Global solvability of the vacuum Einstein equation and the strong cosmic censorship in four dimensions

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Abstract

Let M be a connected, simply connected, oriented, closed, smooth four-manifold which is spin (or equivalently having even intersection form) and put $M^\times := M \setminus \{\text{point}\}$. In this paper we prove that if $X^\times$ is a smooth four-manifold homeomorphic but not necessarily diffeomorphic to $M^\times$ (more precisely, it carries a smooth structure *à la* Gompf) then $X^\times$ can be equipped with a complete Ricci-flat Riemannian metric. As a byproduct of the construction it follows that this metric is self-dual as well consequently $X^\times$ with this metric is in fact a hyper-Kähler manifold. In particular we find that the largest member of the Gompf–Taubes radial family of large exotic $\mathbb{R}^4$'s admits a complete Ricci-flat metric (and in fact it is a hyper-Kähler manifold).

These Riemannian solutions are then converted into Ricci-flat Lorentzian ones thereby exhibiting lot of new vacuum solutions which are not accessable by the initial vaule formulation. A natural physical interpretation of them in the context of the strong cosmic censorship conjecture and topology change is discussed.

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1 Introduction and summary

Solving the *vacuum Einstein equation* globally, or in other words: finding a (pseudo-)Riemannian *Ricci-flat* metric along a differentiable manifold i.e., a metric g which satisfies the second order non-linear partial differential equation

$$\mathrm{Ric}_g = 0$$

over a differentiable manifold M, is a century-old evergreen problem dwelling in the heart of modern differential geometry [3] and theoretical physics [54]. The problem of solvability naturally splits up into *local* and *global* solvability and also depends on the signature of the metric. Let us first consider the *Riemannian* case. Thanks to its non-linearity, solvability of the Ricci-flatness condition is already

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locally problematic; nevertheless exploiting its elliptic character various kinds of local existence results (e.g. [11, 22]) are known at least for the related but in some sense complementary equation $\mathrm{Ric}_g = \Lambda g$ (whose solutions are called *Einstein metrics*) with $\Lambda \neq 0$. As one expects, in these local existence problems the dimension of M plays no special role. However dimensionality issues characteristically enter the game when one considers global solvability. Finding global solutions in four dimensions i.e., when $\dim_{\mathbb{R}} M = 4$ is particularly important from a physical point of view and quite interestingly, from the mathematical viewpoint, precisely this is the dimension where global solvability is the most subtle. As it is well-known, if $\dim_{\mathbb{R}} M < 4$ the vacuum Einstein equation reduces to a full flatness condition on the metric hence it admits only a “few” global solutions; on the contrary, if $\dim_{\mathbb{R}} M > 4$ there are no (known) obstructions for global solvability hence apparently there are “too many” global solutions. A delicate balance is achieved if $\dim_{\mathbb{R}} M = 4$: for instance by a classical result [31, 52] we know that a Riemannian Einstein (hence in particular a Ricci-flat) metric on a compact M can exist only if its Euler characteristic $\chi(M)$ and signature $\sigma(M)$ obey the inequality $\chi(M) \geqq \frac{3}{2}|\sigma(M)|$. This implies for example that the connected sum of at least five copies of complex projective spaces cannot be Einstein. However even in four dimensions if M is non-compact there are no (known) obstruction against the solvability of the vacuum Einstein equation.

Restricting attention to the four dimensional case from now on, the main result of the paper—strongly motivated by [8] and considered as a substantially improved and technically revised and greatly simplified version of our earlier efforts [18, 19]—can be formulated in the *Riemannian* setting as

Theorem 1.1. *Let M be a connected, simply connected, oriented, closed (i.e., compact without boundary), smooth 4-manifold which is spin (or equivalently having even intersection form) and take the punctured space $M^\times := M \setminus \{\text{point}\}$. If $X^\times$ is a smooth 4-manifold homeomorphic but not necessarily diffeomorphic to $M^\times$ such that it carries a smooth structure à la Gompf then $X^\times$ can be equipped with a complete Ricci-flat Riemannian metric.*

As an extreme but important application of Theorem 1.1 we obtain

Corollary 1.1. *Let R^4 be the largest member of the Gomp–Taubes radial family of large exotic $\mathbb{R}^4$'s. Then R^4 carries a complete Ricci-flat Riemannian metric.*

The proof of Theorem 1.1 is based on a successive application of basic results by Gompf [25, 26, 27], Penrose [47], Taubes [50, 51] and Uhlenbeck [53] on exotic smooth structures, twistor theory, self-dual spaces and singularity removal in Yang–Mills fields, respectively. The idea in the spirit of twistor theory is to convert the real-analytic problem of solving $\mathrm{Ric}_g = 0$ on the real 4-space $M^\times$ into a complex-analytic problem on a complex 3-space Z associated to $M^\times$. This is in principle simple and works as follows. Take an arbitrary oriented and closed smooth 4-manifold M. In the first step, following Taubes, by connected summing sufficiently (but finitely) many complex projective spaces to M, we construct a space $\overline{X}_M \cong M\#\mathbb{C}P^2\#\ldots\#\mathbb{C}P^2$ which (with respect to its induced orientation) carries a self-dual metric $\overline{\gamma}$. Then, in the second step following Penrose, we observe that $\overline{Z}$, the *twistor space* of $(\overline{X}_M, \overline{\gamma})$, is a complex 3-manifold. Let $X_M \subset \overline{X}_M$ be the open space obtained by deleting carefully choosen closed subsets, homeomorphic to a projective line, from every $\mathbb{C}P^2$ factor of $\overline{X}_M$ and put $\gamma := \overline{\gamma}|_{X_M}$ and $Z := \overline{Z}|_{X_M}$. Making use of Z we can conformally rescale the incomplete self-dual space (X_M, γ) to a complete Ricci-flat one (X_M, g) if M is connected, simply connected and spin. In the third and last step, by the aid of Uhlenbeck's singularity removal theorem, we remove (or fill in) the extra $\mathbb{C}P^2 \setminus \mathbb{C}P^1 = \mathbb{R}^4$'s along X_M to obtain an open smooth space $X^\times$ which is homeomorphic to the punctured space $M^\times$ however is not necessarily diffeomorphic to it by results of Gompf. The result is a connected, simply connected, open, complete, Ricci-flat Riemannian spin 4-manifold $(X^\times, g)$.

By the conformal invariance of self-duality this technical condition in fact survives the whole procedure. Taking into account that a complete Ricci-flat and self-dual metric on a simply connected 4-manifold always induces a hyper–Kähler structure on it [3, Chapter 13], we can re-formulate the result of our construction as

Theorem 1.2. *The complete Ricci-flat metric of Theorem 1.1 on $X^\times$ with its fixed orientation is self-dual as well consequently $X^\times$ carries a hyper-Kähler structure, too.*

In this way we obtain

Corollary 1.2. *The space R^4 of Corollary 1.1 carries a hyper-Kähler structure.*

Next let consider the analogous problem in *Lorentzian* signature. Surely the most productive—and both mathematically and physically extraordinary important—presently known method to find global solutions of the Lorentzian vacuum Einstein equation is based on the *initial value formulation* [54, Chapter 10] which exploits the hyperbolic character of the Ricci-flatness condition (far from being complete, just for recent results cf. eg. [6, 9, 38, 42]) and the references therein). In this approach one starts with an appropriate initial value data set, subject to the (simpler) vacuum constraint equations, on a *three* dimensional manifold Σ and obtains solutions of the original vacuum Einstein equation on a *four* dimensional manifold M which is *always diffeomorphic* to the smooth product $\Sigma\times\mathbb{R}$ (with the unique smooth structures on the factors) [2, 8]. It is worth calling attention that even if the initial value formulation produces an abundance of solutions from the viewpoint of *global analysis* and *theoretical physics*, it is quite inproductive from the viewpoint of (low dimensional) *differential topology*. To illustrate this, suppose we want to find spaces (M,g) satisfying $\mathrm{Ric}_g=0$ over a connected and simply connected, open four-manifold M. If the initial value formulation is applied, and if in this case we impose a further condition that the corresponding Cauchy surface Σ be compact, then by the Poincaré–Hamilton–Perelman theorem Σ must be homeomorphic hence diffeomorphic to the three-sphere S^3 consequently M is uniquely fixed to be $S^3\times\mathbb{R}$ up to diffeomorphisms (but of course this unique M still can carry lot of non-isometric Ricci-flat metrics g).

However in sharp contrast to this differentio-topological rigidity of initial value formulation in the simply connected setting we obtain

Theorem 1.3. *Consider the space $X^\times$ as in Theorem 1.1 or equivalently, in Theorem 1.2. Then there exists a smooth Lorentzian metric g_L on $X^\times$ such that $(X^\times,g_L)$ is, a perhaps incomplete, Ricci-flat Lorentzian* 4*-manifold.*

To make a comparison, let us indicate the "size" of the set of non-isometric solutions to the Lorentzian vacuum Einstein equation provided by Theorem 1.3. By the fundamental classification result of Freedman [21], connected and simply connected, oriented, closed topological four-manifolds are topologically classified by their intersection form $Q_M: H_2(M;\mathbb{Z})\times H_2(M;\mathbb{Z})\to H_4(M;\mathbb{Z})\cong\mathbb{Z}$. By assumptions in our theorems here, M is spin and smooth hence Q_M must be even hence indefinite taking into account the other fundamental result in this field by Donaldson [14]. Therefore if $\sigma(M)$ denotes the signature and $b_2(M)$ the second Betti number of M then its intersection form looks like

$$Q_M=\frac{1}{8}\sigma(M)\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0\\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0\\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2\end{pmatrix}\oplus\frac{1}{2}\left(b_2(M)-\sigma(M)\right)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$$

hence the simplest examples for M are S^4, $S^2 \times S^2$, the $K3$ surfaces, etc. Consequently, unlike the initial value formulation in the simply connected case, the set of solutions provided by Theorem 1.3 already contains many topologically different underlying spaces. But even more, most of these compact M's themselves carry countable infinitely many different smooth structures, too. Finally, passing to the non-compact punctured spaces $M^\times$, the cardinality of the inequivalent smooth structures $X^\times$ underlying the Ricci-flat solutions in Theorem 1.1 already reaches that of the continuum in ZFC set theory by a theorem of Gompf [27] (recalled as Theorem 2.4 here). Therefore the set of non-isometric Ricci-flat spaces exhibited in Theorem 1.3 is huge indeed. These solutions are not accessible within the initial value formulation because they, compared to the time evolution of typical initial data sets, are "too long" in an appropriate sense (cf. [19, Section 5]). Informally speaking, the vacuum Einstein equation is more tractable in Riemannian signature because of the elliptic nature of the Ricci-flatness condition in contrast to its hyperbolic character in Lorentzian signature: meanwhile solutions in Riemannian signature are protected by elliptic regularity hence "extend well", the regularity profiles of Lorentzian initial data sets quickly get destroyed during their hyperbolic time evolution.

The paper is organized as follows. Section 2 contains the collection of the required background material with rapid discussions of these results from our viewpoint. Sections 3 and 4, respectively, contain the construction in the simpler non-exoting setting i.e., when $X^\times$ is not only homeomorphic but even diffeomorphic to $M^\times \subset M$ and then in the exotic setting with appropriate modifications. In Section 5 we prove Theorem 1.3 by simply recalling [19, Lemma 4.2]. Finally in Section 6 a physical interpretation of these Lorentzian Ricci-flat solutions is discussed. This interpretation places these solutions into the realm of the *strong cosmic censorship conjecture* and gravitational *topology change* processes.

2 Background material

Let us begin with recalling all the powerful results, techniques, tools to be used during the construction of Riemannian Ricci-flat metrics in this paper.

Construction of self-dual spaces. It is well-known that the Fubini–Study metric on the complex projective space $\mathbb{C}P^2$ with orientation inherited from its complex structure is self-dual (or half-conformally flat) i.e., the anti-self-dual part W^- of its Weyl tensor vanishes; consequently the oppositely oriented complex projective plane $(\mathbb{C}P^2)^{op}$ is anti-self-dual. A powerful generalization of this latter classical fact is Taubes' construction of an abundance of anti-self-dual 4-manifolds; firstly we exhibit his result but now in an orientation-reversed form:

Theorem 2.1. (Taubes [51, Theorem 1.1]) *Let M be a connected, compact, oriented smooth 4-manifold. Let $\mathbb{C}P^2$ denote the complex projective plane with its usual orientation and let # denote the operation of taking the connected sum of manifolds. Then there exists a natural number $k_M \geqq 0$ such that for all $k \geqq k_M$ the modified compact manifold*

$$M\#\underbrace{\mathbb{C}P^2\#\ldots\#\mathbb{C}P^2}_{k}$$

admits a self-dual Riemannian metric. $\diamondsuit$

Let us roughly summarize how Taubes' construction works ([51, Section 2]). Take an arbitrary connected, oriented, closed Riemannian 4-manifold (M,g) and consider the density of the anti-self-dual part of the Weyl curvature of g i.e., the pointwise norm $|W^-_g|_g$ along M. If it happens that somewhere around a point $p \in M$ this curvature density is large then take a $\mathbb{C}P^2$ with its usual Fubini–Study metric having zero anti-self-dual Weyl tensor and glue it to a ball $B^4_\varepsilon(p) \subset M$ of sufficiently small radius about the point. The result is a Riemannian metric on $M\#\mathbb{C}P^2$ having a bit smaller anti-self-dual Weyl tensor: this is because while W^-_g is unchanged on $M \setminus \overline{B}^4_\varepsilon(p)$ it is killed in the bulk of $B^4_\varepsilon(p)$ except possibly along an annulus where g and the Fubini–Study metric of $\mathbb{C}P^2$ have been glued together. Repeating this procedure, without doing connected summing on any previously added $\mathbb{C}P^2$ factor, probably very (but surely finitely) many times one comes up with a metric $\overline{\gamma}'$ on $\overline{X}_M := M\#\mathbb{C}P^2\#\dots\#\mathbb{C}P^2$ (regarding the specific notation cf. Sections 3 and 4 below) whose $W^-_{\overline{\gamma}'}$ is already arbitrarily small in e.g. the original L^2-norm. Then, by the aid of the implicit function theorem, one perturbs this metric with a small symmetric tensor field h on $\overline{X}_M$ into a new one $\overline{\gamma} := \overline{\gamma}' + h$ which is already self-dual i.e. having $W^-_{\overline{\gamma}} = 0$ along $\overline{X}_M$. For further rather technical details we refer to [51, Section 2].

