System of Two Falling Balls

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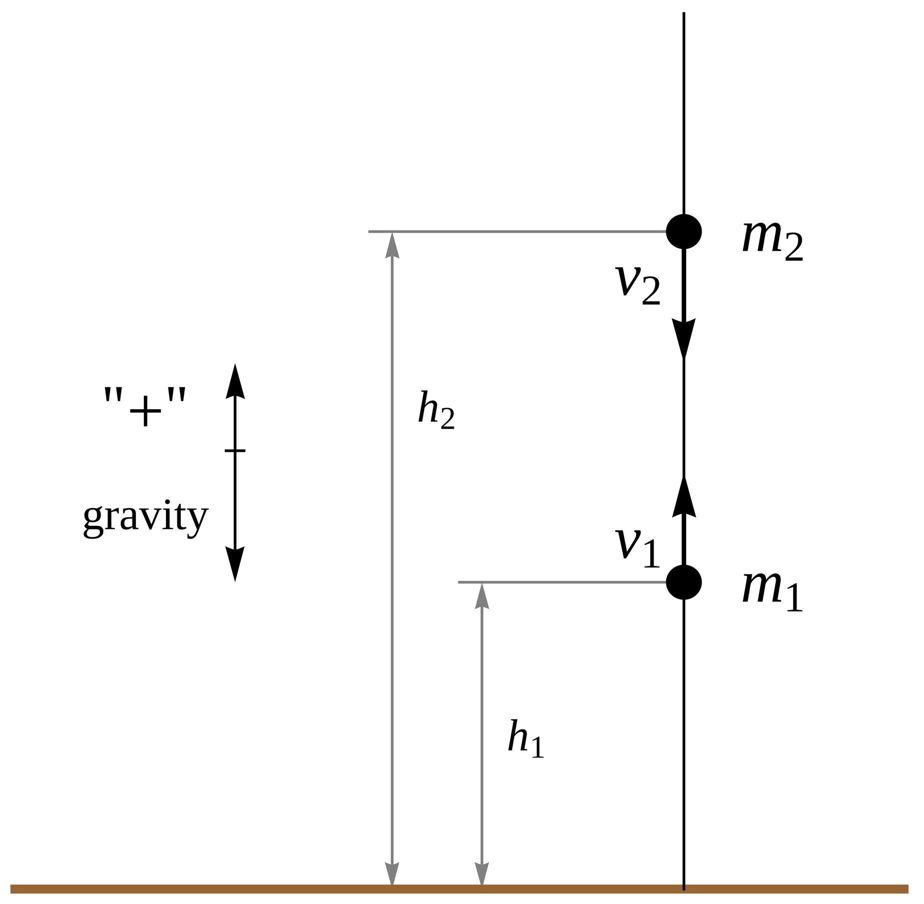
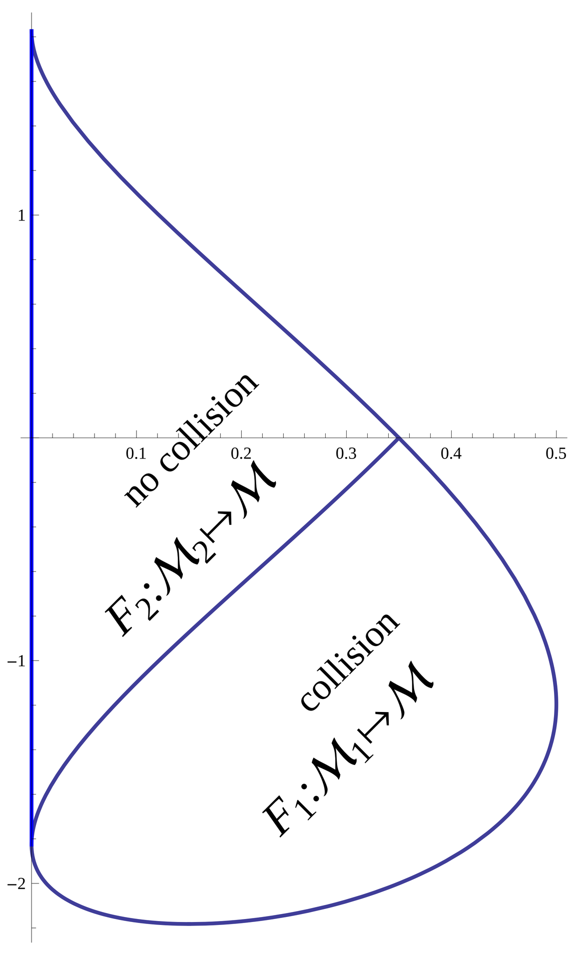
The System

Two infinitesimally small balls move up and down along a vertical half line. The lower ball collides with the upper ball and with the floor in a totally elastic way.



