The topology of asymptotically locally flat gravitational instantons

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August 29, 2011

Abstract

In this letter we demonstrate that the intersection form of the Hausel–Hunsicker–Mazzeo compactification of a four dimensional ALF gravitational instanton is definite and diagonalizable over the integers if one of the Kähler forms of the hyper-Kähler gravitational instanton metric is exact. This leads to their topological classification.

The proof exploits the relationship between L^2 cohomology and U(1) anti-instantons over gravitational instantons recognized by Hitchin. We then interprete these as reducible points in a singular SU(2) anti-instanton moduli space over the compactification providing the identification of its intersection form.

This observation on the intersection form might be a useful tool in the full geometric classification of various asymptotically locally flat gravitational instantons.

Keywords: *Gravitational instantons, Yang–Mills instantons*
PACS numbers: 02.40.Ma; 04.90.+e; 11.15.-q

1 Introduction

By a *gravitational instanton* we mean a connected, four dimensional complete hyper-Kähler Riemannian manifold. In particular these spaces have $SU(2) \cong Sp(1)$ holonomy consequently are Ricci flat hence solve the Euclidean Einstein's vacuum equation. Since the only compact four dimensional hyper-Kähler spaces up to universal covering are diffeomorphic to the flat torus T^4 or a $K3$ surface, for further solutions we have to seek non-compact examples. Compactness in this case is naturally replaced by the condition that the metric be complete and decay to the flat metric somehow at infinity.

Such open examples can be constructed as follows. Consider a connected, orientable compact four-manifold $\overline{M}$ with connected boundary $\partial\overline{M}$ which is a smooth three-manifold. Then the open manifold $M := \overline{M} \setminus \partial\overline{M}$ has a decomposition $M = C \cup N$ where C is a compact subset and $N \cong \partial\overline{M} \times \mathbb{R}^+$ is an open annulus or neck. Parameterize the half-line $\mathbb{R}^+$ by r. Assume $\partial\overline{M}$ admits a smooth fibration $\partial\overline{M} \to B$ over a base manifold B with fibers F. We suppose the complete hyper-Kähler metric g can be written asymptotically and locally as

$$\mathrm{d}r^2 + r^2 g_B + g_F. \tag{1}$$

In other words the base B of the fibration blows up locally in a Euclidean way as $r \to \infty$ while the volume of the fiber remains finite. By the curvature decay g_F must be flat hence F is a connected, compact, orientable flat manifold. Depending on the dimension of F we can introduce several cases of increasing transcendentality, using the terminology of Cherkis and Kapustin [2]:

(i) (M, g) is ALE (asymptotically locally Euclidean) if $\dim F = 0$;

(ii) (M, g) is ALF (asymptotically locally flat) if $\dim F = 1$, in this case necessarily $F \cong S^1$ must hold;

(iii) (M, g) is ALG (this abbreviation by induction) if $\dim F = 2$, in this case $F \cong T^2$;

(iv) (M, g) is ALH if $\dim F = 3$, in this case F is diffeomorphic to one of the six flat orientable three-manifolds.

Due to their relevance in quantum gravity or recently rather in low-energy supersymmetric solutions of string theory and last but not least their mathematical beauty, there has been some effort to classify these spaces over the past decades. Trivial examples for all cases are provided by the spaces $\mathbb{R}^{4-\dim F} \times F$ with their product flat metrics. The first two non-trivial infinite families in a rather explicit form were discovered by Gibbons and Hawking in 1976 [11]. One of these families are the A_k ALE or multi-Eguchi–Hanson spaces. In 1989 Kronheimer gave a full classification of ALE spaces [15] constructing them as minimal resolutions of $\mathbb{C}^2/\Gamma$ where $\Gamma \subset \mathrm{SU}(2)$ is a finite subgroup i.e., Γ is either a cyclic group A_k, $k \geq 0$, dihedral group D_k with $k > 0$, or one of the exceptional groups E_l with $l = 6, 7, 8$.

The other infinite family of Gibbons and Hawking is the A_k ALF or multi-Taub–NUT family. Recently another D_k ALF family has been constructed by Cherkis and Kapustin [5] and in a more explicit form by Cherkis and Hitchin [3].