Tools from twistor theory. Let us now recall Penrose' twistor method [47] to solve the Riemannian vacuum Einstein equation (for a very clear introduction cf. [3, Chapter 13], [32, 33]). Consider the bundle of unit-length anti-self-dual 2-forms $S(\wedge^-\overline{X}_M)$ over a compact oriented space $(\overline{X}_M, \overline{\gamma})$ which is self-dual with respect to its orientation. Since in 4 dimensions $\wedge^-\overline{X}_M$ is a rank 3 real vector bundle over $\overline{X}_M$, its unit-sphere bundle $S(\wedge^-\overline{X}_M)$ is the total space of a smooth S^2-fibration $\overline{p} : S(\wedge^-\overline{X}_M) \to \overline{X}_M$. The Levi–Civita connection of the metric $\overline{\gamma}$ on $\overline{X}_M$ can be used to furnish the real 6-manifold $S(\wedge^-\overline{X}_M)$ with a canonical almost complex structure; the fundamental observation of twistor theory is that this almost complex structure is integrable because $\overline{\gamma}$ is self-dual [3, Theorem 13.46]. The resulting complex 3-manifold $\overline{Z} \cong S(\wedge^-\overline{X}_M)$ is called the *twistor space* while the smooth fibration $\overline{p} : \overline{Z} \to \overline{X}_M$ the *twistor fibration* of $(\overline{X}_M, \overline{\gamma})$. The most important property of a twistor space of this kind is that its twistor fibers $\overline{p}^{-1}(x) \subset \overline{Z}$ for all $x \in \overline{X}_M$ fit into a locally complete complex 4-paremeter family $\overline{X}^{\mathbb{C}}_M$ of projective lines $Y \subset \overline{Z}$ each with normal bundle $H \oplus H$, with H being the dual of the tautological line bundle over $Y \cong \mathbb{C}P^1$. Moreover, there exists a real structure $\overline{\tau} : \overline{Z} \to \overline{Z}$ defined by taking the antipodal maps along the twistor fibers $Y_x := \overline{p}^{-1}(x) \subset \overline{Z}$ for all $x \in \overline{X}_M \subset \overline{X}^{\mathbb{C}}_M$ which are therefore called *real lines* among all the lines Y in $\overline{Z}$. In other words, $\overline{Z}$ is fibered exactly by the real lines Y_x for all $x \in \overline{X}_M$. Hence the real 4 dimensional self-dual geometry has been encoded into a 3 dimensional complex analytic structure in the sense that one can recover $(\overline{X}_M, \overline{\gamma})$ just from $\overline{Z}$ up to conformal equivalence.

One can go further and raise the question how to recover precisely $(\overline{X}_M, \overline{\gamma})$ itself from its conformal class, or more interestingly to us: how to get a Ricci-flat Riemannian 4-manifold (X_M, g) i.e., a solution of the (self-dual) Riemannian vacuum Einstein equation. Not surprisingly, to get the latter stronger structure, one has to specify further data on the twistor space. A fundamental result of twistor theory [47] is that a solution of the 4 dimensional (self-dual) Riemannian vacuum Einstein equation is equivalent to the following set of data (cf. [32, 33]):

* A complex 3-manifold Z, the total space of a holomorphic fibration $\pi : Z \to \mathbb{C}P^1$;
* A complex 4-paremeter family of holomorphically embedded complex projective lines $Y \subset Z$, each with normal bundle $NY \cong H \oplus H$ (here H is the dual of the tautological bundle i.e., the unique holomorphic line bundle on $Y \cong \mathbb{C}P^1$ with $\langle c_1(H), [Y]\rangle = 1$);
* A non-vanishing holomorphic section s of $K_Z \otimes \pi^* H^4$ (here K_Z is the canonical bundle of Z);
* A real structure $\tau : Z \to Z$ such that it coincides with the antipodal map $u \mapsto -\overline{u}^{-1}$ of $\mathbb{C}P^1$ upon restricting to the τ-invariant elements $Y \subset Z$ (called real lines) from the family; moreover these real lines are both sections of π and comprise a fibration of Z.

These data allow one to construct a Ricci-flat and self-dual (i.e., the Ricci and the anti-self-dual Weyl part of the curvature tensor vanishes) solution (X_M, g) of the *Riemannian* Einstein's vacuum equation with vanishing cosmological constant as follows. The holomorphic lines $Y \subset Z$ form a locally complete family and fit together into a complex 4-manifold $X_M^{\mathbb{C}}$. This space carries a natural complex conformal structure by declaring two nearby points $y_1, y_2 \in X_M^{\mathbb{C}}$ to be null-separated if the corresponding lines intersect i.e., $Y_1 \cap Y_2 \neq \emptyset$ in Z. Infinitesimally this intersection condition means that on every tangent space $T_y X_M^{\mathbb{C}} \cong \mathbb{C}^4$ a null cone is specified: using the identification $T_y X_M^{\mathbb{C}} \cong H^0(Y_y; \mathscr{O}(NY_y)) \cong H^0(\mathbb{C}P^1; \mathscr{O}(H \oplus H))$ given by $(a,b,c,d) \mapsto (au+b, cu+d)$, a tangent vector at y is null if and only if its corresponding holomorphic sections have a common zero i.e. $ad - bc = 0$ which is an equation of a cone. Restricting the complex conformal structure to the real lines singled out by τ and parameterized by an embedded real 4-manifold $X_M \subset X_M^{\mathbb{C}}$ we obtain the real conformal class $[g]$ of a Riemannian metric on X_M. The isomorphism $s : \pi^* H^{-4} \cong K_Z$ is essentially uniquely fixed by its compatibility with τ and gives rise to a volume form on X_M this way fixing the metric g in the conformal class. Given the conformal class, it is already meaningful to talk about the unit-sphere bundle of anti-self-dual 2-forms $S(\wedge^- X_M)$ over X_M with its induced orientation from the twistor space and Z can be identified with the total space of $S(\wedge^- X_M)$. This way we obtain a smooth twistor fibration $p : Z \to X_M$ whose fibers are $\mathbb{C}P^1$'s hence $\pi : Z \to \mathbb{C}P^1$ can be regarded as a parallel translation along this bundle over X_M with respect to a flat connection which is nothing but the induced connection of g on $\wedge^- X_M$, cf. [39]. Knowing the decomposition of the Riemannian curvature into irreducible components over an oriented Riemannian 4-manifold [49], this partial flatness of $S(\wedge^- X_M)$ implies that g is Ricci-flat and self-dual. Finally note that, compared with the bare twistor space $\overline{Z}$ of a self-dual manifold $(\overline{X}_M, \overline{\gamma})$ above, the essential new requirement for constructing a self-dual *Ricci-flat* space (X_M, g) is the existence of a holomorphic map π from the twistor space Z into $\mathbb{C}P^1$ which is compatible with the real structure in the above sense. We conclude our summary of the non-linear graviton construction by referring to [32, 33, 36, 39, 56] or [3, Chapter 13] for further details.

Removable singularities in Yang–Mills fields. Next let us refresh Uhlenbeck's by-now classical singularity removal theorem:

Theorem 2.2. (Uhlenbeck [53, Theorem 4.1] or [20, Appendix D])

$*$ *Local version: Let $\nabla^\times$ be a solution of the* $\mathrm{SU}(2)$ *Yang–Mills equations in the open punctured* 4-*ball $B^4 \setminus \{0\}$ with $\|F_{\nabla^\times}\|^2_{L^2(B^4)} = \int_{B^4} |F_{\nabla^\times}|^2 < +\infty$ i.e., having finite energy and $\nabla^\times = \mathrm{d} + A^\times$ such that $A^\times \in L^2_1(B^4 \setminus \{0\})$. Then $\nabla^\times$ is L^2_2 gauge equivalent to a connection ∇ which extends smoothly across the singularity to a smooth connection.*

$*$ *Global version: Let (M, g) be a connected, closed, oriented Riemannian* 4-*manifold and let $\nabla^\times$ be an* $\mathrm{SU}(2)$ *connection on a vector bundle $E^\times$ over $M^\times := M \setminus \{\mathrm{point}\}$ which is a solution of the* $\mathrm{SU}(2)$ *Yang–Mills equations and satisfies $\|F_{\nabla^\times}\|_{L^2(M)} < +\infty$ and there is an $L^2_{1,loc}$ gauge for $\nabla^\times$ around the puncturing of M. Then $\nabla^\times$ is $L^2_{2,loc}$ gauge equivalent to a connection ∇ on a vector bundle E over M i.e., to a connection which extends across the pointlike singularity of the original connection.* $\diamond$

Locally finite energy i.e., $F_{\nabla^\times} \in L^2_{loc}$ does not guarantee the continuity of the gauge transformation hence the topology of $E^\times$ can change i.e., $E^\times$ and E can be different; however if $F_{\nabla^\times} \in L^{2+\varepsilon}_{loc}$ holds then we can assume continuity. Nevertheless the isomorphism class of E is fully determined by the smooth connection ∇ via the numerical value of the integral $-\infty < \frac{1}{8\pi^2} \int_M \mathrm{tr}(F_\nabla \wedge F_\nabla) < +\infty$, the second Chern number of the bundle E.

Exotic stuff. Finally we evoke some results which provide us with a sort of summary of what is so special in four dimensions (i.e., absent in any other ones). First we recall a special class of large exotic (or fake) $\mathbb{R}^4$'s whose properties we will need here are summarized as follows:

Theorem 2.3. (Gompf–Taubes, cf. [28, Lemma 9.4.2, Addendum 9.4.4 and Theorem 9.4.10]) *There exists a pair (R^4, K) consisting of a differentiable 4-manifold R^4 homeomorphic but not diffeomorphic to the standard $\mathbb{R}^4$ and a compact oriented smooth 4-manifold $K \subset R^4$ such that*

* R^4 *cannot be smoothly embedded into the standard* $\mathbb{R}^4$ *i.e.,* $R^4 \not\subseteqq \mathbb{R}^4$ *but it can be smoothly embedded as a proper open subset into the complex projective plane i.e.,* $R^4 \subsetneqq \mathbb{C}P^2$;
* *Take a homeomorphism* $f : \mathbb{R}^4 \to R^4$, *let* $0 \in B^4_t \subset \mathbb{R}^4$ *be the standard open 4-ball of radius* $t \in \mathbb{R}_+$ *centered at the origin and put* $R^4_t := f(B^4_t)$ *and* $R^4_{+\infty} := R^4$. *Then*

$$\left\{ R^4_t \,\middle|\, r \leqq t \leqq +\infty \text{ such that } 0 < r < +\infty \text{ satisfies } K \subset R^4_r \right\}$$

is an uncountable family of nondiffeomorphic exotic $\mathbb{R}^4$*'s none of them admitting a smooth embedding into* $\mathbb{R}^4$ *i.e.,* $R^4_t \not\subseteqq \mathbb{R}^4$ *for all* $r \leqq t \leqq +\infty$. $\diamond$

The fact that any member R^4_t in this family is not diffeomorphic to $\mathbb{R}^4$ implies the counterintuitive phenomenon that $R^4_t \not\cong W \times \mathbb{R}$ i.e., R^4_t does not admit any *smooth* splitting into a 3-manifold W and $\mathbb{R}$ (with their unique smooth structures) in spite of the fact that such *continuous* splittings obviously exist. Indeed, from the contractibility of R^4_t we can see that W must be a contractible open 3-manifold (a so-called *Whitehead continuum* [57]) however, by an early result of McMillen [43] spaces of this kind always satisfy $W \times \mathbb{R} \cong \mathbb{R}^4$ i.e., their product with a line is always diffeomorphic to the standard $\mathbb{R}^4$. We will call this property of (any) exotic $\mathbb{R}^4$ occasionally below as "creased".

