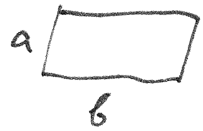
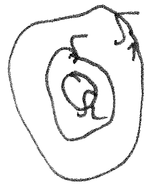
Határozott integrál

Terület tulajdonságai:

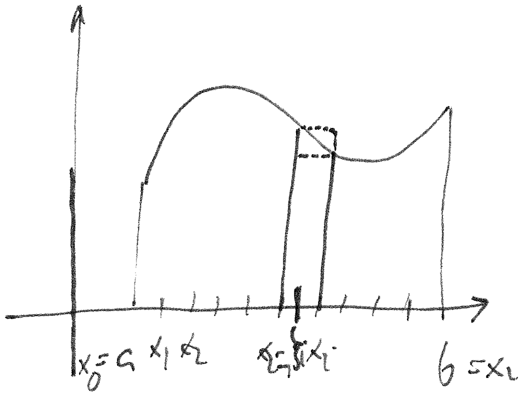
1.) Az a és b oldalú téglalap területe $a \cdot b$



2.) $S_1 \subset S \subset S_2$ esetén $ter(S_1) \le ter(S) \le ter(S_2)$



Hogyan értelmezhetjük az $f(x) \geq 0$, $a \le x \le b$ görbe alatti területet?



Az $[a, b]$ intervallumot felbontjuk az $a \le x_0 < x_1 < x_2 < \dots < x_{i-1} < x_i < \dots < x_n = b$ osztópontok segítségével n db kis intervallumra.

Legyen az $[x_{i-1}, x_i]$ intervallumban az $f(x)$ fv legnagyobb értéke M_i és legkisebb értéke m_i.

Legyen az $[x_{i-1}, x_i]$ felett lévő terület Δter_i, $x_i - x_{i-1} = \Delta x_i$.

Ekkor

$$m_i \Delta x_i \le \Delta ter_i \le M_i \Delta x_i .$$

Így az $f(x)$ $[a, b]$-n értelmezett fv esetén:

$ter = \sum_{i=1}^{n} \Delta ter_i$, ami ha $f(x)$ folytonos és $\xi_i \in [x_{i-1}, x_i]$, akkor $\Delta ter_i \approx f(\xi_i) \Delta x_i$, így $ter \approx \sum_{i=1}^{n} f(\xi_i) \Delta x_i$.

<u>Def</u> Az $f(x)$ Riemann-integrálható az $[a,b]$ intervallumon, ha

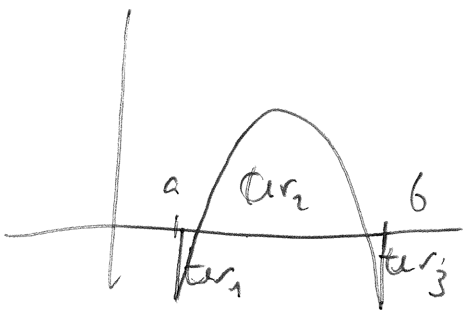
$$\lim_{\substack{n\to\infty \\ \max \Delta x_i \to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$ létezik és független a

ξ_i-k, x_i-k választásától.

Jelölés: $\int_a^b f(x)\,dx$

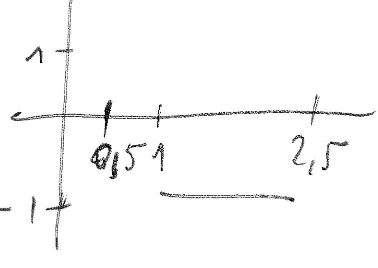
Tehát $$\sum_{i=1}^{n} f(\xi_i)\Delta x_i \approx \int_a^b f(x)\,dx$$

<u>Megj</u> A Riemann-integrál nemlilletesen a görbe és az x tengely közötti előjeles területet jelenti, azaz ami az x tengely felett van, azt $+$-szal, ami alatta van, azt $-$-szal számolja.



$$\int_a^b f(x)\,dx = -ter_1 + ter_2 - ter_3$$

$$f(x) = \begin{cases} 2 & 0{,}5 \le x \le 1 \\ -1 & 1 < x \le 2{,}5 \end{cases}$$



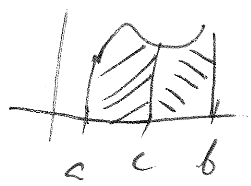
$$\int_{0{,}5}^{2{,}5} f(x)\,dx = (1-0{,}5)\cdot 2 - (2{,}5-1)\cdot 1 = -0{,}5$$

A Riemann-integrál tulajdonságai:

1, $c \in \mathbb{R}$: $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$

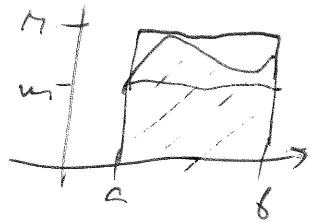
2, $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3, $a<c<b$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



4, Ha az $[a,b]$-ben $m \le f(x) \le M \Rightarrow$

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$



Tétel Ha $f(x)$ folytonos vagy véges sok pont kivételével folytonos az $[a,b]$-ben, akkor ott Riemann-integrálható.

Tétel (Az integrálszámítás középértéktétele)

Ha $f(x)$ folytonos az $[a,b]$-ben, akkor létezik u:

$$f(u)(b-a) = \int_a^b f(x)\,dx.$$

Biz Legyen $m = \min_{a \le x \le b} f(x)$, $M = \max_{a \le x \le b} f(x)$.

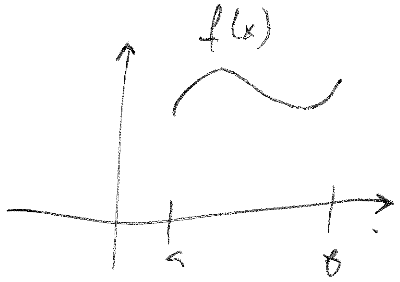
Ekkor $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$, azaz

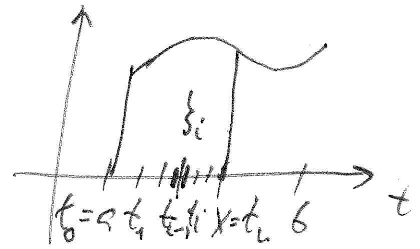
$m \le \frac{\int_a^b f(x)\,dx}{b-a} \le M$, de a Bolzano-tétel szerint $f(x)$ minden m és M közti értéket felvesz $[a,b]$-ben, így $\frac{\int_a^b f(x)\,dx}{b-a}$ értékét is.

