How not to put a picture on the wall

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1 Problem

Usually a picture's frame is fitted with a string and the picture is hung on the wall with a nail. This is presented in the left hand side of the figure bellow.

Figures/pics-base.JPG

I am interested in finding ways to hang the picture using two nails but with the constraint that pulling either one of them will cause the picture to drop. The right hand figure is a wrong solution to this question. Although pulling the right nail will cause the picture to fall, but pulling the left one doesn't.

2 Solution

During a control theory exam I had time to solve the question :) I knew that there exists an algebraic solution but I did not figure that out. Instead I present an other one which I got.

Imagine a vertical line between the nails and start following the string with your finger. Let ℓ_1 be number of times your finger passed above the left nail from left to right minus the number of passings from right to left until you cross the imaginary vertical line. Then start counting the same thing for r_1 until you get back to the left half again. Then start ℓ_2 and so on.

In case of the right hand side figure this gives the following:

$$\ell_1 = 1 \qquad r_1 = 1 \qquad \ell_2 = -1 \qquad r_2 = 0$$

It is easy to see that if a nail is pulled than the total number of crossings for the other nail sum up. The picture falls if this sum adds up to zero. In the previous example pulling the left nail gives $r_1 + r_2 = 1$, the frame does not fall. Pulling the right one gives $\ell_1 + \ell_2 = 0$ and the picture falls.

From the previous logic it flows that solving the following linear system of equations with integer variables gives a good solution.

$$\ell_1 + \ell_2 + \ldots + \ell_n = 0$$

$$r_1 + r_2 + \ldots + r_n = 0$$

ℓ_1 and r_n are allowed to be zero but the other variables should be strictly non-zero. There are $n-1$ crossing from left to right over the virtual line. Each variable represents what the string does during a period on one side of that line.

To simplify even more, lets find a symmetric solution. This means $\ell_i = r_{n-1}$. The equation can be solved for one crossing. Here are two solutions also presented on the figure bellow.

Left: $\ell_1 = -1 \quad \ell_2 = 1 \quad r_1 = 1 \quad r_2 = -1$ Right: $\ell_1 = 1 \quad \ell_2 = -1 \quad r_1 = -1 \quad r_2 = 1$

`Figures/sol.jpg`

To verify the solution in real life, the left solution was tried out using my shoelaces and my phone. You can see it here:

how not to put a picture on the wall

3 Algebraic solution

When I showed the solution to Marci, he said that he knew the problem and he knows the algebraic answer as well. So I give you that as well. The solution uses a non commutative group to solve the problem.

Let ℓ be an element denoting passing over the left nail from the left. Its inverse ℓ^{-1} denotes passing over the left nail from the right. The same for the right nail are r and r^{-1}. The operation is multiplication. Each expression (after all possible simplifications) that does not simplify to one represents a way to hang up the picture in way that it stays there. Pulling the left nail will cause ℓ and ℓ^{-1} to drop from the expression. After this, if all r-s cancel out then the frame drops.

The Right solution from the previous section is $\ell r^{-1} \ell^{-1} r$. Removing the left nail: $r^{-1} r = 1$. Removing the right nail: $\ell \ell^{-1} = 1$.

4 Algebraic topoligy solution

Let $A = (-1, 0) \in \mathbb{R}^2, B = (1, 0) \in \mathbb{R}^2$ and $X = (0, -1) \in \mathbb{R}^2$. Also let $\mathcal{M} = \mathbb{R}^2 \setminus \{A, B\}$ be a pointed topological space (X is the basepoint).

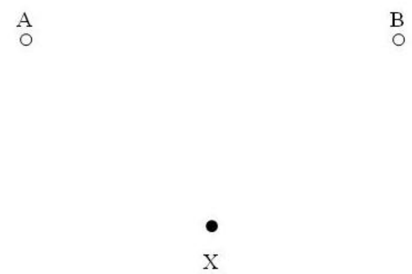


Figure 1: $\mathbb{R}^2$ with two holes

X is the picture, A and B are the nails. The string follows a continuous, closed curve, starting from X, avoiding the points A and B, and ending in X. Per definitionem *every string configuration is a representative of a fundamental group element.*

$$g \in \pi_1(\mathcal{M})$$

The fundamental group of $\mathcal{M}$ is a free group of two letters, since $\mathcal{M}$ is homotopic relative with the figure eight.

Due to the properties of homotopy and the basepoint invariance, the relative position of A, B and X is not relevant. Neither the direction of the gravity.