From Theorem 2.3 we deduce that for all $r < t < +\infty$ there is a sequence of smooth proper embeddings

$$R^4_r \subsetneqq R^4_t \subsetneqq R^4_{+\infty} = R^4 \subsetneqq \mathbb{C}P^2$$

which are very wild in the following sense. The complement $\mathbb{C}P^2 \setminus R^4$ of the largest member R^4 of this family is homeomorphic to S^2 regarded as an only "continuously embedded projective line" in $\mathbb{C}P^2$; therefore we shall denote this complement as $S^2 := \mathbb{C}P^2 \setminus R^4 \subset \mathbb{C}P^2$ in order to distinguish it from the ordinary projective lines $\mathbb{C}P^1 = \mathbb{C}P^2 \setminus \mathbb{R}^4 \subset \mathbb{C}P^2$. If $\mathbb{C}P^2 = \mathbb{R}^4 \cup \mathbb{C}P^1 = \mathbb{C}^2 \cup \mathbb{C}P^1$ is any holomorphic decomposition then $R^4 \cap \mathbb{C}P^1 \neq \emptyset$ (because otherwise $R^4 \subseteqq \mathbb{R}^4$ would hold, a contradiction) as well as $S^2 \cap \mathbb{C}P^1 \neq \emptyset$ (because otherwise $H_2(R^4;\mathbb{Z}) \cong \mathbb{Z}$ would hold since $\mathbb{C}P^1 \subset \mathbb{C}P^2$ represents a generator of $H_2(\mathbb{C}P^2;\mathbb{Z}) \cong \mathbb{Z}$, a contradiction again). Hence an ordinary projective line $\mathbb{C}P^1$ is always intersected by both R^4 and S^2 such that $S^2 \cap \mathbb{C}P^1$ in the worst situation is a Cantor set. These demonstrate that the members of the large radial family "live somewhere between" $\mathbb{R}^4$ and its complex projective closure $\mathbb{C}P^2$. However a more precise identification or location of them is a difficult task because these large exotic $\mathbb{R}^4$'s—although being honest differentiable 4-manifolds—are very transcendental objects, cf. [28, p. 366]: They require infinitely many 3-handles in any handle decomposition (like any other known large exotic $\mathbb{R}^4$) and there is presently[1] no clue as how one might draw explicit handle diagrams of them (even after removing their 3-handles).

We note that the structure of small exotic $\mathbb{R}^4$'s i.e., which admit smooth embeddings into $\mathbb{R}^4$, is better understood and is quite different, cf. [28, Chapter 9]. For instance, unlike the large case, in their corresponding radial family certain (but surely not more than countably many) members are diffeomorphic such that the non-diffeomorphic small exotic $\mathbb{R}^4$'s are parameterized not by an interval but a Cantor set only, cf. [28, Theorem 9.4.12 and its proof].

Our last ingredient is the following *ménagerie* result of Gompf.

[1] More precisely in the year 1999, cf. [28].

Theorem 2.4. (Gompf [27, Theorem 2.1]) *Let X be a connected (possibly non-compact, possibly with boundary) topological 4-manifold and let $X^\times := X \setminus \{\text{point}\}$ be the punctured manifold with a single point removed. Then the non-compact space $X^\times$ admits noncountably many (with the cardinality of the continuum in ZFC set theory) pairwise non-diffeomorphic smooth structures.* $\diamond$

If for instance M is a connected compact smooth 4-manifold then Gompf's construction simply goes as follows: Take R^4 from Theorem 2.3 and put

$$X^\times := M\#R^4$$

which is a smooth 4-manifold obviously homeomorphic to the punctured $M^\times$. More generally, the construction $X_t^\times := M^\times\#R_t^4$ produces uncountably many mutually non-diffeomorphic smooth structures on the unique topological 4-manifold underlying $X_t^\times$.

3 The construction

In this section, which serves as a warming-up for the next one, we construct solutions of the vacuum Einstein equation on punctured 4-manifolds carrying their standard smooth structure. We begin with an application of Theorem 2.1 as follows.

Lemma 3.1. *Out of any connected, closed (i.e., compact without boundary) oriented smooth 4-manifold M one can construct a connected, open (i.e., non-compact without boundary) oriented smooth Riemannian 4-manifold (X_M, γ) which is self-dual but incomplete in general.*

Proof. Pick any connected, oriented, closed, smooth 4-manifold M. Referring to Theorem 2.1 let $k := \max(1, k_M) \in \mathbb{N}$ be a positive integer, put

$$\overline{X}_M := M\#\underbrace{\mathbb{C}P^2\#\dots\#\mathbb{C}P^2}_{k}$$

and let $\overline{\gamma}$ be a self-dual metric on it. Then $(\overline{X}_M, \overline{\gamma})$ is a compact self-dual manifold. Pick a $\mathbb{C}P^2$ factor within $\overline{X}_M$ and any (holomorphically embedded) projective line $\mathbb{C}P^1 \subset \mathbb{C}P^2$ in that factor (avoiding its attaching point to M); then $\mathbb{C}P^1 = \mathbb{C}P^2 \setminus \mathbb{C}^2 \cong \mathbb{C}P^2 \setminus \mathbb{R}^4$ i.e., the line arises as the complement of an $\mathbb{R}^4$ in $\mathbb{C}P^2$. Let $K \subset \mathbb{R}^4$ be any connected compact subset and put

$$X_M := M\#\underbrace{(\mathbb{C}P^2 \setminus \mathbb{C}P^1)\#\dots\#(\mathbb{C}P^2 \setminus \mathbb{C}P^1)}_{k-1}\#_K(\mathbb{C}P^2 \setminus \mathbb{C}P^1) \cong M\#\underbrace{\mathbb{R}^4\#\dots\#\mathbb{R}^4}_{k-1}\#_K\mathbb{R}^4 \cong M^\times\#\underbrace{\mathbb{R}^4\#\dots\#\mathbb{R}^4}_{k-1} \tag{1}$$

where the operation $\#_K$ means that the attaching point $y_0 \in \mathbb{R}^4$ taken to glue a distinguished $\mathbb{R}^4$ with the rest $M\#\underbrace{\mathbb{R}^4\#\dots\#\mathbb{R}^4}_{k-1}$ satisfies $y_0 \in K \subset \mathbb{R}^4$ and $M^\times := M\#_K\mathbb{R}^4 \cong M \setminus \{\text{point}\}$ is the punctured space with its inherited smooth structure from the smooth embedding $M^\times \subset M$. The result is a connected, open 4-manifold X_M (see Figure 1).

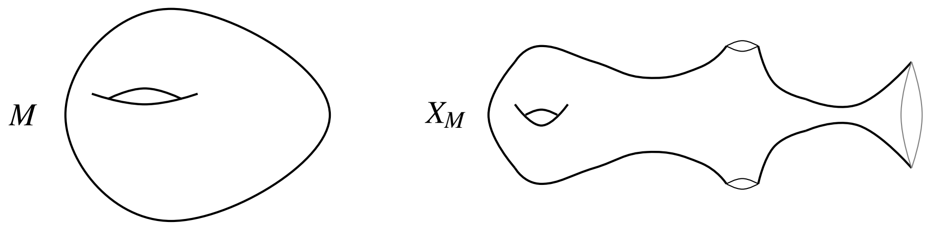

Figure 1. Construction of X_M out of M. The gray ellipse represents a distinguished end diffeomorphic to the complement of a connected compact subset K in $\mathbb{R}^4$.

From the proper smooth embedding $X_M \subsetneqq \overline{X}_M$ there exists a restricted self-dual Riemannian metric $\gamma := \overline{\gamma}|_{X_M}$ on X_M which is however in general non-complete. $\diamond$

Next we improve the incomplete self-dual space (X_M, γ) of Lemma 3.1 to a complete Ricci-flat space (X_M, g) by conformally rescaling γ with a suitable positive smooth function $\varphi : X_M \to \mathbb{R}_+$ which is a "multi-task" function in the sense that it kills both the scalar curvature and the traceless Ricci tensor of γ moreover blows up sufficiently fast along the $\mathbb{R}^4$ ends of X_M to render the rescaled metric g complete. Two classical examples serve as a motivation.

First example. First, let $S^4 \subset \mathbb{R}^5$ be the standard 4-sphere equipped with the standard orientation and round metric inherited from the embedding. Put $\overline{X}_M := S^4$ and $\overline{\gamma}$:=the standard round metric. It is well-known that $(\overline{X}_M, \overline{\gamma}) = (S^4, \overline{\gamma})$ is self-dual and Einstein with non-zero cosmological constant i.e., not Ricci-flat. Put $X_M := S^4 \setminus \{\infty\} = \mathbb{R}^4$; then $\gamma = \overline{\gamma}|_{\mathbb{R}^4}$ thus $(X_M, \gamma) = (\mathbb{R}^4, \gamma)$ is an incomplete self-dual space. But setting $\varphi : \mathbb{R}^4 \to \mathbb{R}_+$ to be $\varphi(x) := (1+|x|^2)^{-1}$, then $g := \varphi^{-2}\gamma$ is nothing but the standard flat metric η on $\mathbb{R}^4$ which is of course complete and Ricci-flat. Hence $(X_M, g) = (\mathbb{R}^4, \eta)$, the conformal rescaling of $(X_M, \gamma) = (\mathbb{R}^4, \gamma)$, is the desired complete Ricci-flat space in this simple case. Note that $(\mathbb{R}^4, \eta)$ is a trivial hyper-Kähler space, too.

It is worth working out here how the corresponding holomorphic map $\pi : Z \to \mathbb{C}P^1$ over the corresponding twistor space arises in this situation (see the summary of twistor theory in Section 2). Consider the smooth twistor fibration $\overline{p} : \overline{Z} \to S^4$. Since $\mathbb{R}^4 \subset S^4$ writing $Z := \overline{Z}|_{\mathbb{R}^4}$ and $p := \overline{p}|_{\mathbb{R}^4}$ we obtain a restricted fibration $p : Z \to \mathbb{R}^4$. Unlike the full twistor fibration over S^4, the restricted one is topologically trivial i.e. Z is homeomorphic to $\mathbb{R}^4 \times S^2$ since $\mathbb{R}^4$ is contractible; consequently Z admits a continuous trivialization over $\mathbb{R}^4$. This is a necessary topological condition for the existence of the map π. Since $\mathbb{R}^4$ with its flat metric is conformally equivalent to $S^4 \setminus \{\infty\}$ with its round metric, Z arises by deleting the twistor line over $\infty \in S^4$ from $\overline{Z}$. However it is well-known that the twistor space $\overline{Z}$ of the round S^4 is $\mathbb{C}P^3$ consequently the twistor space Z of the flat $\mathbb{R}^4$ is simply $\mathbb{C}P^3 \setminus \mathbb{C}P^1$. More explicitly, take a homogeneous coordinate system $[z_0 : z_1 : z_2 : z_3]$ on $\mathbb{C}P^3$ and remove the line $z_0 = z_1 = 0$ from $\mathbb{C}P^3$ to get Z. We wish to define a map $\pi : Z \to \mathbb{C}P^1$ such that its target space is a twistor i.e. a real line in Z. Any line in $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ can be written as $[z_0 : z_1 : az_1 + bz_0 : cz_1 + dz_0]$ with $[z_0 : z_1] \in \mathbb{C}P^1$ and $a, b, c, d \in \mathbb{C}$ being some parameters. Note that the case of $a = b = c = d = 0$ is meaningful and $[z_0 : z_1 : 0 : 0]$ is simply the distinguished line $[z_0 : z_1]$ in $\mathbb{C}P^3 \setminus \mathbb{C}P^1$. Thus

$$\text{The lines in } Z = \{[z_0 : z_1 : az_1 + bz_0 : cz_1 + dz_0] \,|\, [z_0 : z_1] \in \mathbb{C}P^1 \text{ and } a, b, c, d \in \mathbb{C}\}\ . \tag{2}$$

The real structure on $\overline{Z}$ is defined by demanding the fibers of $\overline{p} : \overline{Z} \to S^4$ to be invariant. Under $\overline{Z} \cong \mathbb{C}P^3$ it comes from the identification $\mathbb{C}^4 \cong \mathbb{H}^2$ and has the form $[z_0 : z_1 : z_2 : z_3] \mapsto [\overline{z}_1 : -\overline{z}_0 : \overline{z}_3 : -\overline{z}_2]$. It is compatible with the antipodal map $[z_0 : z_1] \mapsto [\overline{z}_1 : -\overline{z}_0]$ and restricts to a real structure $\tau : Z \to Z$. It then follows that the corresponding real lines have the shape $[z_0 : z_1 : az_1 + \overline{c}z_0 : cz_1 - \overline{a}z_0]$. Consequently the twistor fibration $p : Z \to \mathbb{R}^4$ looks like $[z_0 : z_1 : az_1 + \overline{c}z_0 : cz_1 - \overline{a}z_0] \mapsto (a, c) \in \mathbb{C}^2 \cong \mathbb{R}^4$ and in particular the distinguished line is real and can be identified with the twistor line $p^{-1}(0)$ over the origin. Since every point $z \in Z$ contained in exactly one real line let us define $\pi : Z \to p^{-1}(0)$ by the canonical projection $\pi([z_0 : z_1 : az_1 + \overline{c}z_0 : cz_1 - \overline{a}z_0]) := [z_0 : z_1]$. Upon introducing the projective coordinate $u := \frac{cz_1 - \overline{a}z_0}{az_1 + \overline{c}z_0}$ if $(a,c) \neq (0,0)$ or $u := \frac{z_1}{z_0}$ if $(a,c) = (0,0)$ along the twistor lines in the domain of π the map looks like

$$\pi(u) = \begin{cases} \frac{\overline{c}u + \overline{a}}{-au + c} & \text{if } (a,c) \neq (0,0) \\ u & \text{if } (a,c) = (0,0) \end{cases} \tag{3}$$

which is an obviously holomorphic map since it arises by holomorphic deformations of $p^{-1}(0)$ within Z moreover it is the identity on $p^{-1}(0)$. What we only have to check is that π is compatible with the real structure. This means that we have to demonstrate that all real lines $p^{-1}(x) \subset Z$ are sections of $\pi : Z \to p^{-1}(0)$ or in other words that $\pi|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(0)$ is a holomorphic bijection of $\mathbb{C}P^1$ for every $x \in \mathbb{R}^4$. Assume that this is not true. Since $\pi|_{p^{-1}(x)}$ has the form (3) we can normalize its coefficients such that $|c|^2 + |a|^2 = 1$. However the assumption implies that this rational function is constant in u yielding $|c|^2 + |a|^2 = 0$, a contradiction.