Hogyan számítható ki az $\int_a^b f(x)dx$? ④

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Legyen $T(x) = \int_a^x f(t)dt \approx \sum_{i=1}^{n} f(\xi_i)\Delta t_i$, $\quad a < x < b$





$\Delta t_i = t_i - t_{i-1}$

Legyen $f(x)$ folyt $[a,b]$-ben. Ekkor $\Delta x > 0$ esetén

$$T(x+\Delta x) - T(x) = \int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt =$$

$$\int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt - \int_a^x f(t)dt = \int_x^{x+\Delta x} f(t)dt \approx f(x)\Delta x$$

$$\Rightarrow \frac{T(x+\Delta x) - T(x)}{\Delta x} \approx f(x) \Rightarrow \lim_{\Delta x \to 0} \frac{T(x+\Delta x) - T(x)}{\Delta x} =$$

$T'(x) = f(x)$, azaz $T(x)$ egy primitív fv $f(x)$-nek.

Legyen $F(x)$ egy primitív fve $f(x)$-nek. Ekkor létezik $c \in \mathbb{R}$, hogy $T(x) = F(x) + c$. $\quad c = ?$

$0 = T(a) = F(a) + c \Rightarrow c = -F(a)$, azaz $T(x) = F(x) - F(a)$.

Így $T(b) = \int_a^b f(x)dx = F(b) - F(a)$.

Newton-Leibniz-formula (Integrálszámítás alaptétele)

Legyen $f(x)$ egy folytonos fv $[a,b]$-ben. Ekkor ha $F(x)$ egy primitív fve $f(x)$-nek, akkor

$$\int_a^b f(x)\,dx = F(b)-F(a) = \left[F(x)\right]_a^b .$$

Pl. 1. $\int_0^{\pi/2} \cos^3 x\,dx = \int_0^{\pi/2} \cos^2 x \cdot \cos x\,dx = \int_0^{\pi/2} (1-\sin^2 x)\cos x\,dx =$

$\int_0^{\pi/2} \cos x - \sin^2 x \cdot \cos x\,dx = \left[\sin x - \frac{\sin^3 x}{3}\right]_0^{\pi/2} = \sin\frac{\pi}{2} - \frac{\sin^3\frac{\pi}{2}}{3} -$

$\left(\sin 0 - \frac{\sin^3 0}{3}\right) = 1 - \frac{1}{3} = \frac{2}{3}$

2. $\int_0^1 \underset{u}{x}\,\underset{v'}{e^x}\,dx = \left[xe^x\right]_0^1 - \int_0^1 1\cdot e^x\,dx = e - \left[e^x\right]_0^1 = e-(e-e^0) = 1$

$u'=1 \quad v=e^x$

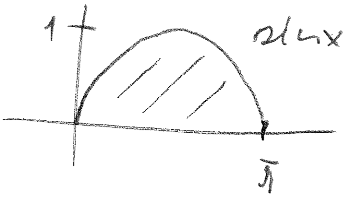
Határozott integrál alkalmazásai

I. Területszámítás

1, $f(x) \geq 0 \quad a \leq x \leq b$

$Ter = \int_a^b f(x)\,dx$

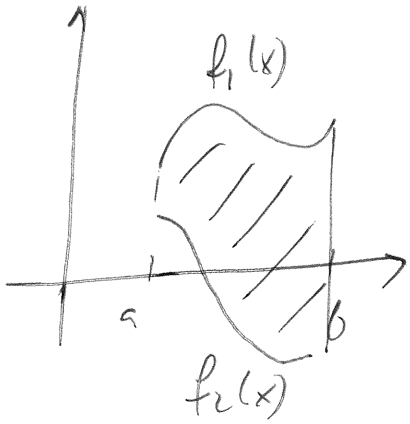
Pl.



$\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} =$

$-\cos\pi - (-\cos 0) = -(-1)+1 = 2$

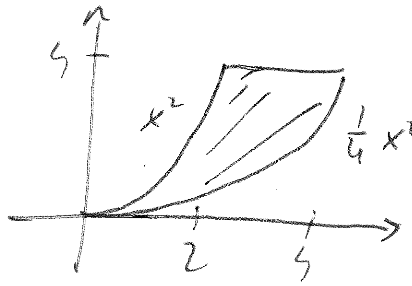
2.



$$ter = \int_a^b f_1(x) - f_2(x)\,dx =$$

$$\int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx$$

Pl.

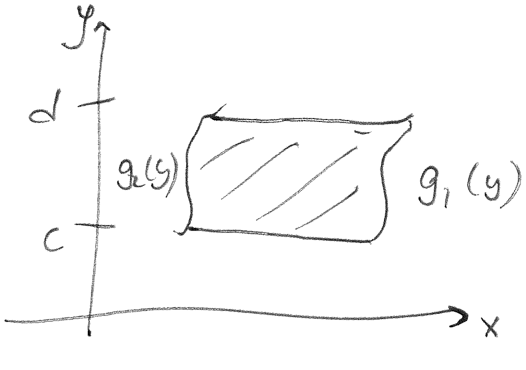


$$ter = \int_0^2 \underbrace{x^2 - \frac{1}{4}x^2}_{\frac{3}{4}x^2}\,dx + \int_2^4 4 - \frac{1}{4}x^2\,dx =$$

$$\left[\frac{3}{4}\cdot\frac{x^3}{3}\right]_0^2 + \left[4x - \frac{1}{12}x^3\right]_2^4 =$$

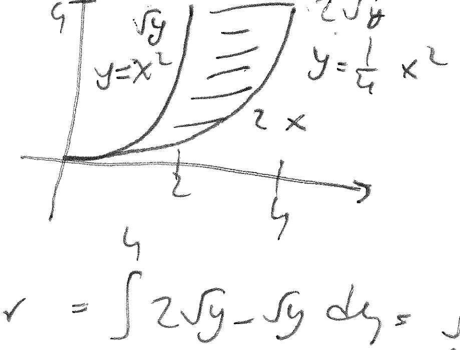
$$2 - 0 + \left(\underbrace{16 - \frac{64}{12}}_{32/3} - \left(\underbrace{8 - \frac{8}{12}}_{22/3}\right)\right) = 2 + \frac{10}{3} = \frac{16}{3}$$

3,



$$ter = \int_c^d g_1(y) - g_2(y)\,dy$$

Pl.