Using string theory motivations, recently Cherkis and Kapustin have been suggesting a classification scheme for ALF cases as well as for ALG and ALH [2]. They claim that the A_k and D_k families with $k \geq 0$ exhaust the ALF geometry (in this enumeration D_0 is the Atiyah–Hitchin manifold) while for ALG it is conjectured that the possibilities are D_k with $0 \leq k \leq 5$ [4] and E_l with $l = 6, 7, 8$. Conjecturally, there is only one non-trivial ALH space. The trouble is that these spaces are more transcendental as $\dim F$ increases hence their constructions, involving twistor theory, Nahm transform, etc. are less straightforward and explicit.

To conclude this brief survey we remark that by the restrictive hyper-Kähler assumption on the metric which appeared to be relevant in the later string theoretic investigations, some examples considered as “gravitational instantons” in the early eighties, have been excluded later. Non-compact examples which satisfy the above fall-off conditions are for instance the Euclidean Schwarzschild solution or the Euclidean Kerr–Newman solution which are complete ALF, Ricci flat but not hyper-Kähler spaces [13]. For a more complete list of such “old” examples cf. [7].

From Donaldson theory we have learned that the moduli spaces of SU(2) instantons over compact four-manifolds encompass lot of information about the original manifold hence understanding SU(2) instantons over gravitational instantons also might be helpful in their classification. The full construction of SU(2) instantons in the ALE case was carried out by Kronheimer and Nakajima in 1990 [16]. However already in the ALF case we run into analytical difficulties and have only sporadic examples of explicit solutions (cf. e.g. [1][8] [9]) not to mention the ALG and ALH geometries.

An intermediate stage between gravitational instantons and their SU(2) instanton moduli is L^2 cohomology because L^2 harmonic 2-forms always can be regarded as U(1) (anti-)instantons hence reducible points in the (anti-)instanton moduli spaces over gravitational instantons. In spite of SU(2) instantons, L^2 harmonic forms are quite well understood mainly due to the recent paper of Hausel–Hunsicker–Mazzeo [12]. Their construction reduces the calculation of the L^2 cohomology groups of (M, g), highly non-trivial analytical objects, to the intersection hence topological cohomology of a certain compactification X of (M, g). A consequence of this is the stability of the L^2 cohomology under compact perturbations of the metric g. We will investigate this space X in more detail below. For more historical remarks on L^2 cohomology we refer to the bibliography of [12].

In this letter we focus our attention to the special case of ALF spaces. For this class X is a connected, compact, orientable, smooth four-manifold without boundary. Put an orientation onto X by extending the canonical orientation of M coming from the hyper-Kähler structure. Assume in this moment for simplicity that the metric also extends conformally to X that is we can construct an oriented Riemannian manifold $(X, \tilde{g})$ whose restriction to M is conformally equivalent to (M, g). Assume furthermore that there exists a Kähler form on (M, g) satisfying $\omega = \mathrm{d}\beta$. Then the ALF condition implies that β has linear growth and L^2 harmonic 2-forms are anti-self-dual by a theorem of Hitchin [14] hence, if suitable normed, represent U(1) anti-instantons over (M, g). Moreover they extend as reducible SU(2) anti-instantons over $(X, \tilde{g})$ by a codimension 2 singularity removal theorem of Sibner and Sibner [18], reporved by Råde [17]. Assume finally that X is simply connected. Then L^2 harmonic 2-forms provide all the theoretically possible reducible points in the moduli space. We will demonstrate here using elements of Hodge theory and Donaldson theory that this implies definiteness and diagonalizability of the intersection form of X over the integers.

This observation leads to the topological classification of ALF gravitational instantons subject to the above technical conditions using Freedman's fundamental theorem [10]. It turns out that if k denotes the dimension of the 2nd L^2 cohomology space of (M, g) then X is homeomorphic either to the four-sphere if $k = 0$ or to the connected sum of $k > 0$ copies of complex projective spaces with reversed orientation.

2 L^2 cohomology and anti-self-duality

Remember that a k-form φ over an oriented Riemannian manifold (M, g) is called an L^2 *harmonic* k*-form* if $\|\varphi\|_{L^2(M,g)} < \infty$ and $\mathrm{d}\varphi = 0$ as well as $\delta\varphi = 0$ where $\delta = - * \mathrm{d} *$ is the formal adjoint of d (and $*$ is the Hodge star operation). The space $\mathcal{H}^k(M, g)$ of all L^2 harmonic k-forms is denoted by $\overline{H}^k_{L^2}(M, g)$ and is called the *kth (reduced)* L^2 *cohomology group of* (M, g). This is a natural generalization of the de Rham cohomology over compact manifolds to the non-compact case since by de Rham's theorem $\mathcal{H}^k(M, g)$ is isomorphic to the corresponding de Rham cohomology group $H^k(M)$ over a compact manifold.