Second example. This time put $\overline{X}_M := \mathbb{C}P^2$ and $\overline{\gamma}$:=Fubini–Study metric. It is well-known that $(\overline{X}_M, \overline{\gamma}) = (\mathbb{C}P^2, \overline{\gamma})$ is self-dual and Einstein with non-zero cosmological constant i.e., not Ricci-flat. Now let $X_M := \mathbb{C}P^2 \setminus \mathbb{C}P^1 = \mathbb{R}^4$; then $\gamma = \overline{\gamma}|_{\mathbb{R}^4}$ and $(X_M, \gamma) = (\mathbb{R}^4, \gamma)$ is an incomplete self-dual space. If $0 \neq (z_0, z_1, z_2) \in \mathbb{C}^3$ and $[z_0 : z_1 : z_2] \in \mathbb{C}P^2$ then take the projective line $\mathbb{C}P^1 \subset \mathbb{C}P^2$ defined by $z_0 = 0$. Introducing $w_i = \frac{z_i}{z_0}$ $(i = 1,2)$ and $w = (w_1, w_2) \in \mathbb{C}^2 = \mathbb{R}^4$, on the complementum $\mathbb{C}P^2 \setminus \mathbb{C}P^1$ the restricted Fubini–Study metric γ looks like

$$\gamma_{ij}(w) = (1 + |w|^2)^{-1}\delta_{ij} - (1 + |w|^2)^{-2}\overline{w}_i w_j$$

and along this local part it already possesses a Kähler potential $K(w) = \log(1 + |w|^2)$. This time define $\varphi : \mathbb{R}^4 \to \mathbb{R}_+$ as $\varphi(w) := \mathrm{e}^{-\frac{3}{4}K(w)} = (1 + |w|^2)^{-\frac{3}{4}}$ which is a non-holomorphic function and consider the conformally rescaled (real) metric $g := \varphi^{-2}\gamma$. One can check that this is a complete Ricci-flat metric on $\mathbb{R}^4$. Hence $(X_M, g) = (\mathbb{R}^4, g)$, the conformal rescaling of $(X_M, \gamma) = (\mathbb{R}^4, \gamma)$ is a complete Ricci-flat space. It is already not flat but note again that nevertheless g indudes a (not asymptotically flat in any sense) hyper-Kähler structure on $\mathbb{R}^4$ because g is a complete, self-dual, Ricci-flat metric on the simply connected space $\mathbb{R}^4$.

Again, the corresponding twistor-theoretic map π arises as follows. Consider the smooth twistor fibration $\overline{p} : \overline{Z} \to \mathbb{C}P^2$. Since $\mathbb{R}^4 \subset \mathbb{C}P^2$, writing $Z := \overline{Z}|_{\mathbb{R}^4}$ and $p := \overline{p}|_{\mathbb{R}^4}$ we obtain a restricted fibration $p : Z \to \mathbb{R}^4$. Unlike the full twistor fibration over $\mathbb{C}P^2$, this restricted one is topologically trivial i.e., Z is homeomorphic to $\mathbb{R}^4 \times S^2$ since $\mathbb{R}^4$ is contractible; consequently Z admits a continuous trivialization over $\mathbb{R}^4$. This is a necessary topological condition for the existence of the map π. It is known that $\overline{Z} \cong P(T\mathbb{C}P^2)$ i.e., the twistor space of the complex projective space can be identified with its projective holomorphic tangent bundle. Consequently $\overline{Z}$ admits a very classical description namely can be identified with the flag manifold $F_{12}(\mathbb{C}^3)$ consisting of pairs (L, P) where $0 \in L \subset \mathbb{C}^3$ is a line (i.e., a point $p \in \mathbb{C}P^2$) and $0 \in L \subset P \subset \mathbb{C}^3$ is a plane containing the line (i.e., a line $p \in \ell \subset \mathbb{C}P^2$ containing the point). Then in the twistor fibration $\overline{p} : \overline{Z} \to \mathbb{C}P^2$ of the complex projective space $(L, P) \in F_{12}(\mathbb{C}^3)$ is sent into the point $x \in \mathbb{C}P^2$ provided by the line $X := L^\perp \cap P \subset \mathbb{C}^3$ where $L^\perp$ is the plane perpendicular to the line L in $\mathbb{C}^3$ with respect to the standard Hermitian scalar product. This is a smooth but not holomorphic fibration over $\mathbb{C}P^2$ with $\mathbb{C}P^1$'s as fibers since $\overline{p}^{-1}(x) = \{(L, P) \,|\, L \subset X^\perp, X \subset P\} = \{(p, \ell) \,|\, x^\perp \cap \ell, x \in \ell\}$ i.e., it consists of all lines $\ell \subset \mathbb{C}P^2$ through $x \in \mathbb{C}P^2$ (a copy of $\mathbb{C}P^1$) and a distinguished point p on each given by its intersection with the line $x^\perp \subset \mathbb{C}P^2$ given by $X^\perp \subset \mathbb{C}^3$. Consider now the restricted twistor fibration $p: Z \to \mathbb{R}^4$. Fix a point $x_0 \in \mathbb{C}P^2 \setminus x_0^\perp = \mathbb{R}^4$ with target space $p^{-1}(x_0) \cong \mathbb{C}P^1$ consisting of terminating pairs $(p_0, \ell_0) \in p^{-1}(x_0) \subset Z$. Take a starting pair $(p, \ell) \in Z$ over a running point $x \in \mathbb{C}P^2 \setminus x_0^\perp$. Our aim is to construct a holomorphic map which associates to (p, ℓ) another pair (p_0, ℓ_0). We construct this $\pi : Z \to p^{-1}(x_0)$ very simply as follows. Consider a starting pair (p, ℓ) and take its line component $\ell \subset \mathbb{C}P^2$. This line has a unique intersecion $p_0 := x_0^\perp \cap \ell$ with the infinitely distant projective line. Then, given the target space $p^{-1}(x_0)$, define the projective line component $\ell_0 \subset \mathbb{C}P^2$ in the terminating pair $(p_0, \ell_0) \in p^{-1}(x_0)$ by taking the unique projective line ℓ_0 connecting p_0 with x_0. In short,

$$\pi((p,\ell)) := (p_0, \ell_0) \text{ where } \quad \begin{array}{l} p_0 \in \mathbb{C}P^2 \text{ satisfies } p_0 := x_0^\perp \cap \ell \text{ and} \\ \ell_0 \subset \mathbb{C}P^2 \text{ satisfies that } \ell_0 \text{ connects } p_0 \text{ with } x_0 \text{ in } \mathbb{C}P^2 \end{array} \tag{4}$$

(see Figure 2 for a construction of this map in projective geometry).

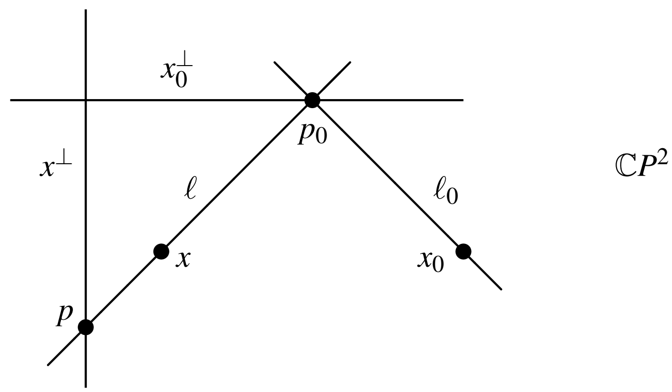

Figure 2. Construction of the map $\pi : Z \to \mathbb{C}P^1$ satisfying $\pi((p,\ell)) = (p_0, \ell_0)$.

It is a classical observation that this map is well-defined on Z and holomorphic; in particular it is the identity on the target space $p^{-1}(x_0)$ i.e., $\pi((p_0,\ell_0)) = (p_0,\ell_0)$.

This globally defined map admits a local description which looks very similar to the *First example*. Given the target point $x_0 \in \mathbb{C}P^2 \setminus x_0^\perp \cong \mathbb{R}^4$, its twistor line can be identified with the infinitely distant line $x_0^\perp \subset \mathbb{C}P^2$ or equivalently, $x_0^\perp \subset Z$. Likewise, if $x \in \mathbb{C}P^2 \setminus x_0^\perp \cong \mathbb{R}^4$ is a nearby point then its twistor line is $x^\perp \subset \mathbb{C}P^2$ or equivalently, $x^\perp \subset Z$. In this picture the map (4) can be described simply as follows: if $x \in \ell \subset \mathbb{C}P^2$ is a line then $\pi(\ell \cap x^\perp) = \ell \cap x_0^\perp$ as ℓ runs over all possibilities. Pick homogeneous coordinates $[z_0 : z_1 : z_2]$ on $\mathbb{C}P^2$ such that $x_0 := [1:0:0]$ hence $x_0^\perp = \{[0 : v_1 : v_2] | [v_1 : v_2] \in \mathbb{C}P^1\}$. Likewise, if $x = [1 : z_1 : z_2]$ is the nearby point then $x^\perp = \{[-\bar{z}_1 w_1 - \bar{z}_2 w_2 : w_1 : w_2] | [w_1 : w_2] \in \mathbb{C}P^1\}$. The affine part of the line ℓ connecting $[1 : z_1 : z_2]$ and $[0 : v_1 : v_2]$ is $\{[t : v_1 + (z_1 - v_1)t : v_2 + (z_2 - v_2)t] \,|\, t \in \mathbb{C}\}$ hence by solving the equation $[t : v_1 + (z_1 - v_1)t : v_2 + (z_2 - v_2)t] = [-\bar{z}_1 w_1 - \bar{z}_2 w_2 : w_1 : w_2]$ for $[w_1 : w_2]$ and upon introducing the projective coordinate $u := \frac{w^1}{w^2}$ along $x^\perp \cong \mathbb{C}P^1$ the map (4) takes the shape

$$\pi(u) = \frac{(1 + |z_1|^2)u - z_1 \bar{z}_2}{-\bar{z}_1 z_2 u + (1 + |z_2|^2)}$$

hence looks like (3) indeed.

*Remark***.** It follows from the description (2) of its holomorphic lines that the twistor space Z of the flat $\mathbb{R}^4$ can be globally holomorphically identified with the total space of the bundle $H \oplus H$ over the distinguished projective line $\mathbb{C}P^1$ parameterized with $[z_0 : z_1]$ in (2) and the map (3) is nothing but the projection $\pi : H \oplus H \to \mathbb{C}P^1$. The point is that this picture on the twistor space continues to hold true in the generic case at least locally. Consider a general twistor space Z with its twistor fibration $p: Z \to X_M$. Take $x \in X_M$ and let $Np^{-1}(x)$ be the normal bundle of the twistor line $p^{-1}(x) \subset Z$. We know (see the summary of twistor theory in Section 2) that the holomorphy type of the normal bundle is fixed in advance and is a very special bundle: it is positive hence admits holomorphic sections such that they parameterize a locally complete family of projective lines $Y \subset Z$ which are small holomorphic deformations of $Y_x = p^{-1}(x)$ inside Z (cf. e.g. [30, Sections III.1 and III.2]) including therefore all nearby real lines as well. Thus there exist small open neighbourhoods $U_x \subset X_M$ of x and $V_x \subset Np^{-1}(x)$ of the zero section with an injection $\Psi_x : p^{-1}(U_x) \to V_x$ that is, Ψ_x maps injectively the twistor fibers over U_x into the space of holomorphic sections of V_x such that this map is onto an appropriately defined subspace of real sections. More explicitly, we know that the normal bundle is always isomorphic to $H \oplus H$ consequently all small holomorphic deformations of a given twistor line within Z can be parameterized by $(a,b,c,d) \in \mathbb{C}^4 \cong H^0(\mathbb{C}P^1; \mathscr{O}(H \oplus H))$ such that the twistor lines satisfy a reality condition implying $ad - bc \neq 0$ (because the real lines never intersect), exactly like in the *First example*.

We can return now to the much more general situation set up in Lemma 3.1; motiveted by the examples, instead of finding conformal rescalings $\varphi : X_M \to \mathbb{R}_+$ directly, we are going to use Penrose' non-linear graviton construction (i.e., twistor theory [47]) to find their holomorphic counterparts $\pi : Z \to \mathbb{C}P^1$. Consider the compact self-dual space $(\overline{X}_M, \overline{\gamma})$ from Lemma 3.1, take its twistor fibration $\overline{p} : \overline{Z} \to \overline{X}_M$ and let $p : Z \to X_M$ be its restriction induced by the smooth embedding $X_M \subsetneqq \overline{X}_M$ i.e., $Z := \overline{Z}|_{X_M}$ and $p := \overline{p}|_{X_M}$. Then Z is a non-compact complex 3-manifold already obviously possessing all the required twistor data except the existence of a holomorphic mapping $\pi : Z \to \mathbb{C}P^1$.

Lemma 3.2. *Consider the connected, open, oriented, incomplete, self-dual space (X_M, γ) as in Lemma 3.1 with its twistor fibration $p: Z \to X_M$ constructed above. If $\pi_1(M) = 1$ and M is spin (or equivalently, having even intersection form) then there exists a holomorphic mapping $\pi : Z \to \mathbb{C}P^1$.*

Proof. Let $x_0 \in X_M$ be a fixed point. Our aim is to construct a holomorphic map

$$\pi : \ Z \longrightarrow p^{-1}(x_0) \cong \mathbb{C}P^1 \tag{5}$$

that we carry out by analytic continuation.