$y = x^2$

$x = \sqrt{y} = g_2(y)$

$y = \frac{1}{4}x^2$

$4y = x^2$

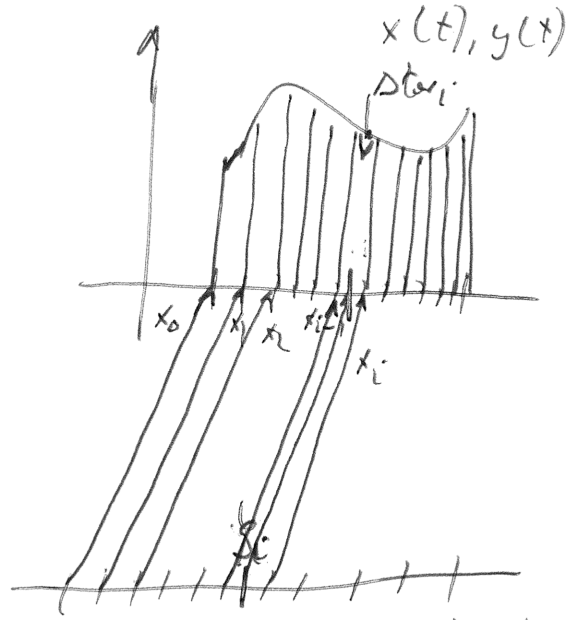
$x = 2\sqrt{y}$

$$ter = \int_0^4 2\sqrt{y} - \sqrt{y}\,dy = \int_0^4 y^{1/2}\,dy = \left(\frac{y^{3/2}}{3/2}\right)_0^4 = \frac{4^{3/2}}{3/2} = \frac{8}{3/2} = \frac{16}{3}.$$

4. Paraméteresen adott görbe alatti terület

$x(t), y(t)$ $\quad \alpha \le t \le \beta$

Tfh $x(t)$ monoton nő



$t_0 = \alpha \; t_1 t_2 \; t_{i-1} t_i \qquad \beta = t_n$

$x(t_i) = x_i$

$$ter = \Delta ter_1 + \Delta ter_2 + \dots + \Delta ter_n = \sum_{i=1}^{n} \Delta ter_i$$

Ha $y(t)$ folyt, akkor

$$\Delta ter_i \approx \underbrace{(x_i - x_{i-1})}_{\Delta x_i} y(\xi_i) =$$

$$x_i - x_{i-1} = x(t_i) - x(t_{i-1})$$

Tudjuk: $\dot{x}(t_{i-1}) = \lim_{t \to t_{i-1}} \frac{x(t) - x(t_{i-1})}{t - t_{i-1}} \approx$

$$\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \Rightarrow x(t_i) - x(t_{i-1}) \approx \dot{x}(t_i)\underbrace{(t_i - t_{i-1})}_{\Delta t}$$

Így $\Delta ter_i \approx \dot{x}(t_i) y(\xi_i) \Delta t_i \approx \dot{x}(\xi_i) y(\xi_i) \Delta t_i$

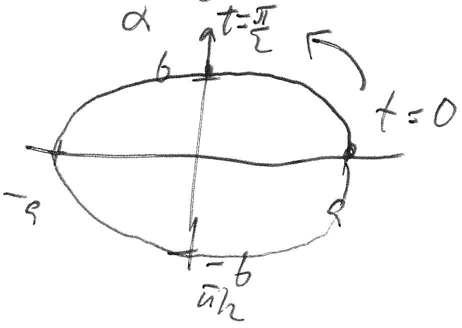
ha $\dot{x}$ folyt

Így $ter \approx \sum_{i=1}^{n} \dot{x}(\xi_i) y(\xi_i) \Delta t_i \approx \int_{\alpha}^{\beta} \dot{x}\, y \, dt$

Tehát ha $x(t)$ monoton nő: ter $= \int_\alpha^\beta \dot{x}\, y\, dt$.

Ha $x(t)$ monoton csökken: ter $= -\int_\alpha^\beta \dot{x}\, y\, dt$.

Pl. 1. Ellipszis: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$x = a\cos t \qquad 0\le t\le \frac{\pi}{2}$

$y = b\sin t$

$$\text{ter} = -4\int_0^{\pi/2} -a\sin t \cdot b\sin t\,dt = 4ab\int_0^{\pi/2}\sin^2 t\,dt = 4ab\int_0^{\pi/2}\frac{1-\cos 2t}{2}\,dt$$

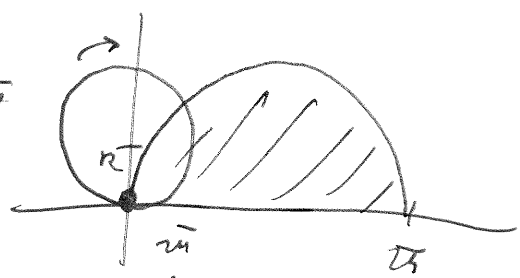
$$=4ab\int_0^{\pi/2}\frac12-\frac12\cos 2t\,dt = 4ab\left[\frac12 t - \frac12\,\frac{\sin 2t}{2}\right]_0^{t=\pi/2} =$$

$$4ab\left(\frac12\cdot\frac{\pi}{2}-\frac12\,\frac{\sin\pi}{2}-\left(\frac12\cdot 0-\frac12\,\frac{\sin 0}{2}\right)\right) = ab\pi$$

2. Ciklois

$x = R(t-\sin t) \qquad 0\le t\le 2\pi$

$y = R(1-\cos t)$

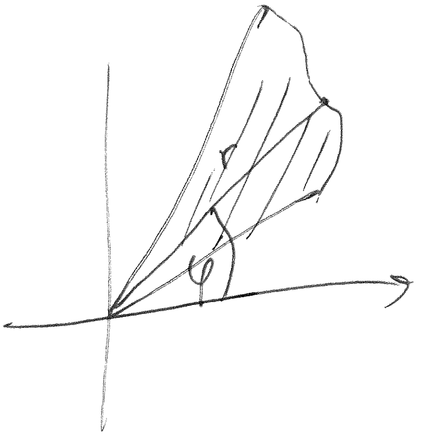


$$\text{ter} = \int_0^{2\pi} R(1-\cos t)\cdot R(1-\cos t)\,dt = R^2\int_0^{2\pi} 1-2\cos t+\underbrace{\cos^2 t}_{\frac{1+\cos 2t}{2}}\,dt =$$

$$R^2\int_0^{2\pi}\frac32-2\cos t+\frac12\cos 2t\,dt =$$

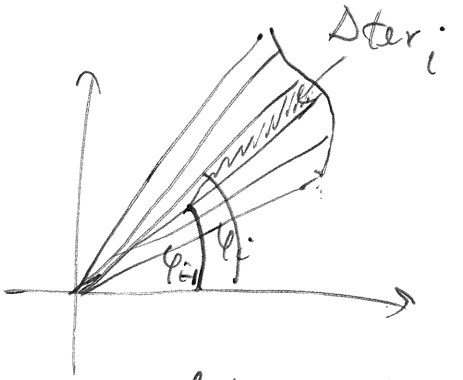
$$R^2\left[\frac32 t-2\sin t+\frac12\cdot\frac{\sin 2t}{2}\right]_0^{2\pi} = R^2\left(\frac32\cdot 2\pi-2\sin 2\pi+\frac12\,\frac{\sin 4\pi}{2}-\left(\frac32\cdot 0-2\sin 0+\frac12\cdot\frac{\sin 0}{2}\right)\right) = 3R^2\pi$$