Consider a non-compact gravitational instanton (M, g) of any type (i)-(iv) above. If one tries to understand the L^2 cohomology of it, it is useful to consider a compactification X of M by collapsing the fibers F_r over all points of B_r as $r \to \infty$ [12] (remember that $\partial\overline{M}_r := \partial\overline{M} \times \{r\}$ is fibered over B_r with fibers F_r, $r \in \mathbb{R}^+$). In general X is not a manifold but a stratified space only with one singular stratum B_∞ and a principal stratum $M = X \setminus B_\infty$. However if F_r is a sphere, the resulting space X will be a connected, compact, oriented, smooth four-manifold without boundary. If (M, g) is an ALF gravitational instanton then $F_r \cong S^1$ therefore X is smooth and M is identified with the complement of the two dimensional submanifold $B_\infty \subset X$. The following theorem is a consequence of Corollary 1 in [12] combined with vanishing theorems of Dodziuk [6] and Yau [19] (also cf. Corollary 9 therein):

Theorem 2.1 (Hausel–Hunsicker–Mazzeo, 2004) *Let (M, g) be a gravitational instanton of ALF type. Then we have the following natural isomorphisms*

$$\overline{H}^k_{L^2}(M, g) \cong \begin{cases} H^k(X) & \text{if } k = 2, \\ 0 & \text{if } k \neq 2 \end{cases}$$

where the cohomology group on the right hand side is the ordinary singular cohomology. $\Diamond$

We see then that for an ALF gravitational instanton L^2 cohomology reduces to degree 2. Hence we focus our attention to this case and re-interpret it as follows (cf. [8]). Over an oriented Riemannian four-manifold if φ is an L^2 harmonic 2-form then so is its Hodge-dual $*\varphi$; consequently $F^\pm := \frac{\mathbf{i}}{2}(\varphi \pm *\varphi)$ are (anti-)self-dual imaginary L^2 harmonic 2-forms. Over a contractible ball $U \subset M$ the condition $\mathrm{d}F^\pm|_U = 0$ implies the existence of a local 1-form $A^\pm_U$ such that $F^\pm|_U = \mathrm{d}A^\pm_U$. Clearly, if $U \cap V \neq \emptyset$ for two balls U, V and $\mathrm{d}A^\pm_U$, $\mathrm{d}A^\pm_V$ are two representatives of $F^\pm$ on $U \cap V$ then $A^\pm_V = A^\pm_U + \mathrm{d}\chi^\pm_{UV}$ holds for an imaginary function $\chi^\pm_{UV}$. If $F^\pm$ represents a non-trivial element of $H^2(M, \mathbb{R})$ then we assume that $\left[\frac{1}{2\pi\mathbf{i}}F^\pm\right] \in H^2(M, \mathbb{Z})$ by linearity yielding $F^\pm$ always can be regarded as the curvature of an (anti)-self-dual U(1) connection $\nabla^\pm$ of finite action with curvature $F_{\nabla^\pm} = F^\pm$ on a complex line bundle $L^\pm$ over M.

Proceeding further $\nabla^\pm$ induces a connection $(\nabla^\pm)^{-1}$ with curvature $-F_{\nabla^\pm}$ on the dual bundle $(L^\pm)^{-1}$ providing a reducible SU(2) connection $\nabla^\pm \oplus (\nabla^\pm)^{-1}$ on the split bundle $E^\pm = L^\pm \oplus (L^\pm)^{-1}$ with curvature $F^\pm \oplus (-F^\pm)$. We fix the normalization constants such that

$$-\frac{1}{8\pi^2}\int\limits_M \mathrm{tr}(F_{\nabla^\pm} \oplus (-F_{\nabla^\pm}) \wedge F_{\nabla^\pm} \oplus (-F_{\nabla^\pm})) = -\frac{1}{4\pi^2}\int\limits_M \mathrm{tr}(F_{\nabla^\pm} \wedge F_{\nabla^\pm}) = \pm 1. \tag{2}$$

Summing up, an L^2 harmonic 2-form φ induces a reducible SU(2) (anti-)instanton $\nabla^\pm \oplus (\nabla^\pm)^{-1}$ on a split SU(2) bundle $E^\pm = L^\pm \oplus (L^\pm)^{-1}$ over (M, g) with second Chern number ± 1.