First, put $\pi|_{p^{-1}(x_0)} := \mathrm{Id}_{p^{-1}(x_0)}$. Secondly, suppose that in $x \in X_M$ the map is already defined i.e. there exists $\pi|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(x_0)$ which is compatible with the real structure on Z hence is a holomorphic bijection between the twistor fibers in question or in other words is a holomorphic bijection of $\mathbb{C}P^1$. Consider a sufficiently small open neighbourhood $U_x \subset X_M$ of x such that $p^{-1}(U_x) \subset Z$ can be holomorphically modeled within the neighbourhood V_x of the zero section of $Np^{-1}(x)$, the normal bundle of the twistor line $p^{-1}(x)$. Define $\rho_x : p^{-1}(U_x) \to p^{-1}(x)$ to be the restriction of the projection $\pi : Np^{-1}(x) \to p^{-1}(x)$ onto the image of the twistor lines of $p^{-1}(U_x)$ within $Np^{-1}(x)$. That is, given a point $z \in p^{-1}(U_x)$ there exists a unique real line passing through it and $\rho_x(z) \in p^{-1}(x)$ simply arises by the projection of this line onto the central twistor line $p^{-1}(x)$. This local map is clearly holomorphic because it stems from holomorphic deformations of $p^{-1}(x)$ inside Z provided by its locally complete family of lines.[2] Moreover ρ_x is the identity on $p^{-1}(x)$. What we have to still check that it is compatible with the real structure on Z i.e. for every $y \in U_x$ the map ρ_x is a holomorphic

[2] For a comparison with the general theory [30, Proposition 1.3] we remark here that although the Griffiths obstruction groups $H^1(p^{-1}(x); \mathscr{O}((\pi|_{p^{-1}(x)})^* Tp^{-1}(x_0) \otimes S^k N^* p^{-1}(x)))$ against the extendibility of $\pi|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(x_0)$ to the k^{th} formal neighbourhoud of $p^{-1}(x) \subset Z$ are non-trivial for $k \geqq 4$, the above construction (or the explicit *Second example*) shows that the corresponding obstruction classes $\omega(\pi_{k-1})$ themselves are nevertheless trivial. This essentially follows from Kodaira's integrability condition $H^1(p^{-1}(x); \mathscr{O}(Np^{-1}(x))) = \{0\}$, cf. [30, Theorem 3.1].

bijection between $p^{-1}(y)$ and $p^{-1}(x)$. Exploiting the isomorphism $Np^{-1}(x) \cong H \oplus H$ (see the summary of twistor theory in Section 2) the map $\rho_x : p^{-1}(U_x) \to p^{-1}(x)$ can be described by the projection $\pi : H \oplus H \to \mathbb{C}P^1$ therefore, upon introducing the projective coordinate u along $p^{-1}(y) \cong \mathbb{C}P^1$

$$\rho_x(u) = \begin{cases} \frac{au+b}{cu+d} & \text{if } (a,b,c,d) \neq (0,0,0,0) \\ u & \text{if } (a,b,c,d) = (0,0,0,0) \end{cases}$$

where $(a,b,c,d) \in H^0(\mathbb{C}P^1; \mathscr{O}(H \oplus H)) \cong \mathbb{C}^4$ are the coefficients of a real line hence satisfy an appropriate reality condition. Fortunately whatever this reality condition is, we surely know that $ad - bc \neq 0$ because real lines never intersect. However this implies that the map $u \mapsto \frac{au+b}{cu+d}$ is not constant in u that is, ρ_x is indeed a holomorphic bijection between $p^{-1}(y)$ and $p^{-1}(x)$ for all $y \in U_x$ (such that it is the identity on $p^{-1}(x)$) hence ρ_x is compatible with the real structure on $p^{-1}(U_x)$ as desired.

Therefore let us define the local extension $\pi|_{p^{-1}(U_x)} : p^{-1}(U_x) \to p^{-1}(x_0)$ by the composition $\pi|_{p^{-1}(U_x)} := \pi|_{p^{-1}(x)} \circ \rho_x$. By assumption $\pi|_{p^{-1}(x)}$ already possesses all the required properties hence is compatible with the real structure therefore it is a holomorphic bijection between the twistor lines $p^{-1}(x)$ and $p^{-1}(x_0)$; consequently, taking a projective coordinate v along $p^{-1}(x) \cong \mathbb{C}P^1$, we know that $\pi|_{p^{-1}(x)}$ also has the form $v \mapsto \frac{a_0v+b_0}{c_0v+d_0}$ with some $a_0, b_0, c_0, d_0 \in \mathbb{C}$ satisfying $a_0d_0 - b_0c_0 \neq 0$. Composing the maps above means that we insert $v = \frac{au+b}{cu+d}$ where u is the projective coordinate along $p^{-1}(y) \cong \mathbb{C}P^1$ as before; thus the local extension looks like

$$\pi|_{p^{-1}(U_x)}(u) = \frac{a_0\frac{au+b}{cu+d} + b_0}{c_0\frac{au+b}{cu+d} + d_0} = \frac{(a_0a + b_0c)u + (a_0b + b_0d)}{(c_0a + d_0c)u + (c_0b + d_0d)} .$$

Since $(a_0a+b_0c)(c_0b+d_0d) - (a_0b+b_0d)(c_0a+d_0c) = (a_0d_0 - b_0c_0)(ad - bc) \neq 0$ it readily follows that it continues to be compatible with the real structure.

Thirdly, since M is connected, simply connected and spin, Z is connected, simply connected and $p : Z \to X_M$ is trivial. These make sure that π extends over Z in a consistent way. $\diamond$

Remark. Note that the reasons for both the local map ρ_x and the non-local one $\pi|_{p^{-1}(x)}$ having the same shape (namely both are fractional linear transformations of $\mathbb{C}P^1$) are quite different. Nevertheless it makes possible to regard $\pi : Z \to \mathbb{C}P^1$ as an action of $\mathrm{SL}(2;\mathbb{C})$ on the target projective line $\mathbb{C}P^1$ via fractional linear transformations which are in turn $\mathrm{SO}(3)$ rotations on S^2 regarded as the unit sphere in the space of anti-self-dual 2-forms provided either by the old or the new metric γ or g, respectively.

It follows that $\pi : Z \to \mathbb{C}P^1$ i.e., the map (5) constructed in Lemma 3.2 is compatible with the real structure $\tau : Z \to Z$ already fixed by the self-dual structure in Theorem 2.1 therefore twistor theory provides us with a Ricci-flat (and self-dual) Riemannian metric g on X_M. We proceed further and demonstrate that, unlike (X_M, γ), the space (X_M, g) is complete.

Lemma 3.3. *The four dimensional connected and simply connected, open, oriented, Ricci-flat Riemannian spin manifold (X_M, g) is complete.*

Proof. Since both γ and this Ricci-flat metric g arise from the same complex structure on the same twistor space Z we know from twistor theory that these metrics are in fact conformally equivalent. That is, there exists a smooth non-constant strictly positive function $\varphi : X_M \to \mathbb{R}_+$ such that $\varphi^{-2}\gamma = g$. Our strategy to prove completeness is to follow Gordon [29] i.e., to demonstrate that an appropriate real-valued function on X_M, in our case $\log \varphi^{-1} : X_M \to \mathbb{R}$, is proper (i.e., the preimages of compact subsets are compact) with bounded gradient in modulus with respect to g implying the completeness.

Referring to (1) the open space X_M arises by deleting one-one projective line from each $\mathbb{C}P^2$ factor, respectively, of the closed space $\overline{X}_M$. First we observe that $\varphi^{-1}: X_M \to \mathbb{R}_+$ is uniformly divergent along these projective lines. Assume that φ^{-1} extends over $\overline{X}_M \supset X_M$ in a uniform continuous manner i.e. $\overline{\varphi}^{-1} \in C^0(\overline{X}_M)$ exists. A general principle based on the twistor construction is that the continuous extendibility of φ^{-1} over $U \subseteqq \overline{X}_M$ implies the extendibility of π i.e. the holomorphic map (5) over $\overline{p}^{-1}(U) \subseteqq \overline{Z}$ too in a manner which is compatible with the real structure on $\overline{p}^{-1}(U)$ i.e. this extension is a trivialization of the real bundle $S(\wedge^- U)$ (see the summary of twistor theory in Section 2). Therefore by our assumption the holomorphic map (5) extends over $\overline{Z}$ as well. However, since $\mathbb{C}P^2$ is not spin $\overline{X}_M = M\#\mathbb{C}P^2\#\dots\#\mathbb{C}P^2$ cannot be spin, too; consequently $S(\wedge^- \overline{X}_M)$ underlying the compact twistor space $\overline{Z}$ of $(\overline{X}_M, \overline{\gamma})$ is topologically not trivial hence its globally trivializing map (5) cannot extend from X_M to $\overline{X}_M$, a contradiction. Assume that φ^{-1} extends over at least one point of $\overline{X}_M \setminus X_M$ continuously. It yet follows that we run into a same type of contradiction. Assume now that φ^{-1} extends over at least one point of $\overline{X}_M \setminus X_M$ in a discontinuous-but-bounded manner. Then we proceed as follows. The conformal scaling function satisfies with respect to γ the following equations on X_M:

$$\left\{\begin{array}{ll} \Delta_\gamma \varphi^{-1} + \frac{1}{6}\varphi^{-1}\mathrm{Scal}_\gamma & = 0 \ \text{(vanishing of the scalar curvature of } g \text{ on } X_M\text{);} \\ \\ \nabla^2_\gamma \varphi - \frac{1}{4}\left(\Delta_\gamma \varphi\right)\gamma + \frac{1}{2}\varphi\,\mathrm{Ric}^0_\gamma & = 0 \ \text{(vanishing of the traceless Ricci tensor of } g \text{ on } X_M\text{).} \end{array}\right. \tag{6}$$

The Ricci tensor Ric_γ of γ extends smoothly over $\overline{X}_M$ because it is just the restriction of the Ricci tensor of the self-dual metric $\overline{\gamma}$ on $\overline{X}_M$. Therefore both its scalar curvature Scal_γ and traceless Ricci part Ric^0_γ extend. Thus from the first equation of (6) we can see that $\varphi\Delta_\gamma\varphi^{-1}$ extends smoothly over $\overline{X}_M$. Likewise, adding the tracial part to the second equation of (6) we get $\varphi^{-1}\nabla^2_\gamma\varphi = -\frac{1}{2}\mathrm{Ric}_\gamma$ hence we conclude that the symmetric tensor field $\varphi^{-1}\nabla^2_\gamma\varphi$ extends smoothly over $\overline{X}_M$ so its trace $\varphi^{-1}\Delta_\gamma\varphi$ as well. Expanding $\Delta_{\overline{\gamma}}(\varphi\varphi^{-1}) = 0$ gives $(\Delta_{\overline{\gamma}}\varphi)\varphi^{-1} + 2\,\overline{\gamma}(\mathrm{d}\varphi\,,\mathrm{d}\varphi^{-1}) + \varphi\Delta_{\overline{\gamma}}\varphi^{-1} = 0$ hence we obtain the pointwise equality

$$\varphi^2\left|\mathrm{d}\varphi^{-1}\right|^2_{\overline{\gamma}} = \frac{1}{2}\left(\varphi\Delta_{\overline{\gamma}}\varphi^{-1} + \varphi^{-1}\Delta_{\overline{\gamma}}\varphi\right) \tag{7}$$

which demonstrates that $\varphi\left|\mathrm{d}\varphi^{-1}\right|_{\overline{\gamma}}$ extends smoothly over $\overline{X}_M$, too. If φ^{-1} was extendible as a discontinuous bounded function over a point of $\overline{X}_M \setminus X_M$ then its gradient $\mathrm{d}\varphi^{-1}$ was divergent in that point; hence from the extendibility of $\varphi|\mathrm{d}\varphi^{-1}|_{\overline{\gamma}}$ we obtain that φ vanishes hence φ^{-1} is unbounded in that point, a contradiction again. We conclude that $\varphi^{-1}: X_M \to \mathbb{R}_+$ is *uniformly divergent along the whole complementum* $\overline{X}_M \setminus X_M$ yielding, on the one hand, that the function $\log\varphi^{-1}: X_M \to \mathbb{R}$ is proper.

As a byproduct the inverse of φ^{-1} is bounded on X_M i.e., $|\varphi| \leqq c_1$ with a finite constant. We already know that $|\varphi\Delta_\gamma\varphi^{-1}| \leqq c_2$ and $|\varphi^{-1}\Delta_\gamma\varphi| \leqq c_3$ with other finite constants as well. Now writing $\varphi|\mathrm{d}\varphi^{-1}|_\gamma = |\mathrm{d}(\log\varphi^{-1})|_\gamma$ and carefully noticing that $|\xi|_g = \varphi|\xi|_\gamma$ on 1-forms we can use (7) and the estimates above to come up with

$$|\mathrm{d}(\log\varphi^{-1})|_g \leqq c_1|\mathrm{d}(\log\varphi^{-1})|_\gamma \leqq c_1\left(\left|\varphi\Delta_\gamma\varphi^{-1}\right| + \left|\varphi^{-1}\Delta_\gamma\varphi\right|\right)^{\frac{1}{2}} \leqq c_1(c_2+c_3)^{\frac{1}{2}} < +\infty$$

and conclude, on the other hand, that $\log\varphi^{-1}: X_M \to \mathbb{R}$ has bounded gradient in modulus with respect to g. Therefore, in light of Gordon's theorem [29], the Ricci-flat space (X_M, g) is complete. $\diamond$

We want to finish the construction by ending up with an open space with a single end, hence we want to remove the extra "non-distinguished" $\mathbb{R}^4$'s from X_M in its decomposition (1) without destroying completeness and Ricci flatness.

