An upper bound for Hilbert cubes

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Abstract

In this note we give a new upper bound for the largest size of subset of $\{1, 2, \ldots, n\}$ not containing a k-cube.

1. Introduction

We call a set H a Hilbert cube of dimension k or simply a k-cube if there are positive integers $a_0, a_1, \ldots, a_k$ such that

$$H = \{a_0 + \sum_{i=1}^{k} \epsilon_i a_i : \epsilon_i \in \{0, 1\}\}.$$

The positive integer k is the dimension of the Hilbert cube. Hilbert originally proved that if the positive integers are colored with finitely many colors then one color class contains a k-cube. The density version of theorem was proved by Szemerédi and has since become known as "Szemerédi's cube lemma" (see e.g. [3]):

Theorem. *Let $k \geq 2$ be a positive integer. If the sequence S_n satisfies $|S_n| \geq (4n)^{1-\frac{1}{2^{k-1}}}$ then S_n contains a k-cube.*

Denote by $H_k(n)$ be the largest size of subset of $\{1, 2, \ldots, n\}$ not containing a k-cube. Gunderson and Rödl improved the above result to $H_k(n) < 2^{1-\frac{1}{2^{k-1}}}(\sqrt{n}+1)^{2-\frac{1}{2^{k-2}}}$ (see [2]).

A sequence S is called Sidon sequence if the sums $s_1 + s_2$, $s_1, s_2 \in S$, $s_1 \leq s_2$ are distinct. Obviously a sequence is Sidon if and only if it does not contain any 2-cubes. It is well known that the maximal size of Sidon sequences can be selected from $\{1, 2, \ldots, n\}$ is at most $n^{1/2} + O(n^{1/4})$ (see [1]), that is $H_2(n) < n^{1/2} + O(n^{1/4})$. A very short proof of this fact was given by Lindström (see [4]). Using his method we get the following result

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Theorem *For every $k \geq 3$ we have $H_k(n) < n^{1-\frac{1}{2^{k-1}}} + O(n^{1-\frac{1}{2^{k-2}}})$, where the constant depends on k.*

2. Proof

We will argue by induction. Let us suppose that either $k = 3$ or $k > 3$ and we have verified the statement for $k-1$, that is $H_{k-1}(n) < n^{1-\frac{1}{2^{k-2}}} + O(n^{1-\frac{1}{2^{k-3}}})$ and we prove the theorem for k. Let us suppose that the sequence $1 \leq a_1 < a_2 < \ldots < a_s \leq n$ does not contain any k-cubes. We have to prove that $s < n^{1-\frac{1}{2^{k-1}}} + O(n^{1-\frac{1}{2^{k-2}}})$. Let $r = H_{k-1}(n)$. We will give lower and upper bound for the sum

$$K = \sum_{1 \leq i-j \leq r} a_i - a_j.$$

First we give a lower bound for K. Since the above sequence does not contain any k-cubes, therefore a difference d occurs at most r-times in this sum. This sum contains $rs - \frac{r(r+1)}{2} = rw$ ($w = s - \frac{r+1}{2}$) terms, hence K is at least r-times of the sum of the first $[\frac{rw}{r}] = [w]$ positive integers. Hence

$$K \geq r\frac{[w]([w]+1)}{2} \geq r\frac{w^2 - 0.25}{2}.$$

In the following we give an upper bound for K. The differences in the sum K can be arranged in sequences of type

$$(a_{u+t} - a_t) + (a_{2u+t} - a_{u+t}) + \cdots + (a_{[\frac{n-t}{u}]u+t} - a_{([\frac{n-t}{u}]-1)u+t}) \leq n,$$

where $1 \leq u \leq r$, $1 \leq t \leq u$. Hence

$$K \leq n\frac{r(r+1)}{2}.$$

Compering the bounds we have $r\frac{w^2-0.25}{2} \leq n\frac{r(r+1)}{2}$, that is $w^2 \leq nr + n + 0.25$. Hence

$$s = w + \frac{r+1}{2} \leq \sqrt{nr + n + 0.25} + \frac{r+1}{2}$$

For $k = 3$ we have $r < n^{1/2} + O(n^{1/4})$ which implies

$$s < n^{0.75} + O(n^{0.5}).$$

For $k > 3$ we have $r < n^{1-\frac{1}{2^{k-2}}} + O(n^{1-\frac{1}{2^{k-3}}})$, thus

$$s \leq \sqrt{n^{2-\frac{1}{2^{k-2}}} + O(n^{2-\frac{1}{2^{k-3}}})} + O(n^{1-\frac{1}{2^{k-2}}}) = n^{1-\frac{1}{2^{k-1}}} + O(n^{1-\frac{1}{2^{k-2}}}),$$

which proves the theorem. ∎

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