In case of an ALF gravitational instanton however, a more intrinsic relationship exists between L^2 cohomology and anti-self-duality which removes the disturbing self-dual-anti-self-dual ambiguity from this construction. To see this we will assume that there is an orthogonal complex structure J on (M, g) whose induced Kähler form, given by

$$\omega(X, Y) = g(JX, Y)$$

for two vector fields, satisfies $\omega = \mathrm{d}\beta$ for a 1-form β. Consider the infinite neck $N \subset M$ and pick up a local unitary orthonormal frame $\{X_1, X_2, X_3, X_4\}$ on $U \subset N$ with respect to J and g that is, $g(X_i, X_j) = \delta_{ij}$ and $X_2 = JX_1$ and $X_4 = JX_3$. Take a diffeomorphism $\phi : N \cong \partial\overline{M} \times \mathbb{R}^+$ and denote by r the coordinate on $\mathbb{R}^+$ as usual. Since (1) describes a metric of asymptotically vanishing curvature, then using the induced frame on $\phi(U) \subset \partial\overline{M} \times \mathbb{R}^+$ satisfying $\phi_* X_1 = \partial/\partial r$

the metric and the almost complex structure locally decay as

$$(\phi^{-1})^*g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + O(1/r^p), \quad (\phi^{-1})^*J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} + O(1/r^q) \qquad (3)$$

for some positive exponents p and q. The induced Kähler form satisfies $\omega_{ij} = J_i^k g_{kj}$ and by integrating (3) the corresponding β has *linear growth* that is,

$$|(\phi^{-1})^*\beta|_g \leq c_0 + c_1 r$$

for positive constants c_0 and c_1 as $r \to \infty$. Since the metric is hyper-Kähler we can apply a theorem of Hitchin (cf. Theorem 4 in [14]):

Theorem 2.2 (Hitchin, 2000) *Let (M, g) be a four dimensional complete hyper-Kähler manifold with induced orientation from the complex structures. Assume one of the Kähler forms satisfies $\omega = \mathrm{d}\beta$ where β has linear growth. Then any L^2 harmonic form is anti-self-dual.* $\diamond$

Consequently over an ALF gravitational instanton satisfying the above technical assumption φ can be identified with the curvature of an U(1) anti-instanton ∇ by writing $F_\nabla = \mathbf{i}\varphi$. We regard this as a reducible SU(2) anti-instanton $\nabla \oplus \nabla^{-1}$ on the split bundle $E = L \oplus L^{-1}$ over (M, g) with second Chern number -1.

To proceed further, let us consider the extendibility of various fields from M over X. The identification of L^2 harmonic 2-forms with Yang–Mills connections can be exploited at this point. First we endow X with an orientation induced by the orientation of M used in Hitchin's theorem. Next consider the metric. Take a smooth positive function $f : M \to \mathbb{R}^+$ of suitable asymptotics, that is $f(r) \sim O(r^{-2})$, then the rescaled metric $\tilde{g} := f^2 g$ extends over X as a symmetric tensor field. In other words, there is a symmetric tensor field $\tilde{g}$ on X which is conformally equivalent to g on $X \setminus B_\infty = M$ and degenerates along B_∞ providing an oriented, degenerate Riemannian manifold $(X, \tilde{g})$.

The Yang–Mills connection $\nabla \oplus \nabla^{-1}$ is also singular along $B_\infty \subset X$ which is a smooth, codimension 2 submanifold. Consider an open neighbourhood V_ε of B_∞ in X. We suppose V_ε is a disk bundle over B_∞ and locally we can introduce polar coordinates (ρ, θ) with $0 \leq \rho < \varepsilon$ and $0 \leq \theta < 2\pi$ on the disks over all points of B_∞. Let f_ε be a smooth positive function on X which vanishes on $X \setminus V_\varepsilon$ and is equal to 1 on $V_{\frac{\varepsilon}{2}}$. Finally choose a smooth Riemannian metric h on X and consider the regularization

