On the rectifiability condition of a second order differential equation in special Finsler spaces

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Abstract

The purpose of this article is the better understanding of the family of geodesics in special Finsler spaces. After the basics in section 1 we introduce the two differential equations (in section 2) which define a geodesic in such spaces. These equations are way too complicate to solve in general, therefore we considered special cases and certain restrictions, detailed in the latter sections. After a brief introduction in section 3 we collect or results in tabulars, sorted by the type of the restrictions.

1 Some introductive notations and theorems

Let M be differentiable manifold of dimension n and $L(\xi, \eta)$ a real function on TM. Then we define the notion of length of a curve. Let $\xi^i = \xi^i(t)$,
$a \leq t \leq b$ be the equations of a segment of a curve c in a coordinate neighborhood U. The length s of the segment is given by the integral

$$s = \int_a^b L\big(\xi(t), \dot{\xi}(t)\big) dt \ . \tag{1}$$

Definition 1.1 *[AIM] The manifold M equipped with such a notion of length is called an n-dimensional Finsler space with the fundamental function L, if L fulfills the conditions below*

1. *The length of any oriented curve does not depend on the choice of parameter. That is the fundamental function L is positively homogeneous of degree one in η.*
2. *The length integral (1) gives rise to the regular variational problem. An important special case is when*

$$g_{ij} := \frac{\partial^2 F}{\partial \eta^i \partial \eta^j} \ , \quad \textit{where } F := L^2/2 \tag{2}$$

has non-zero determinant.

3. *In each tangent space $T_\xi M$ we have a region such that $L(\xi,\eta)$ is differentiable in η^i, where this region does not contain $\eta = 0$ and is a positively conical region, that consists of non-zero tangent vectors η for which py are within that region for any $p > 0$. Thus let TM^* be the domain of the definition of L.*

4. *The fundamental function $L(\xi,\eta)$ is positive-valued for any η belonging to the above mentioned positively conical region.*

Definition 1.2 *[AIM] The extremal of the length integral $s = \int_a^b L\big(\xi(t),\dot{\xi}(t)\big)dt$ is called the geodesic of the space.*

A geodesic is a curve given by the differential equations

$$\frac{d^2\xi^i}{ds^2} + 2G^i(\xi, d\xi/ds) = 0 \ , \tag{3}$$

where s is the normalized parameter, that is the arc-length,

$$2G_j = g_{ij}(\xi,\eta)G^i(\xi,\eta) = \frac{\partial^2 F(\xi,\eta)}{\partial\eta^j\partial\xi^i}\eta^i - \frac{\partial F(\xi,\eta)}{\partial\xi^j} \tag{4}$$

and $F(\xi,\eta) = \big(L(\xi,\eta)\big)^2/2$.

2 On the rectificability condition of a second order ordinary differential equation

The projectively flat spaces are such affine path spaces of which paths are straight. If we are about to determined all the Finsler spaces which admit a path mapping onto projectively flat space, than we come to Hilbert's fourth problem.

Definition 2.1 *[AIM] A Finsler space $F^n = (M^n, L(\xi,\eta))$ is said to be with rectilinear extremals, if M^n is covered by coordinate neighborhoods $\big(U,(\xi^i)\big)$ in which any geodesic is represented by n linear equations $\xi^i = \xi^i_0 + ta^i$ of a parameter t.*

If a Finsler space $F^n = (M^n, L(\xi,\eta))$ is a locally Minkowski space, then we have the covering of M^n by the domains of adapted coordinate systems $\big(U,(\xi^i)\big)$ in which L is a function of η^i alone, the quantities G^i vanish in U from (4) and the equation (3) of geodesics reduces to $d^2\xi/ds^2 = 0$, that is why F^n has the rectilinear extremals (projectively flat). To sum it up a projectively flat Finsler space is projective to a locally Minkowski space.