Polárkoordinátákkal

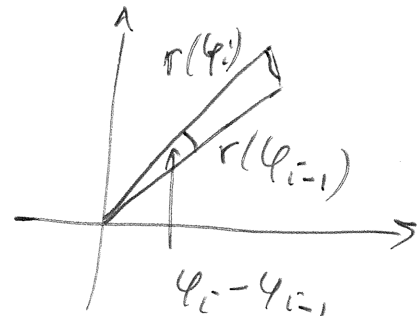


$\varphi \to r, \quad r = r(\varphi) \quad \alpha \le \varphi \le \beta$

$t = ?$



$ter = \sum_{i=1}^{n} \Delta ter_i$



háromszög területe

$$\Delta ter_i \approx \frac{r(\varphi_i)\, r(\varphi_{i-1}) \sin(\varphi_i - \varphi_{i-1})}{2}$$

Ha $r(\varphi)$ folyt, $\varphi_{i-1} \le \xi_i \le \varphi_i \Rightarrow r(\varphi_i) \approx r(\xi_i)$

$r(\varphi_{i-1}) \approx r(\xi_i)$

Tudjuk: $\lim\limits_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \Rightarrow \sin \Delta x \approx \Delta x$, ha Δx kicsi

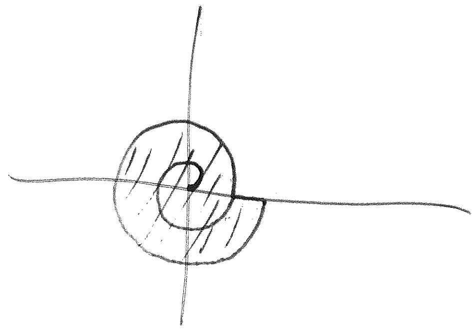
Így $\sin(\varphi_i - \varphi_{i-1}) \approx \varphi_i - \varphi_{i-1} = \Delta \varphi_i$

Tehát $\Delta ter_i \approx \frac{1}{2} r^2(\xi_i) \Delta \varphi_i$, $ter \approx \sum_{i=1}^{n} \frac{1}{2} r^2(\xi_i) \Delta \varphi_i \approx$

$\approx \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\varphi$

$$\text{terület} = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\varphi$$

Pl. $r = R\varphi \qquad 2\pi \le \varphi \le 4\pi$



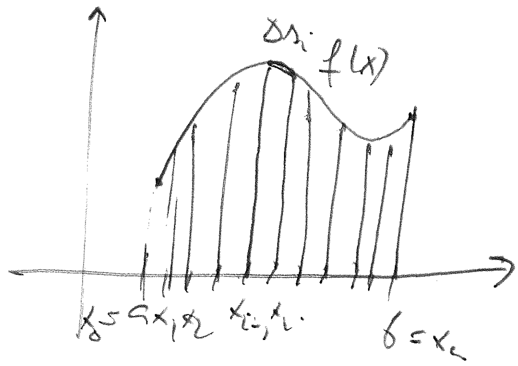
$$T = \frac{1}{2}\int_{2\pi}^{4\pi} r^2\, d\varphi = \frac{1}{2}\int_{2\pi}^{4\pi} R^2\varphi^2\, d\varphi =$$

$$\frac{1}{2}R^2\left[\frac{\varphi^3}{3}\right]_{2\pi}^{\varphi=4\pi} = \frac{1}{2}\frac{R^2}{3}\left((4\pi)^3-(2\pi)^3\right) =$$

$$\frac{28}{3}R^2\pi^3$$

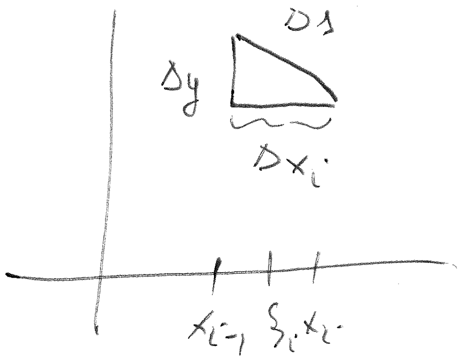
II. Síkgörbe ívhossza

1. Explicit



$s = ?$

$$s = \Delta s_1 + \Delta s_2 + \dots + \Delta s_n = \sum_{i=1}^{n} \Delta s_i$$



$$(\Delta s_i)^2 \approx (\Delta x_i)^2 + (\Delta y_i)^2$$

$$\Delta s \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x_i)^2\left(1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right)}$$

$$\frac{\Delta y_i}{\Delta x_i} \approx f'(x_i) \approx f'(\xi_i)$$

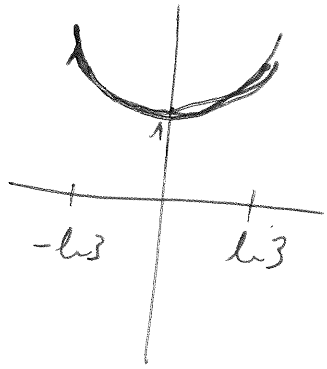
$$\Rightarrow \Delta s_i \approx \sqrt{1+(f'(\xi_i))^2}\,\Delta x_i \Rightarrow s \approx \sum_{i=1}^{n}\sqrt{1+(f'(\xi_i))^2}\,\Delta x_i$$

$$\approx \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Pl. 1. $y = \operatorname{ch} x \qquad -\ln 3 \le x \le \ln 3$