$$\tilde{g}_\varepsilon := f_\varepsilon h + (1 - f_\varepsilon)\tilde{g}.$$

It is clear that $\tilde{g} = \tilde{g}_0$. The action is conformally invariant hence the original assumption $\|F_\nabla \oplus (-F_\nabla)\|_{L^2(M,g)} < \infty$ together with the completeness of g implies that $F_\nabla \in L^2(X, \tilde{g}_\varepsilon)$ hence the perturbed norm $\|F_\nabla \oplus (-F_\nabla)\|_{L^2(V_\varepsilon, \tilde{g}_\varepsilon)}$ is arbitrary small if ε tends to zero. Taking into account (2), which rules out fractional holonomy of the connection along the disks, and referring to a codimension 2 singularity removal theorem of Sibner and Sibner [18] and Råde [17] one can show that the SU(2) anti-instanton $\nabla \oplus \nabla^{-1}$ extends over B_∞ as a smooth SU(2) connection which is of course not anti-self-dual with respect to the regularized metric $\tilde{g}_\varepsilon$. Notice that this connection is independent of the metric $\tilde{g}_\varepsilon$. Then taking the limit $\varepsilon \to 0$ we recover a smooth reducible connection on a smooth SU(2) vector bundle E over X satisfying $c_2(E) = -1$ which is *formally* anti-self-dual over $(X, \tilde{g}_0)$.

Consider the space $\mathcal{B}_E(X) := \mathcal{A}_E(X)/\mathcal{G}_E(X)$ of gauge equivalence classes of smooth SU(2) connections on the fixed bundle E. Let $\mathcal{M}_E(X, \tilde{g}_\varepsilon) \subset \mathcal{B}_E(X)$ be the moduli of smooth SU(2) anti-instantons on E over $(X, \tilde{g}_\varepsilon)$. For a generic smooth $\tilde{g}_\varepsilon$ this is a smooth compact manifold except finitely many points. Then as $\varepsilon \to 0$ the moduli space $\mathcal{M}_E(X, \tilde{g}_0)$ intersects in $\mathcal{B}_E(X)$ the reducible connections constructed above. The singularity of $\tilde{g}_0$ somewhat decreases the overall regularity of the moduli space however this fact will not cause any inconvenience in our considerations ahead. Summing up again it is clear by now that the cohomology class $[\varphi] \in \overline{H}^2_{L^2}(M, g)$ is identified with a reducible point $[\nabla \oplus \nabla^{-1}] \in \mathcal{M}_E(X, \tilde{g}_0)$.

3 Topological classification

Now we are in a position to make two simple observations on the intersection form of X hence the topology of M. We assume that X is simply connected. Recall moreover that X has a fixed orientation.

Lemma 3.1 *We find $b^+(X) = 0$ that is, the intersection form $q_X : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$ of X is negative definite.*

Proof. It follows on the one hand that

$$\left[\frac{1}{2\pi\mathbf{i}} F_\nabla\right] = c_1(L) \in H^2(X, \mathbb{Z}).$$

On the other hand F_∇ is anti-self-dual hence harmonic yielding at the same time

$$\frac{1}{2\pi\mathbf{i}} F_\nabla \in \mathcal{H}^-(X, \tilde{g}_0)$$

where $\mathcal{H}^-(X, \tilde{g}_0)$ contains anti-self-dual harmonic 2-forms. However the decomposition

$$\mathcal{H}^2(X, \tilde{g}_\varepsilon) = \mathcal{H}^+(X, \tilde{g}_\varepsilon) \oplus \mathcal{H}^-(X, \tilde{g}_\varepsilon)$$

into self-dual and anti-self-dual harmonic 2-forms induces $H^2(X, \mathbb{R}) = H^+(X, \mathbb{R}) \oplus H^-(X, \mathbb{R})$ through the Hodge isomorphism for $\varepsilon > 0$ and $\dim H^\pm(X, \mathbb{R}) = b^\pm(X)$. Consequently we get a similar decomposition of $H^2(X, \mathbb{R})$ for $\varepsilon \to 0$.