It is well-known that a Finsler space is projectively flat if and only if its Douglas tensor, Weyl tensor and K curvature tensor vanish identically. (The components of K are $K^i_{hjk} = R^i_{hjk} - C^i_{hr}R^r_{jk}$ in Rund connection.) Most of the papers in this subject are rather long and difficult to understand. On the contrary, Sándor Bácsó and Makoto Matsumoto's method of characterization is more easy to comprehend. A projective change $F^n = \big(M^n, L(\xi,\eta)\big) \longrightarrow \tilde{F}^n = \big(M^n, \tilde{L}(\xi,\eta)\big)$ of the Finsler metric gives rise to various projective invariants. First we have

$$
\begin{aligned}
&Q^0\text{-invariants:} \quad Q^h = G^h - \frac{1}{n+1} G y^h \ , \\
&Q^1\text{-invariants:} \quad Q^h_i = \dot{\partial}_i Q^h = G^h_i - \frac{1}{n+1}\big(G_i y^h + G\delta^h_i\big) \ , \\
&Q^2\text{-invariants:} \quad Q^h_{ij} = \dot{\partial}_j Q^h_i = G^h_{ij} - \frac{1}{n+1}\big(G_{ij} y^h + G_i \delta^h_j + G_j \delta^h_i\big) \ ,
\end{aligned}
$$

where $G = G^r_r$, $G_i = G^r_{ri}$ and $G_{ij} = G^r_{rij}$ is the hv-Ricci tensor of $B\Gamma$.

The Q^2-invariants satisfy the following important identities:

$$Q^h_{ij} = Q^h_{ji} \ , \qquad Q^r_{rj} = 0 \ . \tag{5}$$

Secondly we have a projectively invariant tensor, the Douglas tensor

$$D^h_{ijk} = \dot{\partial}_k Q^h_{ij} \ . \tag{6}$$

Starting from the Q^2-invariants we shall introduce the following quantities in a way similar to constructing the h-curvature tensor: Q^3-invariants

$$Q^h_{ijk} = \delta_k Q^h_{ij} + Q^r_{ij} Q^h_{rk} - \delta_j Q^h_{ik} - Q^r_{ik} Q^h_{rj} \ , \tag{7}$$

where $\delta_k Q^h_{ij} = \partial_k Q^h_{ij} - \big(\dot{\partial}_r Q^h_{ij}\big) G^r_k$.

The Q^3-invariants satisfy the following identities:

$$Q^h_{ijk} + (i,j,k) = 0 \ , \qquad Q^r_{rjk} = 0 \ . \tag{8}$$

Therefore the authors with the help of $Q_{ij} = Q^r_{ijr}$ Ricci-type tensor produce two tensors, which are very important in projective Finsler geometry:

$$\Pi^1\text{-tensor:} \quad \Pi^h_{ijk} = Q^h_{ijk} + \frac{1}{n-1}\big(\delta^h_j Q_{ik} - \delta^h_k Q_{ij}\big) \ , \tag{9}$$

$$\Pi^2\text{-tensor:} \quad \Pi_{ijk} = \delta_k Q_{ij} + Q^r_{ij} Q_{rk} - \delta_j Q_{ik} - Q^r_{ik} Q_{rj} \ . \tag{10}$$

That is how we can obtain a new characteristic property for the flat projectively space:

Theorem 2.2 *[BM3] A Finsler space F^n is projectively flat if and only if F^n is a Douglas space and its characteristic satisfies*

$$(1) \quad n > 2: \quad \Pi^h_{ijk} = 0 \qquad \textit{or} \qquad (2) \quad n = 2: \quad \Pi_{ijk} = 0 \ .$$

In the famous book of Arnold [Arn] we can find the following theorem:

> "An equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$ can be reduced to the form $d^2\bar{y}/d\bar{x}^2 = 0$ if and only if the righthand side is a polinomial in the derivative of order not greater 3 both for the equation and for its dual."

This theorem can be formulated in the following form on the basis of [BM3]:

> "An equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$ can be reduced to the form $d^2\bar{y}/d\bar{x}^2 = 0$ if and only if the path space P^2 (determined by the equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$) is projectively related to a two-dimensional projectively flat Finsler space F^2."

