A note on a conjecture of Erdős-Turán

Csaba Sándor[1]
Department of Stochastics, Budapest University of Technology and Economics, Hungary
csandor@math.bme.hu

Abstract

Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of nonnegative integers. In this note we prove that for $s(x) = \sum_{n=1}^{\infty} x^{a_n}$ and $s^2(x) = \sum_{n=0}^{\infty} b_n x^n$ the condition $\limsup_{n\to\infty} b_n = A$ for some positive integer A implis that $\liminf_{n\to\infty} b_n \le A - 2\sqrt{A} + 1$.

1. Introduction

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers. Let

$$s(x) = \sum_{n=1}^{\infty} x^{a_n}$$

and

$$s(x)^2 = \sum_{n=0}^{\infty} b_n x^n.$$

The sequence $\{a_n\}_{n=1}^{\infty}$ is called additive basis of order two if $b_n > 0$ for every nonnegative integer n and asymptotic additive basis of order two if $b_n > 0$ for every sufficiently large n. The Erdős-Turán conjecture says that for any additive basis of order two $\{a_n\}_{n=1}^{\infty}$ the sequence $\{b_n\}_{n=0}^{\infty}$ is unbounded. This conjecture can be rephrased in number theoretic language: Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers. Denote by $R(n)$ the number of solution $n = a_i + a_j$ i.e.

$$R(n) = \#\{(i,j) : n = a_i + a_j\}.$$

Using this representation function the original Erdős-Turán conjecture can be stated as follows,

Conjecture 1 (Erdős-Turán conjecture for bases of order two) *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers such that $R(n) > 0$ for every nonnegative integer n. Then the sequence $\{R(n)\}_{n=0}^{\infty}$ is unbounded.*

[1]Supported by Hungarian National Foundation for Scientific Research, Grant No T 49693 and 61908
AMS Mathematics Subject Classification: 11B83
Key words and phrases: Representation function, Erdős-Turán conjecture.

Grekos, Haddad, Helou and Pihko [3] proved that $\limsup_{n\to\infty} R(n) \geq 6$ for every basis $\{a_n\}$. Later Borwein, Choi and Chu [1] improved it to $\limsup_{n\to\infty} R(n) \geq 8$. The above conjecture is equivalent to

Conjecture 2 (Erdős-Turán conjecture for asymptotic bases of order two) *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers such that $R(n) > 0$ for every $n \geq n_0$. Then the sequence $\{R(n)\}_{n=0}^{\infty}$ is unbounded.*

The second version can be formulated as

Conjecture 3 (Erdős-Turán conjecture for bounded representation function) *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers and*

$$\limsup_{n\to\infty} R(n) = A$$

for some positive integer A. Then we have

$$\liminf_{n\to\infty} R(n) = 0.$$

In this note we give a non-trivial upper bound for $\liminf_{n\to\infty} R(n)$ if the sequence $\{R(n)\}_{n=0}^{\infty}$ is bounded.

Theorem 1 *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers and*

$$\limsup_{n\to\infty} R(n) = A$$

for some positive integer A. Then we have

$$\liminf_{n\to\infty} R(n) \leq A - 2\sqrt{A} + 1.$$

2. Proof

If $a_n > n^2$ for infinitely many integer n, then $R(n) = 0$ for infinitely many integer n. Hence $\liminf_{n\to\infty} R(n) = 0 \leq A - 2\sqrt{A} + 1$, which proves the theorem. Therefore we may assume that

$$a_n \leq n^2 \qquad \text{for } n \geq n_1. \tag{1}$$

Let us suppose that there exists a strictly increasing sequence of nonnegative integers $\{a_n\}_{n=1}^{\infty}$ such that $\limsup_{n\to\infty} R(n) = A$ but $\liminf R(n) > A - 2\sqrt{A} + 1$. Then there exist an integer n_2 and $0 < \epsilon < \sqrt{A}$ for which $A - 2\sqrt{A} + 1 + \epsilon \leq R(n) \leq A$ for $n \geq n_2$. Set $C = A - \sqrt{A} + \epsilon$. By elementary calculus we have $f(x) = \frac{(x-C)^2}{x} < 1$ for every $x \in [A - 2\sqrt{A} + 1 + \epsilon, A]$, therefore there exists a $\delta > 0$ such that