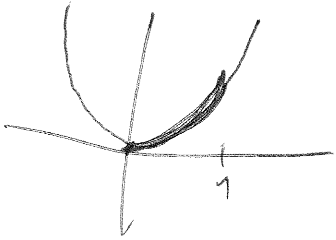
$$s = \int_{-\ln 3}^{\ln 3} \sqrt{1+(\operatorname{sh} x)^2}\, dx = \int_{-\ln 3}^{\ln 3} \operatorname{ch} x \, dx = \left[\operatorname{sh} x\right]_{-\ln 3}^{\ln 3} = \operatorname{sh}(\ln 3) - \operatorname{sh}(-\ln 3)$$

$y' = \operatorname{sh} x \qquad 1 + \operatorname{sh}^2 x = \operatorname{ch}^2 x$



$$= \frac{e^{\ln 3} - e^{-\ln 3}}{2} - \frac{e^{-\ln 3} - e^{-(-\ln 3)}}{2} = 3 - \frac{1}{3} = \frac{8}{3}$$

2. $y = x^2 \qquad 0 \le x \le 1$



$$s = \int_0^1 \sqrt{1+(2x)^2}\, dx =$$

$2x = \operatorname{sh} t \quad \Rightarrow \quad t = \operatorname{arsh}(2x)$

$x = \frac{1}{2} \operatorname{sh} t$

$dx = \frac{1}{2} \operatorname{ch} t \, dt$

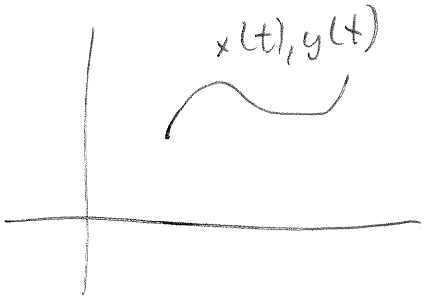
$$\int \sqrt{1+(2x)^2}\, dx = \int \underbrace{\sqrt{1+\operatorname{sh}^2 t}}_{\operatorname{ch} t} \frac{1}{2} \operatorname{ch} t \, dt = \int \frac{1}{2} \operatorname{ch}^2 t \, dt = \frac{1}{2} \int \frac{1+\operatorname{ch} 2t}{2} dt =$$

$$\frac{1}{4} \int 1 + \operatorname{ch} 2t \, dt = \frac{1}{4}\left(t + \frac{\operatorname{sh} 2t}{2}\right) = \frac{1}{4} t + \frac{1}{8} \underbrace{\operatorname{sh} 2t}_{2 \operatorname{sh} t \operatorname{ch} t = 2 \operatorname{sh} t \sqrt{1+\operatorname{sh}^2 t}} = \frac{1}{4} \operatorname{arsh}(2x) + \frac{1}{4} x \sqrt{1+(2x)^2}$$

$\Rightarrow$

$$\left[\frac{1}{4} \operatorname{arsh}(2x) + \frac{1}{4} x \sqrt{1+(2x)^2}\right]_0^1 = \frac{1}{4} \operatorname{arsh} 2 + \frac{1}{4} \sqrt{5}$$

Paraméteresen adott görbe ívhossza



$\alpha \le t \le \beta$

$$s = \int_{\alpha}^{\beta} \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt$$

Pl. 1. Ciklois

$x = R(t - \sin t)$ $\qquad \dot{x} = R(1 - \cos t)$ $\qquad 0 \le t \le 2\pi$

$y = R(1 - \cos t)$ $\qquad \dot{y} = R \sin t$

$$(\dot{x})^2 + (\dot{y})^2 = R^2(1-\cos t)^2 + R^2(\sin t)^2 = R^2(1 - 2\cos t + \cos^2 t + \sin^2 t) =$$

$$R^2(2 - 2\cos t) = 2R^2\left(1 - \cos 2 \cdot \frac{t}{2}\right) = 2R^2\left(\cos^2\frac{t}{2} + \sin^2\frac{t}{2} - \left(\cos^2\frac{t}{2} - \sin^2\frac{t}{2}\right)\right)$$

$$= 4R^2 \sin^2\frac{t}{2} \Rightarrow \sqrt{4R^2\sin^2\frac{t}{2}} = 2R\sin\frac{t}{2}$$

$0 \le \frac{t}{2} \le \pi$

$$s = \int_0^{2\pi} 2R \cdot \sin\frac{t}{2}\, dt = 2R \cdot \left[-\frac{\cos t/2}{1/2}\right]_0^{2\pi} = -4R\overbrace{\cos\pi}^{-1} - \left(-4R\overbrace{\cos 0}^{1}\right) =$$

$$= 8R$$

2. Ellipszis: $\quad x = a\cos t$ $\qquad 0 \le t \le \pi$

$\qquad\qquad\quad y = b \sin t$

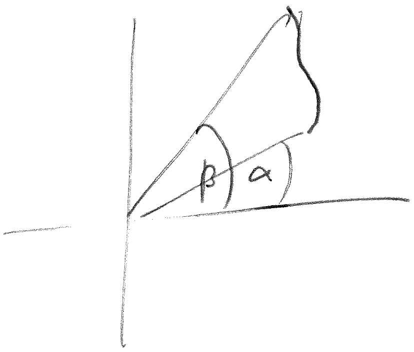
$$s = \int_0^{2\pi} \underbrace{\sqrt{a^2\sin^2 t + b\cos^2 t}}_{\text{nincs szép primitív fv}}\, dt$$

Vannak olyan fv-ek, aminek nincs elp primitív fv-e: 13.

$\int \sqrt{a^2 \sin^2 x + b^2 \cos^2 x}\, dx$, $\int \sqrt{1+x^3}\, dx$, $\int \frac{1}{\log x}\, dx$, $\int e^{x^2} dx$,

$\int \frac{\sin x}{x}\, dx$

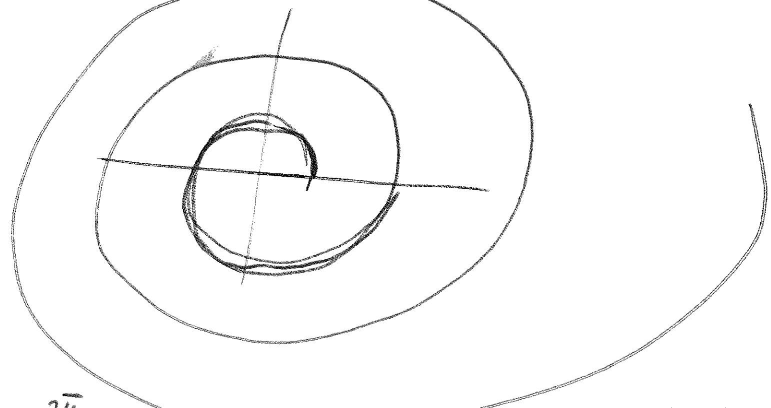
Polárkoordinátával adott görbe ívhossza:



$$s = \int_{\alpha}^{\beta} \sqrt{(r(\varphi))^2 + (r'(\varphi))^2}\, d\varphi$$

Pl. Logaritmikus spirál.