The differential equations of geodesic curves of F^n:

$$d^2x^i/dt^2 = -2G^i(x, \dot{x}) \ .$$

The integral curves of this second order differential equation are called paths. A path space P^n and a Finsler space F^n are projectively related to each other, if any path of P^n is a geodesic curve of F^n and vice versa.

From the previous theorems and definitions we obtain:

Theorem 2.3 *[BOSZ] In a Douglas space*

$$\begin{aligned}\Pi_{112} = &- \frac{\partial^2 f(x,y)}{\partial y^2} + \frac{2}{3}\frac{\partial^2 g(x,y)}{\partial x \partial y} - \frac{1}{3} g(x,y)\frac{\partial g(x,y)}{\partial y} + f(x,y)\frac{\partial h(x,y)}{\partial y} + \\ &+ h(x,y)\frac{\partial f(x,y)}{\partial y} - \frac{1}{3}\frac{\partial^2 h(x,y)}{\partial x^2} + \frac{1}{3} g(x,y)\frac{\partial h(x,y)}{\partial x} + \\ &+ k(x,y)\frac{\partial f(x,y)}{\partial x} - \frac{2}{27} g^2(x,y)h(x,y) + \frac{2}{3} g(x,y)k(x,y)h(x,y) - \\ &- \frac{2}{3} f(x,y)h(x,y)k(x,y) \ ;\end{aligned} \tag{11}$$

$$\begin{aligned}\Pi_{212} = &- \frac{1}{3}\frac{\partial^2 g(x,y)}{\partial y^2} + \frac{2}{3}\frac{\partial^2 h(x,y)}{\partial x \partial y} - \frac{1}{3} h(x,y)\frac{\partial g(x,y)}{\partial y} + f(x,y)\frac{\partial k(x,y)}{\partial y} + \\ &+ 2k(x,y)\frac{\partial f(x,y)}{\partial y} - \frac{\partial^2 k(x,y)}{\partial x^2} + \frac{2}{3} h(x,y)\frac{\partial h(x,y)}{\partial x} + \\ &+ \frac{2}{3} k(x,y)\frac{\partial h(x,y)}{\partial x} - \frac{4}{27} g(x,y)h^2(x,y) + \frac{2}{3} h(x,y)\frac{\partial h(x,y)}{\partial x} - \\ &- \frac{1}{3} g(x,y)\frac{\partial k(x,y)}{\partial x} - \frac{1}{3} k(x,y)\frac{\partial g(x,y)}{\partial x} - \frac{2}{9} h(x,y)g(x,y)k(x,y) + \\ &+ \frac{2}{9} g^2(x,y)k(x,y) \ .\end{aligned} \tag{12}$$

3 New special solutions

According to the results above we considered special solutions of (11, 12). In section 4 and 5 we obtain two non zero functions with the following restrictions.

- the functions may depend only on one variable
- both functions have a form of $F_x(x) + F_y(y)$
- both functions have a form of $F_x(x) \cdot F_y(y)$

In section 6 we studied other simple cases.

Our notations

- $c_1, c_2, c_3 \dots, a, b, c \dots$: arbitrary constants (independent of x and y). Same characters in one row means same constants, but different from the constants of the other rows.
- $C_1, C_2 \dots, A, B \dots$: arbitrary functions with one variable
- in the tabular the variable name below the function means that the function depends only on that variable
- in the same place "x, y" means that there is no restriction for that function
- "$x + y$" means that the function has a form of $F_x(x) + F_y(y)$
- "$x * y$" means that the function has a form of $F_x(x) \cdot F_y(y)$

The best case is when only f and g are non zero functions. In this case you can derive the fully general solution from an ordinary differential equation. It's easy to see that the x depending of the functions are almost arbitrary. The solution is the following [BOSZ]

$$g = A(x) + yB(x),$$

$$f = -\frac{1}{18}y^3 B(x)^2 + \frac{1}{6}y^2 \left(2B'(x) - A(x)B(x)\right) + yC(x) + D(x).$$

4 Single variable cases

If the functions are single variabled then the conditions result ordinary differential equations and also algebraic equations. We consequently went through all of the possibilities and could solve almost all of them.