If $g = 0 \in \pi_1(\mathcal{M})$ then the picture falls, this is the exact meaning of the indentity element. Any non-zero group element represents a confuguration in which the picture hangs still. Every group element has the following form:

$$g = \ell^{n_1} \cdot r^{n_2} \cdot \ldots \ell^{n_{p-1}} \cdot r^{n_p}$$

where $n_i \in \mathbb{Z}$, $i = 1 \ldots p$.

How to find the group element corresponding to a certain string configuration?

1. Collect the cords between the two nails with your hand and pull them down to the picture frame. With this you formed the cross point of the figure eight.

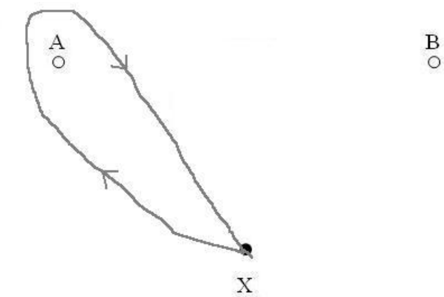
2. Now start counting: follow the string from the left, clockwise, you will go around a nail (A or B) and go back to the frame (you pulled down the cords earlier).

3. If you saw a loop around A, write letter ℓ, otherwise write r. If you took a clockwise roundabout, set the exponent 1, otherwise -1.

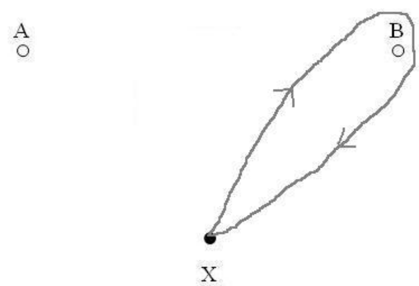
4. Repeat the former step until the string ends.

With this method you end up with a word, like $\ell\ell r\ell^{-1}r^{-1}\ldots$. You can simplify the word by summing the neighboring exponents of the same base, but mind the non-commutativity.

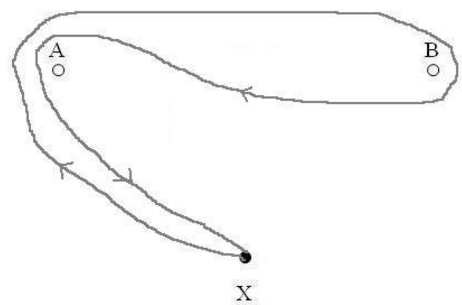
(a) Identity element, or 0



(b) ℓ



(c) r



(d) $\ell r \ell^{-1}$

Figure 2: Some elements of the free group

How to perform the configuration of a certain group element? Let $g = \ell^{n_1} \cdot r^{n_2} \cdot \ldots$.

1. Attach the string to the left hand side of the frame.

2. Loop the nail A n_1 times and go back to the top of the frame.

3. Loop the nail B n_2 times and go back to the top of the frame.

4. Repeat until the word ends, and attach the string to the right hand side of the frame.

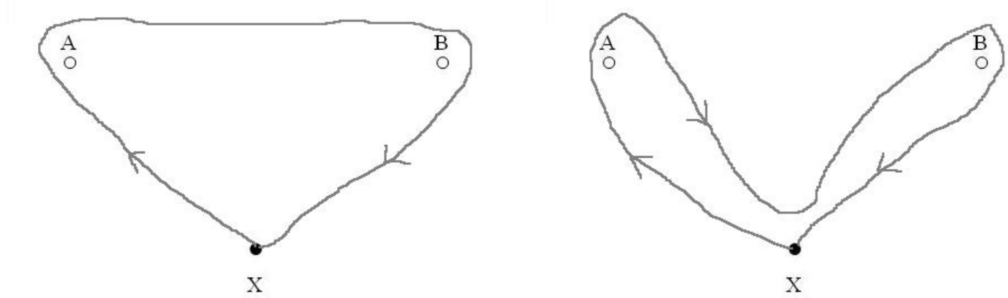
Mind that positive exponent means clockwise loop, negative exponent counter clockwise.