$r = e^{a\varphi}$ $\quad \varphi \geq 0$



$0 \leq \varphi \leq 2\pi$

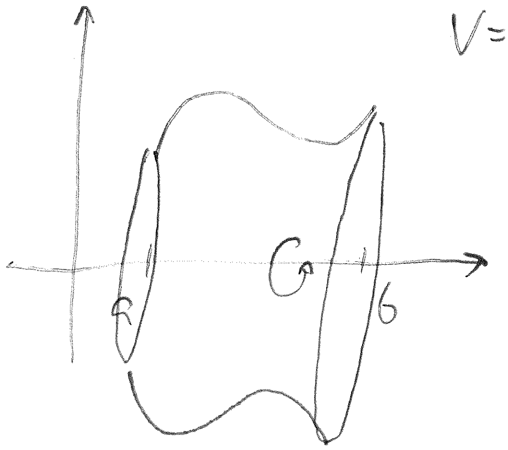
$$s = \int_0^{2\pi} \sqrt{e^{2a\varphi} + (a e^{a\varphi})^2} =$$

$$\int_0^{2\pi} \sqrt{(1+a^2)\, e^{2a\varphi}}\, d\varphi = \sqrt{1+a^2} \int_0^{2\pi} e^{a\varphi} d\varphi = \sqrt{1+a^2} \left[\frac{e^{a\varphi}}{a} \right]_0^{\varphi = 2\pi} =$$

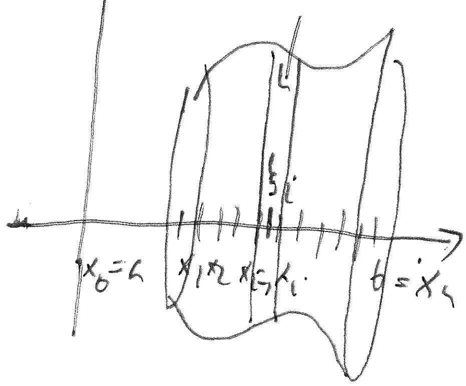
$$\frac{\sqrt{1+a^2}}{a} \left(e^{2\pi a} - 1 \right).$$

3. Forgástest térfogata

1. $y = f(x)$



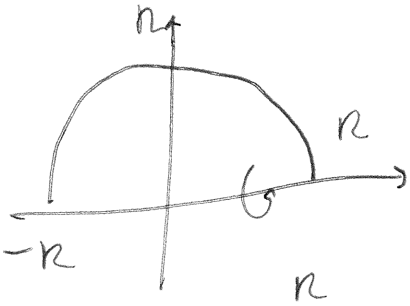
ΔV_i térfogat



$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} f(\xi_i)^2 \pi \Delta x_i \approx \pi \int_a^b f(x)^2 dx$$

$$\Delta V_i \approx f(\xi_i)^2 \pi \underbrace{\Delta x_i}_{x_i - x_{i-1}}$$ (korong térfogata)

Pl. 1. R sugarú gömb térfogata

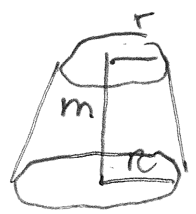


$x^2+y^2=R^2$
$y^2=R^2-x^2$
$y=\sqrt{R^2-x^2}=f(x)$

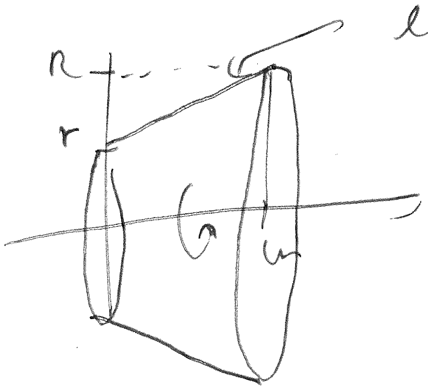
$$V_{gömb} = \pi \int_{-R}^{R} \left(\sqrt{R^2-x^2}\right)^2 dx = \pi \int_{-R}^{R} R^2-x^2 \, dx =$$

$$\pi \left[R^2 x - \frac{x^3}{3}\right]_{-R}^{R} = \pi\left(R^3 - \frac{R^3}{3} - \left(-R^3 - \frac{(-R)^3}{3}\right)\right) = \frac{4R^3\pi}{3}$$

2. Csonkakúp térfogata = ?



egyenes egyenlete:



$$y = \frac{R-r}{m}x + r$$

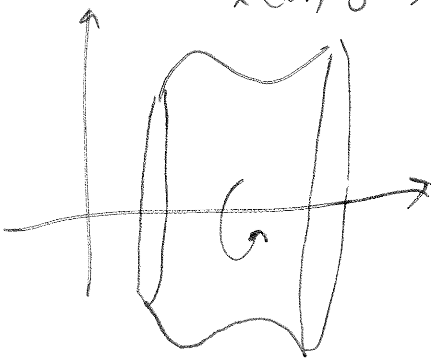
$$V = \pi \int_0^m \left(\frac{R-r}{m}x + r\right)^2 dx = \pi \left[\frac{\left(\frac{R-r}{m}x + r\right)^3}{3 \cdot \frac{R-r}{m}} \right]_0^m$$

$$= \frac{\pi}{3} \cdot \frac{m}{R-r} \cdot \left(\left(\frac{R-\cancel{r}}{\cancel{m}} \cdot \cancel{m} + \cancel{r} \right)^3 - r^3 \right) = \frac{\pi}{3} \frac{m(R^3 - r^3)}{R-r} =$$

$$\frac{\pi m (R-r)(R^2 + rR + r^2)}{3(R-r)} = \frac{\pi m (R^2 + rR + r^2)}{3}$$

2. Paraméteresen adott

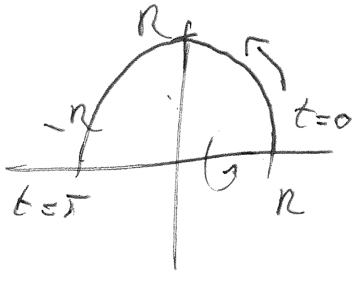
$x(t), y(t)$ $\qquad \alpha \le t \le \beta$



$$V = \pi \int_\alpha^\beta \dot{x} y^2 \, dt \quad \text{ha } x(t) \text{ nő}$$

$$V = -\pi \int_\alpha^\beta \dot{x} y^2 \, dt \quad \text{ha } x(t) \text{ csökken}$$