It follows that the subspace $\mathcal{H}^-(X, \tilde{g}_0) \cong H^-(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$ depends on the metric $\tilde{g}_0$, has codimension $b^+(X)$ and contains as many as $b^2(X) = \dim H^2(X, \mathbb{R})$ linearly independent points of $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$ taking into account Theorem 2.1. Notice that $H^2(X, \mathbb{Z})$ is a lattice of rank $b^2(X)$ because X is simply connected. However $\mathcal{H}^-(X, \tilde{g}_0) \cap H^2(X, \mathbb{Z})$ cannot contain $b^2(X)$ linearly independent points if $b^+(X) > 0$. Hence $b^+(X) = 0$ must hold that is, q_X is negative definite. $\diamond$

Secondly we claim that

Lemma 3.2 *The intersection form q_X of X diagonalizes over the integers.*

Proof. Fix an SU(2) vector bundle E over X satisfying $c_2(E) = -1$. The gauge equivalence classes of reducible anti-self-dual connections on E are in one-to-one correspondence with the possible decompositions $E = L \oplus L^{-1}$ i.e., writing $c_1(L) = \alpha$, with the set of topological reductions of E over X:

$$\mathcal{R}_E(X) = \{(\alpha, -\alpha) \mid \alpha \in H^2(X, \mathbb{Z}),\ \alpha^2 = -1\}.$$

Consider the set $\{(\alpha_1, -\alpha_1), \dots, (\alpha_k, -\alpha_k)\}$ of all possible topological reductions of E. First of all, we can see that $\alpha_1, \dots, \alpha_k$ are linearly independent in $H^2(X, \mathbb{Z})$. Indeed, taking into account that the intersection form is negative definite, from the triangle inequality we get $-1 < q_X(\alpha_i, \alpha_j) < 1$ consequently α_i is perpendicular to α_j if $i \neq j$ hence linear independence follows.

Furthermore, notice that we have identified the reducible points with L^2 harmonic 2-forms therefore Theorem 2.1 shows $k = \dim H^2(X, \mathbb{Z})$ and since $q_X(\alpha_i, \alpha_j) = -\delta_{ij}$ in this basis q_X is diagonal. $\diamond$

This leads to our main theorem:

Theorem 3.1 *Let (M, g) be an ALF gravitational instanton satisfying $\omega = \mathrm{d}\beta$ for one of its Kähler form. Let X be the Hausel–Hunsicker–Mazzeo compactification of (M, g). Assume X is simply connected and is endowed with an orientation induced by any of the complex structures on (M, g). Let $k = \dim \overline{H}^2_{L^2}(M, g)$. Then there is a homeomorphism of oriented topological manifolds*

$$X \cong S^4$$

if $k = 0$ or

$$X \cong \underbrace{\overline{\mathbb{C}P}^2 \# \overline{\mathbb{C}P}^2 \# \dots \# \overline{\mathbb{C}P}^2}_{k}$$

if $k > 0$ where $\overline{\mathbb{C}P}^2$ is the complex projective space with reversed orientation (compared with the orientation induced by its standard complex structure).

Proof. X is simply connected and its intersection form is negative definite by the previous lemma consequently $q_X \cong 0$ if $k = 0$. Hence q_X is isomorphic to the intersection form of S^4. The theorem for $k = 0$ then follows from Freedman's classification of simply connected topological four-manifolds [10]. Similarly, $q_X \cong < -1 > \oplus \dots \oplus < -1 >$ (k times) if $k > 0$. Hence q_X is isomorphic to the intersection form of the connected sum of k copies of $\overline{\mathbb{C}P}^2$'s hence q_X is of odd type. But note that X has a smooth structure as well as the connected sum of k copies of $\overline{\mathbb{C}P}^2$'s. Applying again Freedman's result we obtain the theorem for the remaining cases. $\diamond$

Remark. The case $X \cong S^4$ is realized by the flat space $\mathbb{R}^3 \times S^1$ while the compactification of the multi-Taub–NUT space with as many as k NUTs provides an example for the remaining case.

The hyper-Kählerity of the metric is important in our construction. For example, in case of the Euclidean Schwarzschild manifold there are two independent L^2 harmonic 2-forms, one is self-dual while the other is anti-self-dual [8]. Anti-self-dualizing them one cancels out hence cannot be detected in the anti-instanton moduli space. Notice that this geometry is not hyper-Kähler and the intersection form of its compactification, which is $S^2 \times S^2$, is indefinite.

4 Concluding remarks

We conjecture that all ALF gravitational instantons satisfy the technical condition $\omega = \mathrm{d}\beta$ (in fact it is true for all known examples) however we cannot prove this in this moment.

Acknowledgement. The work was supported by OTKA grants No. T43242 and No. T046365.

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