Lemma 3.4. *Consider the space* (X_M, g) *as in Lemma 3.3. Then the orientation and the complete Ricci-flat metric* g *on* X_M *descend to the punctured space* $M^\times \subset M$ *with its inherited smooth structure, rendering it a connected and simply connected, open, oriented, complete, Ricci-flat Riemannian spin* 4*-manifold* $(M^\times, g)$.

Proof. It is clear from (1) that $M^\times$ arises from X_M by filling in the "centers" of the finitely many non-distinguished $\mathbb{R}^4$ summands with one-one point, respectively (see Figure 3).

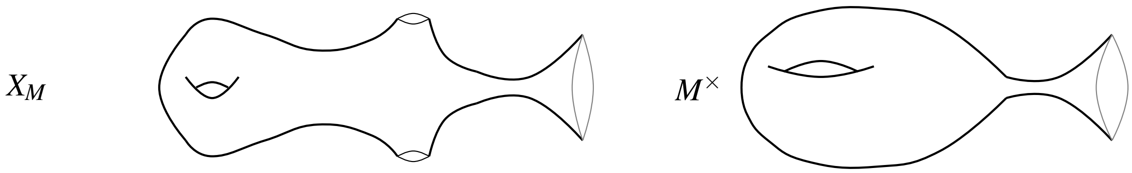

Figure 3. Construction of $M^\times$ out of X_M by filling in the extra $\mathbb{R}^4$'s.

Given this set-up, our strategy to prove the lemma is as follows: First apply Uhlenbeck's singularity removal theorem at each $\mathbb{R}^4$ summand to get rid of the corresponding singularity of the Levi–Civita connection of g—which is certainly an obstacle against the extension of the metric over the "center" of this $\mathbb{R}^4$ summand in the intermediate manifold—and in this way extend the connection to $M^\times$. Finally around each former singular point use a geodesic normal coordinate system adapted to this extended smooth connection on $M^\times$ to conclude that the metric g on X_M smoothly extends over the singularities, too. If this procedure works then the result is a smooth complete Ricci-flat metric on $M^\times$. However, as we shall see shortly, the non-existence of a spin structure on the original compact M plays the role of an (and the only one) obstruction against the feasibility of this procedure.

So let us take a fixed $\mathbb{R}^4$ summand in $X_M = M^\times \# \mathbb{R}^4 \# \ldots \# \mathbb{R}^4$. Since X_M locally looks like a punctured M around this summand i.e., a point $p \in M$ removed, we can *diffeomorphically* model $M^\times \# \mathbb{R}^4 \# \ldots \# \mathbb{R}^4$ around this $\mathbb{R}^4$ summand by an open punctured ball in some local modeling $\mathbb{R}^4$. More precisely let $p \in M$ be a point, $p \in U \subset M$ a neighbourhood containing the point and consider a local coordinate system $(U, y_1, \ldots, y_4)$ centered at p i.e., satisfying $y_1(p) = 0, \ldots, y_4(p) = 0$. Identifying this local coordinate system with $(x_1, \ldots, x_4)$ about the origin of the modeling $\mathbb{R}^4$ implies that p is mapped to $0 \in \mathbb{R}^4$ having coordinates $(x_1, \ldots, x_4) = (0, \ldots, 0)$ and our model for the vicinity of the given $\mathbb{R}^4$ summand in X_M then looks like

$$\left(B_r^{4\times}(0)\,, x_1, \ldots, x_4\right) \tag{8}$$

i.e., a coordinatized open punctured ball $B_r^{4\times}(0) := B_r^4(0) \setminus \{0\}$ about $0 \in \mathbb{R}^4$ of (Euclidean) radius $r > 0$. (In this picture the "infinity" of the $\mathbb{R}^4$ summand corresponds to the the center of the ball.) Consider the restricted tangent bundle $TB_r^{4\times}(0) := TX_M|_{B_r^{4\times}(0)}$; using the restrictions of the orientation on X_M and the metric g, we can render it a real four-rank $\mathrm{SO}(4)$ vector bundle over the punctured ball $B_r^{4\times}(0)$. We claim that $TB_r^{4\times}(0)$ in fact can be reduced to a complex two-rank $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ vector bundle over the annulus. We can see this by exploiting the so far unmentioned feature of our construction namely that as a "byproduct" the space (X_M, g) of Lemma 3.3 carries a compatible hyper-Kähler structure, too. Since the original compact space $(\overline{X}_M, \overline{\gamma})$ of Lemma 3.1 was oriented and self-dual with both properties being conformally invariant, (X_M, g) is in fact a connected, simply connected, oriented, complete self-dual and Ricci-flat space or in other words: A hyper-Kähler 4-manifold [3, Chapter 13]. This implies among other things that the holonomy group of the Levi–Civita connection of g hence the structure group of TX_M reduces to $\mathrm{SU}(2) \subset \mathrm{SO}(4)$. Consider the Levi–Civita connection of (X_M, g). We can therefore suppose that its restriction to $TB_r^{4\times}(0) \subset TX_M$ is an $\mathrm{SU}(2)$ connection $\nabla^\times$ suffering from a singularity at the origin. We know moreover that being $\nabla^\times$ self-dual, it solves the $\mathrm{SU}(2)$ Yang–Mills equations. Moreover $\nabla^\times$ has finite energy over $B_r^{4\times}(0)$. This is because g is Ricci-flat and self-dual so the curvature of $\nabla^\times$ coincides with the self-dual Weyl component W_g^+ of g only; however being conformally invariant, $W_g^+ = W_\gamma^+ = W_{\overline{\gamma}}^+|_{X_M}$ that is, the curvature tensor of $\nabla^\times$ is just the restriction of the Weyl tensor of the original smooth metric $\overline{\gamma}$ on $\overline{X}_M$. Consequently it is a smooth bounded tensor field on $B_r^{4\times}(0)$ implying finite local energy. This also yields that, if $0 < r$ is sufficiently small, $\nabla^\times$ admits an L_1^2 gauge along $B_r^{4\times}(0)$ as well. Therefore, by Uhlenbeck's singularity removal theorem (see Theorem 2.2) there exists an L_2^2 gauge transformation on $TB_r^{4\times}(0)$ such that the gauge transformed connection extends across the singularity to a smooth $\mathrm{SU}(2)$ connection ∇ on the trivial bundle $TB_r^4(0)$. Consequently, switching to the global picture, the singularity of the Levi–Civita connection around the fixed $\mathbb{R}^4$ summand of X_M can be removed hence the corresponding $\mathbb{R}^4$ summand can be deleted from (1) according to our original plan. Repeating this procedure around all the finitely many $\mathbb{R}^4$ summands of X_M we finally come up with a smooth $\mathrm{SU}(2)$ connection over $M^\times$.

However there is an important topological subtlety here. For notational simplicity suppose that $X_M = M^\times \# \mathbb{R}^4$ i.e., possesses one non-distinguished $\mathbb{R}^4$ summand only. Then the singularity removal procedure carried out above convinces us that the original singular Levi–Civita connection defined on the *tangent* bundle $T(M^\times \# \mathbb{R}^4)$, regarded as an $\mathrm{SU}(2)$ bundle, indeed extends to a non-singular $\mathrm{SU}(2)$ connection on *some* $\mathrm{SU}(2)$ bundle $E^\times$ over $M^\times$ i.e., it indeed smoothly exists somewhere which is however not necessarily the tangent bundle of $M^\times$. For instance, as we emphasized in the discussion after Theorem 2.2, the singularity-removing-gauge-transformation is not continuous in general hence the original global vector bundle carrying the singular connection may change topology during the singularity removal procedure. However, we know the following two things. On the one hand complex two-rank $\mathrm{SU}(2)$ vector bundles over $M^\times$, like the $E^\times$ above carrying the non-singular connection, are classified by various characteristic classes taking values in the groups $H^i(M^\times; \pi_{i-1}(\mathrm{SU}(2))$ with $i = 1, \dots, 4$. Knowing the first three homotopy groups of $\mathrm{SU}(2)$ and taking into account the non-compactness of $M^\times$ these cohomology groups are all trivial consequently we know that $E^\times$ is necessarily isomorphic to the trivial bundle over $M^\times$. On the other hand, real rank-four $\mathrm{SO}(4)$ vector bundles over $M^\times$, like the tangent bundle $TM^\times$ carrying an orientation and a Riemann metric, are classified by characteristic classes taking values in $H^i(M^\times; \pi_{i-1}(\mathrm{SO}(4))$. Again recalling the first three homotopy groups of the non-simply connected group $\mathrm{SO}(4)$ and still keeping in mind that $M^\times$ is non-compact, the only potentially non-trivial group here is $H^2(M^\times; \mathbb{Z}_2)$ demonstrating that vector bundles of this type over $M^\times$ are classified by a single element and this is nothing but their second Stiefel–Whitney class. Consequently if M is spin or equivalently $w_2(TM) = 0 \in H^2(M; \mathbb{Z}_2)$ then by the injection $M^\times \subset M$ we find $w_2(TM^\times) = 0 \in H^2(M^\times; \mathbb{Z}_2)$ as well showing that $TM^\times$ *is isomorphic to the trivial bundle*, too. Therefore we conclude that whenever M is spin, we can identify the vector bundle $E^\times$ carrying the non-singular $\mathrm{SU}(2)$ connection over $M^\times$ with its tangent bundle $TM^\times$.

Having understood this, we can finish the proof by extending the metric itself through the singularities. Fortunately this is simple. Consider the restricted connection ∇ about one singular point p. This is now an overall (i.e., including the singular point) smooth connection. Therefore there exists a $\delta(p) > 0$ such that we can suppose without loss of generality that the coordinate system (8) we take about this singular point with $0 < r < \delta(p)$ is a geodesic normal coordinate system with respect to ∇. This implies that the Christoffel symbols $\nabla_{\partial_i}\partial_j = \sum\limits_k \Gamma_{ij}^k \partial_k$ all vanish in the center i.e., $\Gamma_{ij}^k(0, \dots, 0) = 0$

for all $i, j, k = 1, \ldots 4$. Then the well-known compatibility equations

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{4}\left(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\right) g^{lk}$$

imply in a well-known way that in this gauge g extends over the origin, too, such that $g_{ij}(0,\ldots,0) = \delta_{ij}$ and $\partial_k g_{ij}(0,\ldots,0) = 0$ for all $i, j, k = 1, \ldots, 4$. The further differentiablity i.e., the smoothness of g at the origin follows from the smoothness of the Christoffel symbols there. That g is Ricci-flat is a trivial consequence of the same property of the original metric. $\diamond$

4 Construction in the exotic setting

In this section we shall sink into the bottomless sea of four dimensionality, called Exotica, and repeat the procedure performed in Section 3. That is, we shall construct solutions of the vacuum Einstein equation on the smooth 4-manifold $X^\times$ which is only *homeomorphic* but not *diffeomorphic* to the punctured manifold $M^\times$ appeared in Section 3. This construction basically goes along the lines of that presented in Section 3 with minor technical differences. Consequently, those steps which require new tools will be worked out in detail while those which are basically the same as the corresponding ones in Section 3 will be sketched only.

To begin with, we compose Theorems 2.1, 2.3 and 2.4 together as follows.

Lemma 4.1. *Out of any connected, closed (i.e., compact without boundary) oriented smooth* 4*-manifold M one can construct a connected, open (i.e., non-compact without boundary) oriented smooth Riemannian* 4*-manifold (X_M, γ) which is self-dual but incomplete in general.*

Proof. Pick any connected, oriented, closed, smooth 4-manifold M. Referring to Theorem 2.1 let $k := \max(1, k_M) \in \mathbb{N}$ be a positive integer, put

$$\overline{X}_M := M\#\underbrace{\mathbb{C}P^2\#\ldots\#\mathbb{C}P^2}_{k}$$

and let $\overline{\gamma}$ be a self-dual metric on it. Then $(\overline{X}_M, \overline{\gamma})$ is a compact self-dual manifold. Pick one $\mathbb{C}P^2$ factor within $\overline{X}_M$ and denote by $S^2 := \mathbb{C}P^2 \setminus R^4$ the complement of the largest exotic $\mathbb{R}^4$-space $R^4 \subset \mathbb{C}P^2$, considered as an only "continously embedded projective line" in that factor, as in the discussion after Theorem 2.3 (we can suppose that the closed subspace $S^2 \subset \mathbb{C}P^2$ avoids the attaching point of $\mathbb{C}P^2$ to M). Let $K \subset R^4$ be the connected compact subset as in part (ii) of Theorem 2.3 and put

$$X_M := M\#\underbrace{(\mathbb{C}P^2\setminus\mathbb{C}P^1)\#\ldots\#(\mathbb{C}P^2\setminus\mathbb{C}P^1)}_{k-1}\#_K(\mathbb{C}P^2\setminus S^2) \cong M\#\underbrace{\mathbb{R}^4\#\ldots\#\mathbb{R}^4}_{k-1}\#_K R^4 \cong X^\times\#\underbrace{\mathbb{R}^4\#\ldots\#\mathbb{R}^4}_{k-1} \tag{9}$$

where the operation $\#_K$ means that the attaching point $y_0 \in R^4$ used to glue R^4 with $M\#\mathbb{R}^4\#\ldots\#\mathbb{R}^4$ satisfies $y_0 \in K \subset R^4$ and $X^\times := M\#_K R^4$ is a smooth manifold homeomorphic but not diffeomorphic to the puncturation $M^\times$ of the original manifold (see Theorem 2.4). The result is a connected, open 4-manifold X_M (see Figure 4).