Pl. 1. Gömb térfogata:



$x(t) = R\cos t$ $\qquad 0 \le t \le \pi$

$y(t) = R\sin t$

$x(t)$ monoton csökken

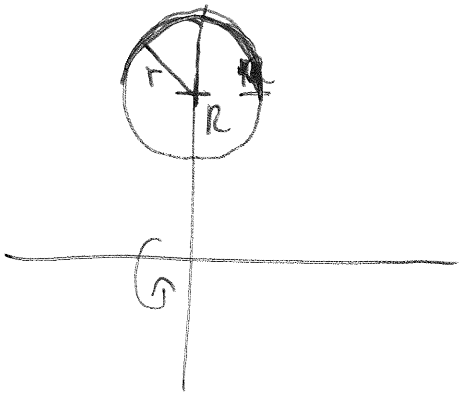
$$V = -\pi \int_0^{\pi} -R\sin t \cdot R^2 \sin^2 t \, dt = \pi R^3 \int_0^{\pi} \underbrace{\sin^3 t}\, dt =$$

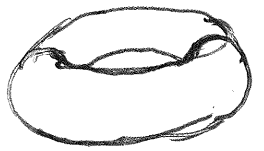
$$\sin t \cdot \sin^2 t = \sin t\,(1-\cos^2 t)$$

$$\pi R^3 \int_0^{\pi} \sin t - \sin t \cos^2 t \, dt = \pi R^3 \left[-\cos t + \frac{\cos^3 t}{3} \right]_{0}^{t=\pi} =$$

$$\pi R^3 \left(-\overbrace{\cos \pi}^{-1} + \frac{\overbrace{\cos(-\pi)}^{-1}}{3} - \left(-\overbrace{\cos 0}^{1} + \frac{\overbrace{\cos 0}^{1}}{3} \right) \right) = \frac{4R^3\pi}{3}$$

2. Tórusz („úszógumi")





$$x = r\cos t \qquad 0 \le t \le 2\pi$$

$$y = r\sin t + R$$

$$V = -\pi \int_0^{\pi} (-r\sin t)(r\sin t + R)^2 \, dt - \pi \int_{\pi}^{2\pi} (-r\sin t)(r\sin t + R)^2 \, dt$$

$$= \pi r \int_0^{\pi} \sin t \left(r^2 \sin^2 t + 2rR\sin t + R^2 \right) dt +$$

$$\pi r \int_{\pi}^{2\pi} \sin t \left(r^2 \sin^2 t + 2rR\sin t + R^2 \right) dt =$$

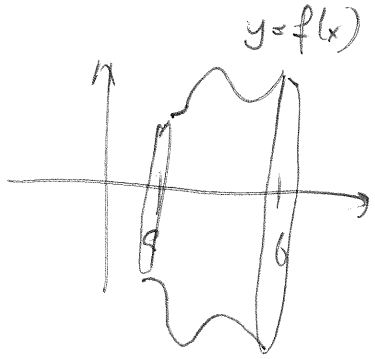
$$= \pi r \int_0^{2\pi} \underbrace{r^2 \sin^3 t}_{\sin t - \sin t \cos^2 t} + 2rR \overbrace{\sin^2 t}^{\frac{1-\cos 2t}{2}} + R^2 \sin t \, dt =$$

$$\pi r \int_0^{2\pi} r^2 \sin t - r^2 \sin t \cos^2 t + rR - rR\cos 2t + R^2 \sin t \, dt =$$

$$\pi r \left[-r^2 \cos t + r^2 \frac{\cos^3 t}{3} + rRt - \frac{rR\sin 2t}{2} - R^2 \cos t \right]_0^{2\pi} = 2Rr^2\pi^2$$

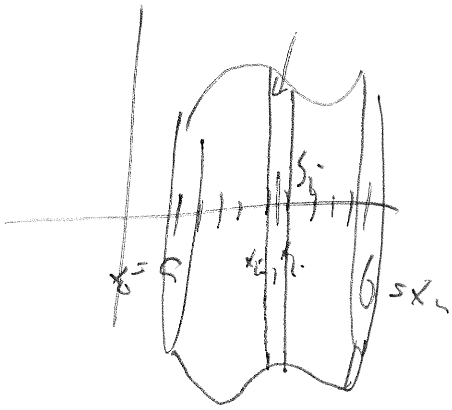
4. Forgástest felszíne

1.



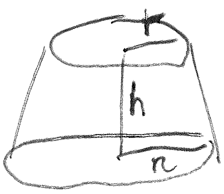
$F = ?$

kis felszín: ΔF_i



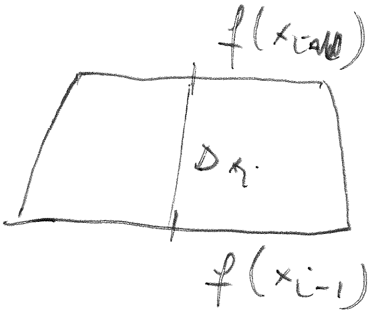
$F = \Delta F_i$

csonkakúp



palást felszín: $\pi\sqrt{h^2+(R-r)^2}\,(R+r)$

ΔF_i:



$$\Delta F_i \approx \pi\sqrt{(\Delta x_i)^2+(f(x_i)-f(x_{i-1}))^2}\,\left(f(x_i)+f(x_{i-1})\right)$$