The total energy of the balls is an **integral of motion** and assumed to be 1/2 and we set $1 - m_2 = m_1 = m$. The dynamics is discretized with a **Poincaré section**: $h_1 = 0, v_1 > 0$. These result in a **two dimensional phase space**.

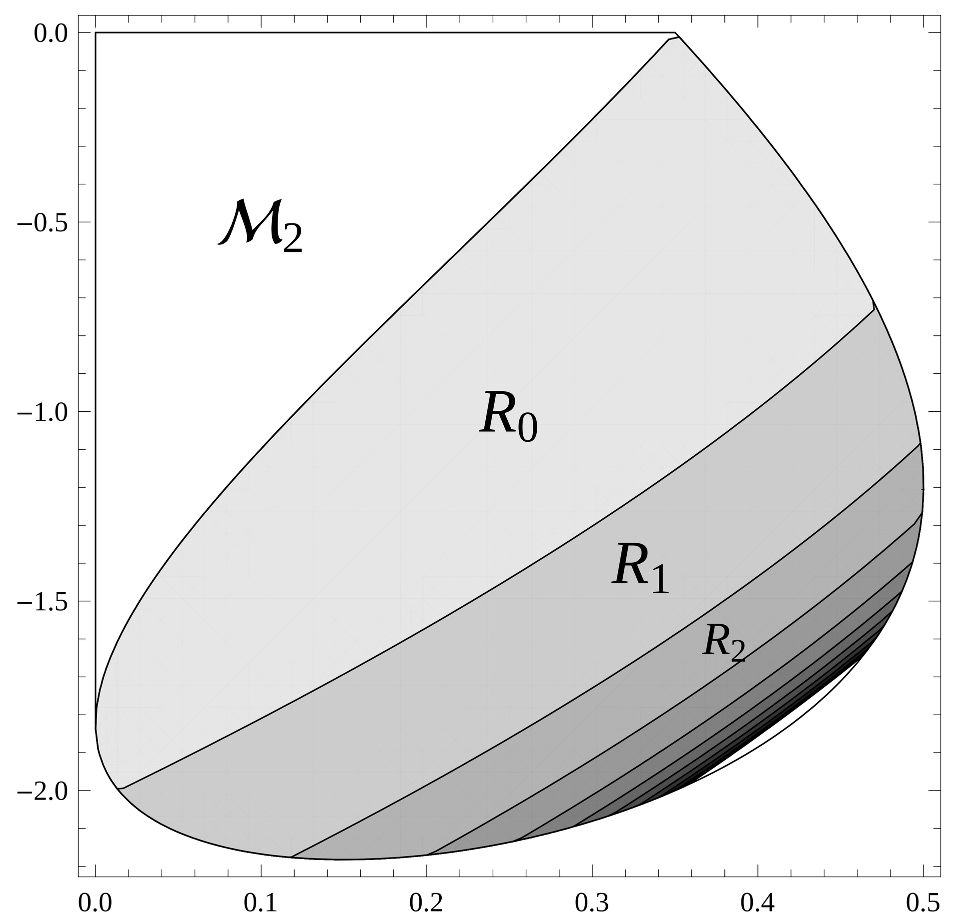
Interesting Phase Space



Wojtkowski in [1] introduced a useful pair of coordinates with which the calculations are simpler and the Lebesgue measure (μ) is invariant.
Considering the collisions there are two scenarios: the balls can collide before the lower one returns to the floor or the lower ball drops back, avoiding the upper one. Hence the phase space is divided into two subsets and the dynamics is piecewise continuous. $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$

First Return Map

In order to have uniform hyperbolicity we introduce the first return map of the set $\mathcal{M}_1$. This means that the lower ball bounces alone for a while until it collides with the upper ball. The new map $T : \mathcal{M}_1 \mapsto \mathcal{M}_1$ has more singularities.