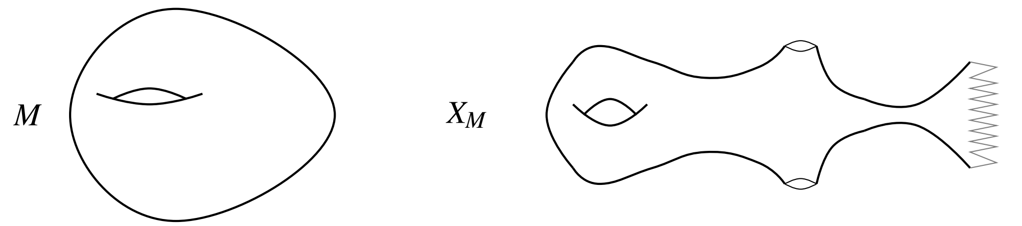

Figure 4. Construction of X_M out of M in the exotic setting. The gray zig-zag represents a "creased end" diffeomorphic to the complement of a connected compact subset K in the exotic R^4.

From the proper smooth embedding $X_M \subsetneqq \overline{X}_M$ there exists a restricted self-dual Riemannian metric $\gamma := \overline{\gamma}|_{X_M}$ on X_M which is however in general non-complete. $\diamond$

In the case of our situation set up in Lemma 4.1 twistor theory works as follows. Consider the compact self-dual space $(\overline{X}_M, \overline{\gamma})$ from Lemma 4.1, take its twistor fibration $\overline{p} : \overline{Z} \to \overline{X}_M$ and let $p : Z \to X_M$ be its restriction induced by the smooth embedding $X_M \subsetneqq \overline{X}_M$ i.e., $Z := \overline{Z}|_{X_M}$ and $p := \overline{p}|_{X_M}$. Then Z is a non-compact complex 3-manifold already obviously possessing all the required twistor data except the existence of a holomorphic mapping $\pi : Z \to \mathbb{C}P^1$.

Lemma 4.2. *Consider the connected, open, oriented, incomplete, self-dual space (X_M, γ) as in Lemma 4.1 with its twistor fibration $p : Z \to X_M$ constructed above. If $\pi_1(M) = 1$ and M is spin (or equivalently, having even intersection form) then there exists a holomorphic mapping $\pi : Z \to \mathbb{C}P^1$.*

Proof. Let $x_0 \in X_M$ be an arbitrary fixed point of X_M in (9). Our aim is to construct a holomorphic map

$$\pi : \ Z \longrightarrow p^{-1}(x_0) \cong \mathbb{C}P^1$$

that we carry out exactly the same way as in the proof of Lemma 3.2 hence we do not repeat it here. $\diamond$

It also follows that $\pi : Z \to \mathbb{C}P^1$ i.e., the map constructed in Lemma 4.2 is compatible with the real structure $\tau : Z \to Z$ already fixed by the self-dual structure in Theorem 2.1 therefore twistor theory provides us with a Ricci-flat (and self-dual) Riemannian metric g on X_M. We proceed further and demonstrate that, unlike (X_M, γ), the space (X_M, g) is complete.

Lemma 4.3. *The four dimensional connected and simply connected, open, oriented, Ricci-flat Riemannian spin manifold (X_M, g) is complete.*

Proof. The metrics γ and g originate from the same twistor space again hence they are conformally equivalent consequently there exists a smooth function $\psi : X_M \to \mathbb{R}_+$ satisfying $g = \psi^{-2}\gamma$. Taking into account that the steps in the proof of Lemma 3.3 have been insensitive for the particular construction of the complementum $\overline{X}_M \setminus X_M$ we can simply repeat them here. Hence we find again, on the one hand, that ψ^{-1} blows up uniformly along the whole $\overline{X}_M \setminus X_M$ this time consisting of the disjoint union of "ordinary" i.e. holomorphically embedded projective lines $\mathbb{C}P^1 = \mathbb{C}P^2 \setminus \mathbb{R}^4$ and the distinguished "continuously embedded projective line" $S^2 = \mathbb{C}P^2 \setminus R^4$ in the distinguished factor in (9); consequently $\log \psi^{-1}$ is proper. Moreover, on the other hand, recalling the steps of Lemma 3.3 we see that $\log \psi^{-1}$ has bounded gradient in modulus with respect to g, too. Consequently (X_M, g) is complete as in the proof of Lemma 3.3 hence the details are omitted. $\diamond$

*Remark***.** For clarity we remark that comparing the proofs of Lemmata 3.3 and 4.3 one cannot conclude that the resulting complete spaces (X_M, g) in the non-exotic and exotic situations are conformally equivalent. This is because (see the discussion after Theorem 2.3) the locations of X_M in the two cases within their common closure $\overline{X}_M$ are different such that even the former cannot be mapped into the latter by any diffeomorphism of $\overline{X}_M$. Consequently taking the pointwise product of the scaling function φ^{-2} in Lemma 3.3 with the inverse one ψ^2 from Lemma 4.3 to obtain a conformal rescaling between the corresponding metrics makes no sense.

Finally we cut down the standard $\mathbb{R}^4$'s from X_M to obtain $X^\times$ as in Lemma 3.4.

Lemma 4.4. *Consider the space (X_M, g) as in Lemma 4.3. Then the orientation and the complete Ricci-flat metric g on X_M descend to the punctured space $X^\times$ (which is homeomorphic but not diffeomorphic to the corresponding space $M^\times$ of Lemma 3.4) with its inherited smooth structure, rendering it a connected and simply connected, open, oriented, complete, Ricci-flat Riemannian spin 4-manifold $(X^\times, g)$.*

Proof. Taking into account that filling in the standard $\mathbb{R}^4$'s in the decomposition (9) of X_M is a completely local procedure (see Figure 5)

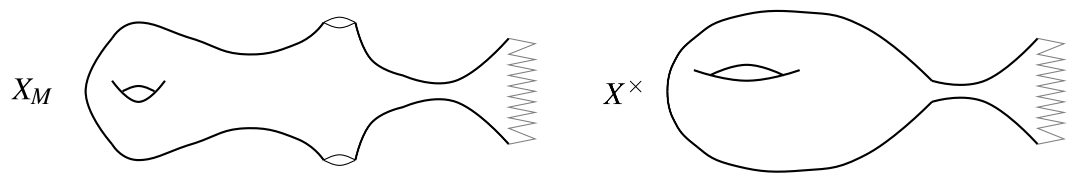

Figure 5. Construction of $X^\times$ out of X_M by filling in the extra $\mathbb{R}^4$'s.

the proof of this lemma is verbatim the same as the proof of Lemma 3.4 hence is omitted. $\Diamond$

Proof of Theorems 1.1 or 1.2. Collecting all the ruslts of Sections 3 and 4 together the desired statements are obtained. $\Diamond$

*Remark***.** Before proceeding further let us note that Theorems 1.1 or 1.2 correspond to the case when the creased end of $X^\times$ is diffeomorphic to the largest member $R^4 = R^4_{+\infty}$ in the radial family of Theorem 2.3 i.e. $X^\times = M\#R^4_{+\infty}$. It would be interesting to understand whether or not a similar construction works for the intermediate members of the family i.e. for $X^\times_t = M\#R^4_t$.

5 Lorentzian solutions

In the previous sections we have produced an immense class of Ricci flat Riemannian spaces $(X^\times, g)$ which are non-compact but complete. In this section we convert all of them i.e. the spaces in Theorem 1.1 or equivalently Theorem 1.2 into Ricci-flat Lorentzian ones as formulated in Theorem 1.3 by (essentially verbatim) recalling [19, Lemma 4.2]. Conversion is in principle possible because all the underlying manifolds $X^\times$ are non-compact hence there is no topological obstruction against Lorentzian structure.

Proof of Theorem 1.3. By virtue of its global triviality (cf. Lemma 3.4 or 4.4), $TX^\times$ admits a nowhere vanishing smooth section yielding a splitting $TX^\times = L \oplus L^\perp$ into a real line bundle $L \subset TX^\times$ spanned the section and its g-orthogonal complementum subbundle $L^\perp \subset TX^\times$. Take the complexification $T^{\mathbb{C}}X^\times := TX^\times \otimes_{\mathbb{R}} \mathbb{C}$ of the real tangent bundle as well as the complex bilinear extension of the Riemannian Ricci-flat metric g found on $TX^\times$ to a Ricci-flat metric $g^{\mathbb{C}}$ on $T^{\mathbb{C}}X^\times$. This means that if $v^{\mathbb{C}}$ is a complexified tangent vector then both $v^{\mathbb{C}} \mapsto g^{\mathbb{C}}(v^{\mathbb{C}}, \cdot) := g(v^{\mathbb{C}}, \cdot)$ and $v^{\mathbb{C}} \mapsto g^{\mathbb{C}}(\cdot, v^{\mathbb{C}}) := g(\cdot, v^{\mathbb{C}})$ are declared to be $\mathbb{C}$-linear and of course $\mathrm{Ric}_{g^{\mathbb{C}}} = \mathrm{Ric}_g = 0$. There is an induced splitting

$$T^{\mathbb{C}}X^{\times} = L \oplus L^{\perp} \oplus \sqrt{-1}\,L \oplus \sqrt{-1}\,L^{\perp} \tag{10}$$

over $\mathbb{R}$ of the complexification i.e., if $T^{\mathbb{C}}X^{\times}$ is considered as a real rank-8 bundle over $X^{\times}$. Define a metric on the real rank-4 sub-bundle $L^{\perp} \oplus \sqrt{-1}\,L \subset T^{\mathbb{C}}X^{\times}$ by taking the restriction $g^{\mathbb{C}}|_{L^{\perp}\oplus\sqrt{-1}L}$. It readily follows from the orthogonality and reality of the splitting that this is a non-degenerate real-valued $\mathbb{R}$-bilinear form of Lorentzian type on this real sub-bundle. To see this, we simply have to observe that taking real vector fields $v_1, v_2 : X^{\times} \to L$ and $w_1, w_2 : X^{\times} \to L^{\perp}$ we can exploit the $\mathbb{C}$-bilinearity of $g^{\mathbb{C}}$ to write

$$g^{\mathbb{C}}|_{L^{\perp}\oplus\sqrt{-1}L}(\sqrt{-1}\,v_1, \sqrt{-1}\,v_1) = g^{\mathbb{C}}(\sqrt{-1}\,v_1, \sqrt{-1}\,v_1) = -g^{\mathbb{C}}(v_1, v_1) = -g(v_1, v_1)$$

and

$$g^{\mathbb{C}}|_{L^{\perp}\oplus\sqrt{-1}L}(\sqrt{-1}\,v_1, w_1) = g^{\mathbb{C}}(\sqrt{-1}\,v_1, w_1) = \sqrt{-1}\,g^{\mathbb{C}}(v_1, w_1) = \sqrt{-1}g(v_1, w_1) = 0$$

and finally

$$g^{\mathbb{C}}|_{L^{\perp}\oplus\sqrt{-1}L}(w_1, w_2) = g^{\mathbb{C}}(w_1, w_2) = g(w_1, w_2)\ .$$

Consider the $\mathbb{R}$-linear bundle isomorphism $W_L : T^{\mathbb{C}}X^{\times} \to T^{\mathbb{C}}X^{\times}$ of the complexified tangent bundle defined by, with respect to the splitting (10), as

$$W_L(v_1, w_1, \sqrt{-1}\,v_2, \sqrt{-1}\,w_2) := (v_2, w_1, \sqrt{-1}\,v_1, \sqrt{-1}\,w_2)\ .$$

Obviously $W_L^2 = \mathrm{Id}_{T^{\mathbb{C}}X^{\times}}$ or more precisely W_L is a *real reflection* with respect to $g^{\mathbb{C}}$ making the diagram

$$\begin{array}{ccc} T^{\mathbb{C}}X^{\times} & \xrightarrow{W_L} & T^{\mathbb{C}}X^{\times} \\ \downarrow & & \downarrow \\ X^{\times} & \xrightarrow{\mathrm{Id}_{X^{\times}}} & X^{\times} \end{array}$$

commutative. In particular it maps the real tangent bundle $TX^{\times} = L \oplus L^{\perp} \subset T^{\mathbb{C}}X^{\times}$ onto the real bundle $L^{\perp} \oplus \sqrt{-1}\,L \subset T^{\mathbb{C}}X^{\times}$ and *vice versa*. Consequently with arbitrary two tangent vectors $v, w : X^{\times} \to TX^{\times}$

$$g_L(v, w) := g^{\mathbb{C}}(W_L v\,,\, W_L w)$$

satisfies $g_L(v, w) = g^{\mathbb{C}}|_{L^{\perp}\oplus\sqrt{-1}L}(W_L v\,,\, W_L w)$ i.e., obtain a non-degenerate real-valued $\mathbb{R}$-bilinear form of Lorentzian type hence a smooth Lorentzian metric g_L on the original real tangent bundle $TX^{\times}$.