Theorem 4.1 *If in the equations (11), (12) are only two nonzero functions and both of them depends only on one variable, then the solutions are the followings.*

f	h	f	h
x	x	arbitrary	c
x	y	arbitrary	c
y	x	$\frac{c_1 e^{ay}}{a} + c_2$	a
y	y	arbitrary	$\frac{c}{f(y)} + \frac{f'(y)}{f(y)}$

f	k	f	k
x	x	c	$ax + b$
x	y	a	b
y	x	$ay + b$	$c_1 e^{\sqrt{2a}x} + c_2 e^{-\sqrt{2a}x}$
y	y	$ay + b$	$\frac{c}{(ay+b)^2}$

g	h	g	h
x	x	$\frac{9a}{2(ax+b)}$	$ax+b$
x	y	$h=0$ or $g=0,$ not a new case	
y	x	$h=0$ or $g=0,$ not a new case	
y	y	$b(11y-a)^{2/11}$	$\frac{9}{a-11y}$

The differential equation in the following table can be solved easily if you fixate k. However when you fixate g you have a linear, second order equation with nonlinear coefficients. We thought it's more useful in this form.

g	k	g	k
x	x	$kg' + gk' + 3k'' = \frac{2g^2k}{3}$	
x	y	$-\frac{3}{3c+2x}$	arbitrary
y	x	a	$b\,e^{-\frac{2ax}{3}} + c\,e^{\frac{ax}{3}}$
y	y	$g=c, k=0$ or $g=0$ and k is arbitrary	

In the following case the conditions can easily satisfied unless both functions depends on x. Here we found only a special solution.

h	k	h	k
x	x	$ax+b$	$d\,e^{\frac{ax^2}{3}+\frac{2bx}{3}}+\frac{1}{12\sqrt{a}}e^{-\frac{1}{3}x(ax+2b)+\frac{ax^2}{3}+\frac{2bx}{3}} *$ $\left(\sqrt{3\pi}\left(3a-2b^2+6c\right)e^{\frac{(ax+b)^2}{3a}}\mathrm{Erf}\left(\frac{ax+b}{\sqrt{3a}}\right)-6\sqrt{a}(ax+b)\right)$
x	y	c	$-c$
y	x	$\frac{3c}{2}$	$\frac{a\ e^{cx}}{c}+b$
y	y	arbitrary	arbitrary

Here Erf is defined as

$$\mathrm{Erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\,d\xi.$$

5 Multivariable cases

The most general way was that we separated the variables. For example if you have a product of two functions which are depending on different variables, and this expression must depend only on one variable, then one of the functions is constant or zero. We tried to arrange functions with one particular variable to the same side of the equation, and tried to reduce the possibilities how the other side could depend only on that variable.

Theorem 5.1 *If in the equations (11), (12) are only two nonzero functions and at least one of them is multivariable, then the solutions are the followings.*

f	h	f	h
x, y	x	$C_2(x)e^{C_3(x)y} + C_1(x)$	c
x, y	x, y	It's a hard case, but here it is a special solution: $-\frac{2c_1^2 \tanh(c_1x+c_2y+c_3)}{9c_2}$	$-\frac{9c_2^2 f}{c_1^2}$
$x * y$	$x * y$	$f = 0$ or h depends only on y or $\frac{A(x)\big((cy+d)^2+1\big)}{2(cy+d)}$	$\frac{c}{-cy-d}$
$x + y$	$x + y$	f and h depends only on y or $ae^{cy} + A(x)$	c

With the f and k being non zero surprisingly the heat equation occurs. With less restriction for the form of f and g heat equation also appears, but with non constant coefficients and in an unhomogeneous way.