For the configurations in R_n the lower particle bounces $n + 1$ times on the floor until the next collision. We proved the following tail bound:

$$\mu(R_n) \leq \text{const} \cdot \frac{1}{n^4}$$

Result: rate of mixing and CLT

From [1] we know, that the system is ergodic and mixing if the lower particle is heavier ($m > \frac{1}{2}$). That is

$$\lim_{n\to\infty} |\mathbb{E}((f \circ T^n)g) - \mathbb{E}(f)\mathbb{E}(g)| = 0.$$

for every $f, g \in L^2_\mu(\mathcal{M})$. We define the rate of mixing as the decay rate in the above formula for Hölder continuous functions, and distinguish polynomial and exponential tail bounds. We proved that the dynamics mixes with a polynomial (summable) rate if the mass parameter is in a certain open set.
We also proved the Central Limit Theorem (CLT), that is, for Hölder continuous f with $\mathbb{E}f = 0$:

$$\lim_{n\to\infty} \mu\left(\frac{f + f \circ T + \ldots f \circ T^{n-1}}{\sqrt{n}\sigma} \leq z\right) = \Phi(z)$$

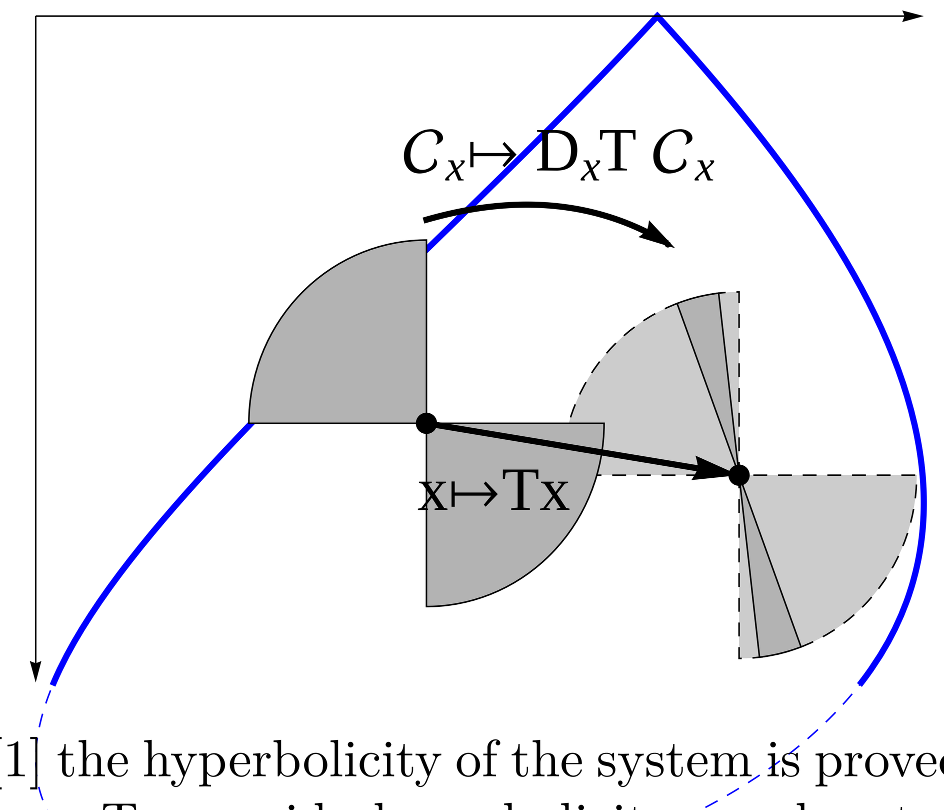
where Φ is the Gauss distribution function and σ depends on the sum of the autocorrelations of f (hence summability of the mixing rate is needed for a finite sigma).

About the Method

We prove the polynomial mixing by combining the *exponential mixing* of the first return map and the estimation of the *first return times* ($\frac{1}{n^4}$). This method of Chernov and Zhang was introduced in [2] based on [3] and [4].
First we prove exponential mixing for the first return map with Young towers ([3]). Secondly we use the estimation of the return times and the a later result of Young ([4]) to prove polynomial mixing for the original map. The final mixing rate was calculated from the first return times:

$$(\log n)^3/n^2$$

Hyperbolicity



In [1] the hyperbolicity of the system is proved via cones. To provide hyperbolicity one has to find an unstable cone field $\{\mathcal{C}_x \subset \mathcal{T}_x\mathcal{M}\}_{x\in\mathcal{M}}$ which satisfies $D_x\, T(\mathcal{C}_x) \subset C_{Tx}$ for every $x \in \mathcal{M}$.
Our phase space (as a manifold) is embedded in $\mathbb{R}^2$, so is the tangent space. Therefore it's easy to define cones. The proper unstable cone field in our system is a constant one: $\{(a, b) \in \mathbb{R}^2 | ab \leq 0\} \subset \mathcal{T}_x\mathcal{M}_1 = \mathbb{R}^2$ for every $x \in \mathcal{M}_1$. If the tangent vector of a curv is lying in the cone field, we call it unstable curv. These curves expand under the action of T.