$$(R(n) - C)^2 \leq (1-\delta)^2 R(n) \qquad \text{for } n \geq n_2. \tag{2}$$

Let

$$F(z)=\sum_{n=1}^{\infty} z^{a_n}.$$

Then

$$F^2(z)=\sum_{n=0}^{\infty} R(n)z^n.$$

Let

$$z=(1-\frac{1}{N})e^{2\pi i\alpha}=re^{2\pi i\alpha},$$

where N is a large integer. We give an upper an lower bound for the integral

$$\int_0^1 |F^2(z)-\sum_{n=0}^{\infty} Cz^n|d\alpha \tag{3}$$

to reach a contradiction. We get an upper bound for (3) by Cauchy's inequality, Parseval's formula and (2)

$$\int_0^1 |F^2(z)-\sum_{n=0}^{\infty} Cz^n|d\alpha=\int_0^1 |\sum_{n=0}^{\infty}(R(n)-C)z^n|d\alpha\le\left(\int_0^1 |\sum_{n=0}^{\infty}(R(n)-C)z^n|^2\right)^{1/2}=$$

$$\left(\sum_{n=0}^{\infty}(R(n)-C)^2r^{2n}\right)^{1/2}\le\left(c_1+(1-\delta)^2(\sum_{n=0}^{\infty}R(n)r^{2n})\right)^{1/2}\le c_2+(1-\delta)F(r^2) \tag{4}$$

Now the lower bound for (3). Obviously,

$$\int_0^1 |F^2(z)-\sum_{n=0}^{\infty} Cz^n|d\alpha\ge\int_0^1 |F^2(z)|d\alpha-\int_0^1 |\sum_{n=0}^{\infty} Cz^n|d\alpha, \tag{5}$$

where by Parseval's formula

$$\int_0^1 |F^2(z)|d\alpha=\sum_{n=1}^{\infty} r^{2a_n}=F(r^2). \tag{6}$$

Moreover

$$\int_0^1 |\sum_{n=0}^{\infty} Cz^n|d\alpha=C\int_0^1 \frac{1}{|1-z|}d\alpha=2C\int_0^{1/2}\frac{1}{|1-z|}d\alpha.$$

Since

$$|1-z|^2=(1-r\cos 2\pi\alpha)^2+(r\sin 2\pi\alpha)^2=(1-r)^2+2r(1-\cos 2\pi\alpha)=(1-r)^2+2r\sin^2\pi\alpha,$$

therefore $|1-z|\ge max\{\frac{1}{N},\alpha\}$ for every $0<\alpha<\frac{1}{2}$. Hence

$$\int_0^1 |\sum_{n=0}^{\infty} Cz^n|d\alpha\le 2C(\int_0^{1/N} Nd\alpha)+\int_{1/N}^{1/2}\frac{1}{\alpha}d\alpha)\le c_3\log N \tag{7}$$

for some $c_3 > 0$. By (4), (6) and (7) we have

$$F(r^2) - c_3 \log N \leq \int_0^1 |F^2(z) - \sum_{n=0}^{\infty} Cz^n| d\alpha \leq (1-\delta)F(r^2) + c_2,$$

therefore

$$\delta F(r^2) < c_2 + c_3 \log N,$$

but in view of (1)

$$F(r^2) = \sum_{n=1}^{\infty} r^{2a_n} \geq \sum_{n=n_1}^{\sqrt{N}} (1 - \frac{1}{N})^{2a_n} > c_4\sqrt{N}$$

for some positive c_4, which is a contradiction to (8) if N is large enough. ■

References

[1] P. Borwein, S. Choi and F. Chen, *An old conjecture of Erdős-Turán on additive bases,* Math. Comp.

[2] P. Erdős and P. Turán, *On a problem of Sidon in additive number theory and some related problems,* J. London Math. Soc., 16:212–215, 1941.

[3] G. Grekos, L. Haddad, C. Helou and J. Pihko, *On zhe Erdős-Turán conjecture,* J. Number Theory, 102(2): 339–352, 2003.