f	k	f	k
x, y	y	f depends only on y (not a new case)	
x	x, y	c_1	$k'_y(x,y) = \frac{1}{c_1}k''_{xx}(x,y)$
$x + y$	$x + y$	$k = 0$ or $ay + b$	$\frac{c_1}{(ay+b)^2} + c_3e^{\sqrt{2a}x} + c_4e^{-\sqrt{2a}x} - \frac{c_2}{a}$

g	k	g	k
y	x, y	c	$e^{-\frac{2cx}{3}} A(y) + e^{\frac{cx}{3}} B(y)$
$x + y$	$x + y$	g and k depends only on x, or $-\frac{3}{c+2x}$	$A(y) + \frac{1}{3} b(c + 2x)^{3/2} + d$

h	k	h	k
y	x, y	$\frac{3C_1(y)}{2}$	$\frac{C_2(y) e^{x C_1(y)}}{C_1(y)} + C_3(y)$
x, y	y	h depends only on y (not a new case)	

With h, k there are some more difficult cases.

The first when there is no restriction for h and k, it's also just a special solution

$$h = A(y)x + B(y),$$

$$k = C_2(y) e^{\frac{Ax^2}{3} + \frac{2Bx}{3}} + \frac{e^{-\frac{1}{3}x(Ax+2B) + \frac{Ax^2}{3} + \frac{2Bx}{3}}}{12 A^{3/2}} \Big(-6 A^{\frac{3}{2}} B - 6 A^{\frac{5}{2}} x +$$
$$3\sqrt{3\pi} A^2 e^{\frac{(Ax+B)^2}{3A}} \operatorname{Erf}\left(\frac{Ax + B}{\sqrt{3A}} \right) - 2\sqrt{3\pi} A e^{\frac{(Ax+B)^2}{3A}} \left(B^2 - 3C_1(y) \right) \operatorname{Erf}\left(\frac{Ax + B}{\sqrt{3A}} \right)$$
$$-12\sqrt{A} A' - 4\sqrt{3\pi} B A' e^{\frac{(Ax+B)^2}{3A}} \operatorname{Erf}\left(\frac{Ax + B}{\sqrt{3}\sqrt{A}} \right) \Big).$$

The second is when $h = h_x(x) + h_y(y)$ and $k = k_x(x) + k_y(y)$. Then the solution is one of the followings

- $h_x = 0$ and $k_x = 0$,
- $h = -2A(y) - 2bx - 2c$ and $k = c + bx + A(y)$,
- $h = b$ and $k = \frac{3c_1 e^{\frac{2bx}{3}}}{2b} + c_2$,
- $h = ax + b$, $k_y = 0$ and $\frac{3}{2} k'' - (b + ax)k' - ak = ab + a^2 x$.

The third is when $h = h_x(x) \cdot h_y(y)$ and $k = k_x(x) \cdot k_y(y)$. Then the solution is one of the followings

- h is constant ,
- $h_x = 1$ and $k_x = 1$,
- $h_x = ax + b$ and $k_y = \frac{2ah_y(y)^2(ax+b)+2ah'_y(y)}{3k''_x(x)-2h_y(y)((ax+b)k'_x(x)+ak_x(x))}$ (but it must depend only on y) .

6 Other types

We examined other types too, but we couldn't treat most of them. Here are the simplest cases.

Theorem 6.1 *If in the equations (11), (12) k is zero and f, g, h depend only on x, then the solution is the following.*

$$f \text{ is arbitrary},$$
$$g = \frac{9a}{2(ax+b)},$$
$$h = ax + b.$$

Theorem 6.2 *If in the equations (11), (12) k is zero and g, h depend only on x and f depends on both x and y, then the solution is the following.*

$$h = h(x) \text{ (arbitrary)},$$
$$f = C_1(x)e^{yh(x)} + C_2(x) + \frac{yh''(x)}{h(x)},$$
$$g = \frac{9h'(x)}{2h(x)}.$$

Theorem 6.3 *If in the equations (11), (12) f, g, h and k are constants then there are three different solutions.*

- $f = g = 0$, h and k are arbitrary
- any two of f, h, k are zero, the others are arbitrary
- $k = \frac{2h^2}{3g-3h}$ and $f = \frac{g^3-g^2h+6gh^2}{6h^2}$ and $g \neq h$.

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