What if I pull out one of the nails? The space $\mathcal{M} = \mathbb{R}^2 \setminus \{A, B\}$ transform to $\mathbb{R}^2 \setminus \{A\}$ or $\mathbb{R}^2 \setminus \{B\}$, depending on the missing nail.

$$\mathbb{R}^2 \setminus \{A\} \cong \mathbb{R}^2 \setminus \{B\} \cong S^1$$

$$\pi_1(S^1) = \mathbb{Z}$$

If I pulled out nail A, then I should erase all letter ℓ. If I pulled out B, then all letter r disappear. Both case, what I get is a word containing one letter, with various exponents. I can sum up the expoents of the same letter, hence an integer



(a) What I see. (b) The group representation is ℓr

Figure 3: A safe configuration

number remains. Like before, zero group element means that the picture fall.

If we understood the formalism, then the problem reads: **find a non-zero element in the two letter free group, such that the exponents of the same letter sum up to zero (in both letters)**.

A solution is $\ell r \ell^{-1} r^{-1}$.

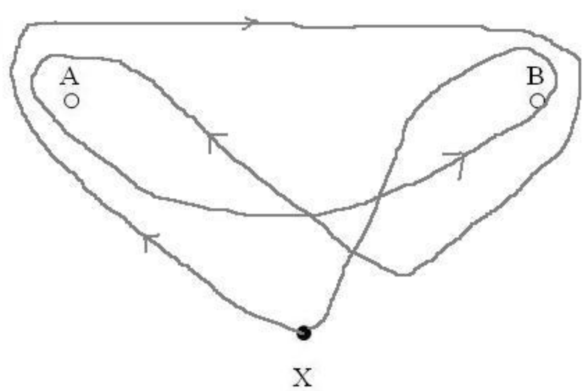


Figure 4: A solution

5 Outlook

Now let us take 3 nails. One would like to hang up the picture for these nails, but in an unsafe way (like metioned above). The picture has to hang still, until any of the nails break or get removed.

Mind the following element of the free group of 3-letters (a, b, c):

$$c^1 b^1 a^1 b^{-1} a^{-1} c^{-1} a^1 b^1 a^{-1} b^{-1} \tag{1}$$

or shortly

$$c^1x^1c^{-1}x^{-1} \tag{2}$$

where x is the solution of the two-nails problem.

If you read the Algebraic topoligy section, then the formula (1) clearly solves the problem. If not, see the video.

Furthermore, one can see, that the problem can be solved for any number of nails in a recursive way, using formula (2). If one has already solved the problem for n nails with the knot x, then let c be the $n+1^{\text{th}}$ nail and form the knot according to (2).

But what is the smallest possible knot, which has this easy-to-fall property? For two nails, one can see (it's realy fun to check) that there is no 3-letters solution i.e. $a^{n_1}b^{n_2}a^{n_3}$ can not have the above property for $n_i \in \mathbb{N}, i = 1, 2, 3$. But is (1) the smallest knot for 3 nails?

Definition 1. *Let us take a fixed, given number $n \in \mathbb{N}$. We call an element from the n-letter free group (F_n) i-*risky *if erasing the i^{th} letter from it makes it collapse into the identity.*

Notice, that the identity is alway i-risky in this definition for any $i = 1 \dots N$. For example any element in the one-letter free group ($\mathbb{Z}$) is a risky element (with respect to the one and only letter).

Definition 2. *Mind the free group of n letters (F_n). We define the group homomorphism*

$$\Pi_i : F_n \mapsto F_n$$

such that Π_i errases the i^{th} letter from a group element.

It is an easy excercise to check that this projection commutes with the group multiplication (which is the concatenation of words), thus it is a group homomorphism. Moreover $\Pi_i(F_n) \cong F_{n-1}$.

Corollary 1. *The above mentioned set F_n^i is the kernel of Π_i*

$$F_n^i = \{\, g \in F_n \,|\, \Pi_i g = 1 \text{ (empty word)}\}$$

, thus it is also a normal subgroup

$$F_n^i \triangleleft F_n$$

Now let us define $F_n^{i,j} = F_n^i \bigcap F_n^j$. The elements of $F_n^{i,j}$ collapses after deleteing either the i^{th} letter or the j^{th} letter. They are more unsafe knots. Since the intersection of normal subgroups forms a normal subgroup $F_n^{i,j} \triangleleft F_n$ also holds. Moreover one can define the solution set of the risky-hanging problem (for arbitrary number of letters/nails).

Definition 3.

$$U_n := \bigcap_{i=1}^{n} F_n^i$$

Because of the normal-group-intersection-property $U_n \triangleleft F_n$. The elements in U_n collapses into identity after ***errasing any letter*** *(not one single letter, but every occurance of one given letter).*