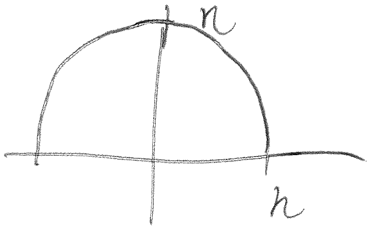
$$\pi\,\Delta x_i\sqrt{1+\underbrace{\left(\frac{f(x_i)-f(x_{i-1})}{\Delta x_i}\right)^2}_{\approx f'(x_i)\approx f'(\xi_i)}}\;\underbrace{\left(f(x_i)+f(x_{i-1})\right)}_{\approx 2f(\xi_i)}$$

$$\Rightarrow \Delta F_i \approx 2\pi\sqrt{1+(f'(\xi_i))^2}\,f(\xi_i)$$

$$\Rightarrow F \approx \sum_{i=1}^{n} 2\pi\sqrt{1+(f'(\xi_i))^2}\,f(\xi_i) \approx 2\pi\int_a^b \sqrt{1+(f'(x))^2}\,f(x)\,dx$$

$$F = 2\pi\int_a^b \sqrt{1+(f'(x))^2}\,f(x)\,dx$$

Pl. Gömb felszíne:



$$x^2+y^2=R^2$$

$$y^2=\sqrt{R^2-x^2}=f(x)$$

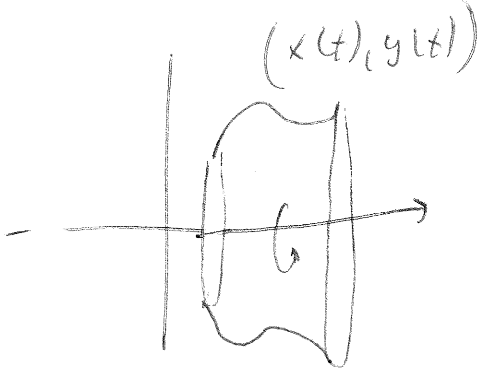
$$f'(x)=\frac{-2x}{2\sqrt{R^2-x^2}}=-\frac{x}{\sqrt{R^2-x^2}}$$

$$F=2\pi\int_{-R}^{R}\sqrt{1+\left(-\frac{x}{\sqrt{R^2-x^2}}\right)^2}\cdot\sqrt{R^2-x^2}\,dx=2\pi\int_{-R}^{R}\sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}}\sqrt{R^2-x^2}\,dx$$

$$2\pi\int_{-R}^{R}\frac{R}{\sqrt{R^2-x^2}}\sqrt{R^2-x^2}\,dx=2\pi\left[Rx\right]_{-R}^{R}=2\pi\left(R^2-(-R^2)\right)=$$

$$4R^2\pi$$

2. Paraméteresen adott görbe esetén



$$\alpha\le t\le\beta$$

$$F=2\pi\int_{\alpha}^{\beta}y\sqrt{(\dot x)^2+(\dot y)^2}\,dt$$

Pl. tórusz felszíne $\quad x=r\cos t,\quad y=R+r\sin t \quad 0\le t\le 2\pi$

$$\dot x=-r\sin t,\quad \dot y=r\cos t \Rightarrow \sqrt{(\dot x)^2+(\dot y)^2}=\sqrt{r^2\sin^2 t+r^2\cos^2 t}=r$$

$$F=2\pi\int_{-\pi}^{\pi}(R+r\sin t)\,r\,dt=2\pi\int_{0}^{2\pi}Rr+r^2\sin t\,dt=$$

$$2\pi\left[Rrt-r^2\cos t\right]_0^{2\pi}=4\pi^2 rR$$

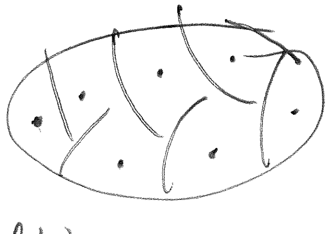
Súlypont

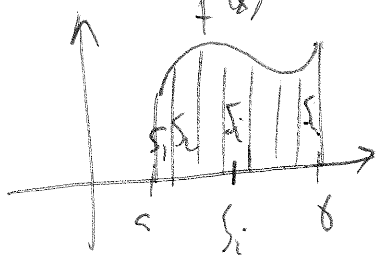
Legyen $f(x) \geq 0$. Mi az $f(x)$ görbe alatti váltomány homogén (1 sűrűségű) lemez súlypontja?

Súlypont tulajdonságai:

1. Ha a $\underline{v}_1 = (x_1, y_1)$ pontban m_1 tömegpont
$\underline{v}_2 = (x_2, y_2)$ m_2
$\vdots$
$\underline{v}_n = (x_n, y_n)$ m_n
van, akkor ennek súlypontja $\dfrac{\sum_{i=1}^n m_i \underline{v}_i}{\sum_{i=1}^n m_i}$

2. Egy test súlypontját úgy számolhatjuk, hogy részekre osztjuk és egy részt helyettesítünk a rész súlypontjában levő megfelelő súlyú tömegponttal.





$\underline{s}$?

S_i súlypontja $\approx \left(\xi_i, \dfrac{f(\xi_i)}{2}\right) = \underline{v}_i$

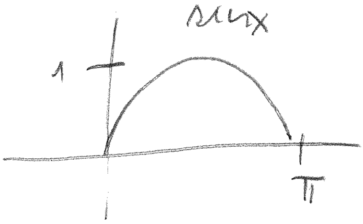
tömege $= m_i \approx \Delta x_i f(\xi_i)$

$$\underline{s} = \frac{\sum_{i=1}^m m_i \underline{v}_i}{\sum_{i=1}^n m_i} = \frac{\sum_{i=1}^n \left(\xi_i, \frac{1}{2} f(\xi_i)\right) f(\xi_i) \Delta x_i}{\sum_{i=1}^n f(\xi_i) \Delta x_i} =$$

$$\frac{\left(\sum_{i=1}^n \xi_i f(\xi_i) \Delta x_i, \sum_{i=1}^n \frac{1}{2} f^2(\xi_i) \Delta x_i\right)}{\sum_{i=1}^n f(\xi_i) \Delta x_i} \approx \left(\frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}, \frac{\frac{1}{2}\int_a^b f^2(x)\,dx}{\int_a^b f(x)\,dx}\right)$$

Tehát $\underline{s} = \left(\dfrac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx} , \dfrac{\frac{1}{2}\int_a^b f^2(x)\,dx}{\int_a^b f(x)\,dx} \right)$

Pl.



súlypont

$$\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = -(\cos\pi) - (-\cos 0) = 2$$

$$\int_0^{\pi} \underset{u}{x}\,\underset{v'}{\sin x}\,dx = \left[-x\cos x\right]_0^{\pi} + \int_0^{\pi} +\cos x\,dx = -\pi\cos\pi + \left(\sin x\right]_0^{\pi} = \pi$$

$u' = 1 \quad v = -\cos x$

$$\int_0^{\pi} \sin^2 x\,dx = \int_0^{\pi} \frac{1-\cos 2x}{2}\,dx = \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2x\,dx = \left[\frac{1}{2}x - \frac{1}{4}\,\frac{\sin 2x}{2}\right]_0^{\pi} =$$

$\frac{1}{2}\pi$

$\Rightarrow \underline{s} = \left(\frac{\pi}{2} , \frac{\pi}{8} \right)$