References

[1] M. P. Wojtkowski, *A system of one dimensional balls with gravity*, Comm. Math. Phys. **126** (1990), 507–533.

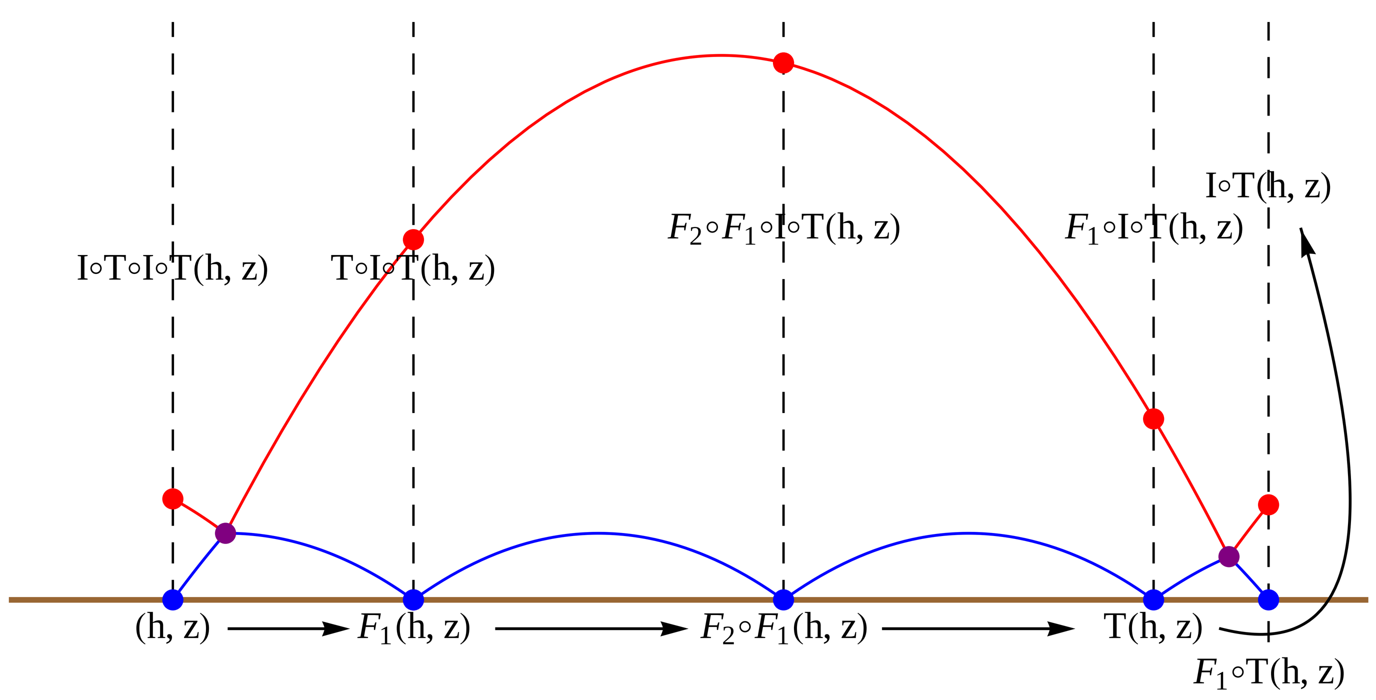
[2] N. Chernov, H.-K. Zhang, *Billiards with polynomial mixing rates*, Nonlinearity 18 (2005), no. 4, 1527–1553.

[3] L.-S. Young, *Statistical properties of systems with some hyperbolicity including certain billiards*, Ann. Math., **147** (1998), 585–650.

[4] L.-S. Young, *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153–188.

Involution

In billiards the involution is a common technique used to handle the inverse of the dynamics. In our proof it helped to reduce the regularity of the stable directions (directions in a backward invariant cone field) to the unstable ones.
We call the map $I : \mathcal{M}_1 \mapsto \mathcal{M}_1$ involution if $I = I^{-1}$ and $T^{-1} = I \circ T \circ I$. In our system the involution is the composition of F_1 and an operator which reverses the velocity of the upper ball.



The Second Iterated Map

In the conditions of Chernov and Zhang one has to treat the expansions of the unstable curves. Hyperbolicity expands these curves but the singularities break them apart. The first return map did not expand enough to overcome the bad effect of the singularities. However we could work with the second iterate of T that has fewer singularities compared to its expansion.