

Consider now the 2×2 matrix $U_2 := 2^{-1/2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. It is unitary with inverse $U_2^* = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. If $z = c + id$, then we have $U_2^* diag(z, \overline{z}) U_2 = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = J(c, d)$. Define the n-term direct sum matrix

$$U := U_2 \oplus U_2 \oplus \cdots \oplus U_2.$$

It is unitary, and we obtain that $U^*[T^c|(Q(z)+Q(\overline{z}))H^c]_f U$ is the $n \times n$ *block* matrix with 2×2 entries: the main block diagonal is $diag[J(c, d), \ldots, J(c, d)]$, and the first block bydiagonal upwards is $\{E_2, E_3, \ldots, E_n\}$, where the 2×2 matrix E_j is defined as $diag(\epsilon_j, \epsilon_j)$. It follows that E_j is the 2×2 identity I_2 if $\epsilon_j = 1$, or else is the 2×2 zero matrix 0_2 if $\epsilon_j = 0$.

On the other hand, the change of basis rule shows that $U^*[T^c|(Q(z)+Q(\overline{z}))H^c]_f U = [T^c|(Q(z)+Q(\overline{z}))H^c]_u$, where the new basis $u = \{u_1, \dots u_{2n}\} \subset (Q(z)+Q(\overline{z}))H^c$ has the property that

$$\{[u_1]_f, \dots, [u_{2n}]_f\} = U$$

or, equivalently, $U^* = \{[f_1]_u, \dots, [f_{2n}]_u\}$. Clearly, the basis u of $(Q(z)+Q(\overline{z}))H^c$ is orthonormal. Further,

$$u_{2k-1} = 2^{-1/2}(b_k + \overline{b_k}) \in H \cap [(Q(z)+Q(\overline{z}))H^c],$$

$$u_{2k} = 2^{-1/2}(-ib_k + i\overline{b_k}) \in H \cap [(Q(z)+Q(\overline{z}))H^c] \quad (k = 1, 2, \dots, n)$$

shows that the basis u is in H. Summarizing, we have shown that in the orthonormal basis $u \subset H$ the matrix $[T^c|(Q(z)+Q(\overline{z}))H^c]_u$ is a real Jordan canonical form.

We can proceed in the same way for any nonreal eigenvalue $z = z_m \in \mathbf{C} \setminus \mathbf{R}$ of T and obtain a corresponding orthonormal basis $u(m) \subset H$ for $(Q(z_m)+Q(\overline{z_m}))H^c$. For real eigenvalues $z = z_j$ of T Theorem 1 shows that there are corresponding orthonormal bases $v(j) \subset H$ for $Q(z_j)H$. Clearly, we can complete each basis vector in each of these bases of subspaces by adding the vector 0 from the orthogonally complementing subspace. Retaining the notation for these completed basis vectors, the union of all the bases $u(m)$ and $v(j)$ is an orthonormal basis in H in which the matrix of the operator T^c is a real Jordan canonical matrix. Since all the basis vectors are elements of H, the matrix of T is identical with the matrix of T^c, hence a real Jordan canonical matrix. △

Theorem 4. *Let H be a finite dimensional real Hilbert space and $T \in L(H)$. If there is an (ordered) orthonormal basis b in which the matrix $[T]_b$ has a real Jordan form, then the spectrum of the self-commutator $TT^* - T^*T$ satisfies*

$$\sigma(TT^* - T^*T) \subset \{0, 1, -1\},$$

and the multiplicities of -1 and 1 are identical nonnegative integers. This multiplicity m is equal to the sum of the number of Jordan blocks of T with real eigenvalues of order not less than 2, plus twice the number of real Jordan blocks of T corresponding to conjugate complex eigenvalues. In the converse direction: if the multiset Σ with basic set $\{0, 1, -1\}$ has the multiplicity property above, i.e.,

$$m(0) + 2m = dim(H) = n,$$

then there is $T \in L(H)$ and an (ordered) orthonormal basis b in which the matrix $[T]_b$ has a real Jordan form, and $s(TT^* - T^*T) = \Sigma$.

Proof. Let b be an (ordered) orthonormal basis in H with the stated property, i.e.,

$$[T]_b = J(z_1; k_1) \oplus \cdots \oplus J(z_p; k_p) \oplus J(c_1, d_1; g_1) \oplus \cdots \oplus J(c_q, d_q; g_q), \quad [k_1 + \cdots + k_p + 2(g_1 + \cdots + g_q) = n].$$

where $z_1, \dots z_p, c_1, \dots c_q, d_1, \dots d_q \in \mathbf{R}$, and the numbers d_j are not zero. Here the terms $J(z_r; k_r)$ are upper type Jordan blocks as in Section 2, and the real Jordan blocks $J(c_s, d_s; g_s)$ are as above. An application of Lemma 1 shows that

$$[TT^* - T^*T]_b = [QQ^* - Q^*Q]_b,$$

where $[Q]_b$ is the nilpotent matrix part of $[T]_b$, i.e.,

$$[Q]_b = J(0; k_1) \oplus \cdots \oplus J(0; k_p) \oplus J(0, 0; g_1) \oplus \cdots \oplus J(0, 0; g_q).$$

Since b is orthonormal and $[Q]_b$ is a real matrix, the matrix $[Q^*]_b$ is the transpose of $[Q]_b$, i.e.,

$$[Q^*]_b = L(0; k_1) \oplus \cdots \oplus L(0; k_p) \oplus J(0, 0; g_1)^t \oplus \cdots \oplus J(0, 0; g_q)^t,$$

where the L are, as before, lower type Jordan blocks, and ^{t} denote transposes. Taking the corresponding pair of blocks, we see that the classical Jordan blocks (belonging to the real eigenvalues z_r) behave as before, hence for these pairs

$$J(0; k_r)L(0; k_r) - L(0; k_r)J(0; k_r) = diag_{k_r}(1, 0, \dots, 0, -1).$$

Similarly we obtain

$$J(0,0;g)J(0,0;g)^t - J(0,0;g)^t J(0,0;g) = diag_g(1,1,0,\ldots,0,-1,-1),$$

where there are $2(g-2)$ zeros in the middle. It follows that

$$[TT^* - T^*T]_b = diag_{k_1}(1,0,\ldots,0,-1) \oplus \cdots \oplus diag_{k_p}(1,0,\ldots,0,-1) \oplus$$

$$\oplus diag_{g_1}(1,1,0,\ldots,0,-1,-1) \oplus \cdots \oplus diag_{g_q}(1,1,0,\ldots,0,-1,-1).$$

The proof for the converse direction can be done in the same style as for the case of Theorem 1. △

Remark. An alternative way of proof could be to apply the technique of the beginning of the proof of Theorem 3 and reduce the problem to the complexification T^c. Application of Theorem 2 would then finish the proof. We feel that the proof described above is more straightforward, not longer, and gives the explicit form of $[TT^* - T^*T]_b$.

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