Concerning the Ricci tensor of g_L, the Levi–Civita connections ∇^L of g_L and $\nabla^{\mathbb{C}}$ of $g^{\mathbb{C}}$ satisfy

$$\begin{aligned} g_L(\nabla^L_u v, w) + g_L(v, \nabla^L_u w) &= \mathrm{d}g_L(v, w)u \\ &= \mathrm{d}g^{\mathbb{C}}(W_L v, W_L w)u \\ &= g^{\mathbb{C}}(\nabla^{\mathbb{C}}_u(W_L v)\,,\, W_L w) + g^{\mathbb{C}}(W_L v\,,\, \nabla^{\mathbb{C}}_u(W_L w)) \\ &= g^{\mathbb{C}}(W_L^2 \nabla^{\mathbb{C}}_u W_L v\,,\, W_L w) + g^{\mathbb{C}}(W_L v\,,\, W_L^2 \nabla^{\mathbb{C}}_u W_L w) \\ &= g_L((W_L \nabla^{\mathbb{C}}_u W_L)v\,,\, w) + g_L(v\,,\, (W_L \nabla^{\mathbb{C}}_u W_L)w) \end{aligned}$$

yielding $\nabla^L = W_L \nabla^{\mathbb{C}} W_L$ (this is an $\mathbb{R}$-linear operator). Consequently the curvature Riem_{g_L} of g_L is

$$\mathrm{Riem}_{g_L}(v, w)u = \left[\nabla^L_v\,,\, \nabla^L_w\right] u - \nabla^L_{[v,w]} u = W_L(\mathrm{Riem}_{g^{\mathbb{C}}}(v, w) W_L u)\ .$$

Let $\{e_0, e_1, e_2, e_3\}$ be a real orthonormal frame for g_L at $T_pX^\times$ satisfying $g_L(e_0, e_0) = -1$ and $+1$ for the rest; then $W_Le_0 = \sqrt{-1}\, e_0$ and $W_Le_j = e_j$ for $j = 1, 2, 3$ together with the definition of g_L imply that

$$g_L(\mathrm{Riem}_{g_L}(e_0, v)w, e_0) = g^{\mathbb{C}}(W_L(\mathrm{Riem}_{g_L}(e_0, v)w), W_Le_0) = g^{\mathbb{C}}\left(\mathrm{Riem}_{g^{\mathbb{C}}}(e_0, v)W_Lw\,, \sqrt{-1}e_0\right)$$

and likewise

$$g_L(\mathrm{Riem}_{g_L}(e_j, v)w\,,\, e_j) = g^{\mathbb{C}}(W_L(\mathrm{Riem}_{g_L}(e_j, v)w), W_Le_j) = g^{\mathbb{C}}(\mathrm{Riem}_{g^{\mathbb{C}}}\left(e_j\,,\, v\right) W_Lw\,, e_j)\ .$$

Using an orthonormal frame $\{f_1, \dots, f_m\}$ for a metric h of any signature, its Ricci tensor looks like $\mathrm{Ric}_h(v, w) = \sum\limits_{k=1}^{m} h(f_k, f_k)h(\mathrm{Riem}_h(f_k, v)w\,,\, f_k)$; hence

$$\begin{aligned}
\mathrm{Ric}_{g_L}(v,w) &= g_L(e_0,e_0)g_L(\mathrm{Riem}_{g_L}(e_0,v)w,e_0) + \sum_{j=1}^{3} g_L(e_j,e_j)g_L(\mathrm{Riem}_{g_L}(e_j,v)w,e_j)\\
&= g^{\mathbb{C}}(\sqrt{-1}e_0,\sqrt{-1}e_0)g^{\mathbb{C}}(\mathrm{Riem}_{g^{\mathbb{C}}}(e_0,v)W_Lw,\sqrt{-1}e_0) + \sum_{j=1}^{3} g^{\mathbb{C}}(e_j,e_j)g^{\mathbb{C}}(\mathrm{Riem}_{g^{\mathbb{C}}}(e_j,v)W_Lw,e_j)\\
&= (-\sqrt{-1}-1)g^{\mathbb{C}}(e_0,e_0)g^{\mathbb{C}}(\mathrm{Riem}_{g^{\mathbb{C}}}(e_0,v)W_Lw,e_0) + \mathrm{Ric}_{g^{\mathbb{C}}}(v\,,\, W_Lw)\\
&= (-1+\sqrt{-1})g_L(\mathrm{Riem}_{g_L}(e_0,v)w,e_0)
\end{aligned}$$

and we also used $\{e_0, e_1, e_2, e_3\}$ as a complex orthonormal basis for $g^{\mathbb{C}}$ on $T_p^{\mathbb{C}}X^\times$ to write

$$\sum_{j=0}^{3} g^{\mathbb{C}}(e_j,e_j)g^{\mathbb{C}}(\mathrm{Riem}_{g^{\mathbb{C}}}(e_j,v)W_Lw,e_j) = \mathrm{Ric}_{g^{\mathbb{C}}}(v,W_Lw) = 0\ .$$

Being the left hand side in $\mathrm{Ric}_{g_L}(v, w) = (-1 + \sqrt{-1})g_L(\mathrm{Riem}_{g_L}(e_0, v)w, e_0)$ real, the right hand side must be real as well for all $v, w \in T_pX^\times$ which is possible if and only if both sides vanish. This demonstrates that g_L is indeed Ricci-flat. $\diamond$

6 Physical interpretation

In this closing section we discuss the physical interpretation of the Lorentzian Ricci-flat geometries found in Theorem 1.3. We believe that an interpretation is necessary because there are many known physically irrelevant solutions of the vacuum Einstein equation and our solutions as presented in Theorem 1.3 are admittedly very implicit and transcendental hence their physical significance, if any, is unclear yet. The offered interpretation fits well into the context of the celebrated *strong cosmic censorship conjecture* in its usual broad formulation (**SCCC** for short) which is a hot topic recently (far from being complete cf. [4, 5, 10, 12, 13, 18, 19, 23, 34, 41, 45]; for historical accounts see [7, 37, 48]) and the so far hypothetical *topology changing* phenomena (again far from being complete, cf. e.g. [15, 24, 35]).

The current situation of the **SCCC** can perhaps be best summarized as a *puzzling dichotomy*: although there are some signs or hints for its (in)validity in *physically relevant* situations (like various black holes in asymptotically flat or de Sitter space-times filled with vacuum or various matter fields, etc. [4, 5, 10, 12, 13, 23, 34, 41, 45]), these are still not sharp enough to decide the status of the **SCCC** in these important cases. On the other hand there exists an superabundance of "exotic" smooth solutions in which the **SCCC** clearly fails [18, 19] (namely the ones exhibited in Theorem 1.3) however the physical meaning of these quite *purely mathematical* solutions is not clear yet. The reason for this latter issue is that, although being smooth solutions of the vacuum Einstein equation hence apparently relevant, the **SCCC** violating properties of these "exotic" solutions rest neither on some physical phenomenon nor on standard analytico-geometric properties of Lorentzian metrics; but rather based on subtle novel differentio-topological features (often called *exotica*) of four dimensional manifolds which have gradually been recognized in the underlying mathematical model of physical space-times from the 1980's onwards (cf. [19, Introduction]). Despite that no *a priori* principle has been introduced so far to exclude these curious and apparently fundamental mathematical discoveries from the game, they have not found their right places in theoretical physics yet [1].

The aim of this section is an effort to fill in this gap by offering a plausible and simple physical interpretation of the new **SCCC** violating solutions [18, 19] (i.e. the spaces exhibited in Theorem 1.3). As an interesting observation it will turn out that, meanwhile the aforementioned classical situations in which **SCCC** breakdown has been examined belong to the well known static or stationary regime of general relativity, the new **SCCC** violating solutions are related with the yet unexplored deep dynamical regime of general relativity describing spatial topology changes as will be explained shortly. We also find that this dynamics appears as a *cosmologial redshift* for late time internal observers within these space-times. Therefore, quite unsurprisingly, one is tempted to say that as one moves from the static towards the dynamical regime, **SCCC** violating phenomena become more and more relevant in general relativity.

Take any connected, simply connected, closed spin 4-manifold M and form the connected sum $X^\times := M\#R^4$ as before (see Figure 5). It is easy to see (cf. the summary of the exotic stuff in Section 2) that $X^\times$ is homeomorphic to the punctured space $M^\times = M\setminus\{\text{point}\}$ however cannot be diffeomorphic to it (with its usual inherited smooth structure from the smooth embedding $M^\times\subset M$) since $M^\times$ is diffeomorphic to $M\#\mathbb{R}^4$ meanwhile $X^\times$ by construction is diffeomorphic to $M\#R^4$ hence the ends of the two open spaces, although homeomorphic, are not diffeomorphic. Actually, from a general viewpoint, the appearance of non-compact 4-manifolds carrying smooth structures like $X^\times$ i.e. having a "creased end" is much more typical. Theorem 1.3 then says that $X^\times$ always carries a Ricci-flat Lorentzian metric g_L. Having $X^\times$ a creased end implies that it surely cannot be written as a smooth product $\Sigma\times\mathbb{R}$ where Σ is a 3-manifold and $\mathbb{R}$ is the real line (with their unique smooth structures); however the existence of such a smooth splitting is a necessary condition of global hyperbolicity [2]. Consequently we arrive at a sort of heavy breakdown of the **SCCC** (in its usual broad formulation, cf. e.g. [18, 19]), namely

$\overline{\textbf{SCCC}}$. *The smooth Ricci-flat Lorentzian 4-manifold $(X^\times, g_L)$ in Theorem 1.3 is not globally hyperbolic and no (sufficiently large in an appropriate topological sense) perturbation of it can be globally hyperbolic.*

Furthermore, Theorem 1.1 says that $X^\times$ carries a complete Ricci-flat Riemannian metric g, too. As a by-product of the construction we have seen that fixing an appropriate orientation on $X^\times$ the metric g is self-dual, too. However a simply-connected, complete Riemannian 4-manifold which is both Ricci-flat and self-dual is in fact, as formulated in Theorem 1.2, hyper-Kähler (cf. e.g. [3, Chapter 13]). Physically speaking the Riemannian 4-manifolds $(X^\times, g)$ exhibited in Theorem 1.1 or equiavlently, in Theorem 1.2 are therefore examples of *gravitational instantons*. Consequently, even if these Riemannian (or Euclidean) vacuum spaces might not play any role in classical general relativity, they are not negligable in any quantum theory perhaps lurking behind classical general relativity.

After these introductory or general remarks let us move towards a suggested physical interpretation of the **SCCC** breaking but otherwise regular geometry $(X^\times, g_L)$. The conversion procedure in Theorem

1.3 rests on a nowhere-vanishing vector field

$$v \in C^\infty(X^\times; TX^\times \setminus \{0\}) \tag{11}$$

along $X^\times$ whose choice was otherwise arbitrary. Therefore, taking into account the global triviality of the tangent bundle $TX^\times$ (cf. Lemmata 3.4 and 4.4), we have a great freedom in specifying it what we now exploit as follows. Consider the original simply connected and closed M used in Theorem 1.1. Simply connectedness implies the vanishing of the first de Rham cohomology of M therefore if we put any Riemannian metric onto M and consider the corresponding Laplacian on 1-forms, its kernel is trivial. The Hodge decomposition theorem then says that any 1-form ξ on M uniquely splits as $\xi = \mathrm{d}f + \mathrm{d}^*\eta$ where f is a function and η a 2-form on M. The corresponding dual decomposition of a smooth vector field v on M therefore looks like $v = \mathrm{grad}f + \mathrm{div}T$ where T is a $(2,0)$-type tensor field.

Motivated by this, consider now the space $X^\times$ of Theorem 1.3 and recall that it is homeomorphic to $M^\times$ consequently has vanishing first de Rham cohomology, too. Therefore, as a first and naive choice, we set the nowhere vanishing vector field (11) used to construct the Ricci-flat Lorentzian metric g_L on $X^\times$ out of the Ricci-flat Riemannian one g to be of the form

$$v := \mathrm{grad}f \tag{12}$$

where $f : X^\times \to (-\infty, 0]$ is a Morse function (to be defined shortly) on $X^\times$ such that $f^{-1}(-\infty)$ corresponds to the creased end of $X^\times$ while $f^{-1}(t) \subset X^\times$ are compact level sets for all $-\infty < t \leqq 0$ and in particular the point $f^{-1}(0)$ is the "top" of $X^\times$ (see Figure 6).

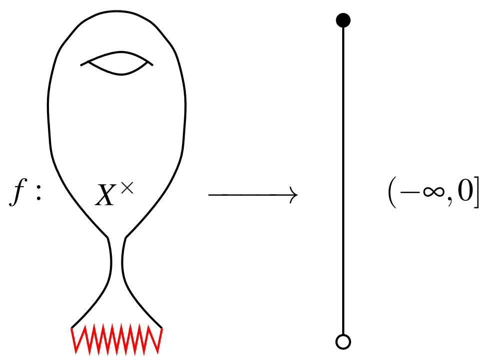

Figure 6. The manifold $X^\times$ with a zig-zag representing its creased end and a Morse function $f : X^\times \to (-\infty, 0]$ on it.

Moreover $\mathrm{grad}f$ in (12) is defined by $\mathrm{d}f = g(\mathrm{grad}f\,,\,\cdot\,)$ to be the dual vector field of the 1-form $\mathrm{d}f$ with respect to the original *Riemannian* metric g on $X^\times$. If the choice in (12) is possible then we gain a very nice picture on the vacuum space-time $(X^\times, g_L)$. Namely, $\mathrm{grad}f : X^\times \to L \subset L \oplus L^\perp = TX^\times$ is a vector field such that for a generic $t \in (-\infty, 0]$ it does not vanish and the level set $f^{-1}(t) \subset X^\times$ is a 3 dimensional closed (i.e., compact without boundary) submanifold with $Tf^{-1}(t) = L^\perp \subset L \oplus L^\perp = TX^\times$. Hence with respect to g_L we find that $\mathrm{grad}f$ is a timelike and by definition future-directed vector field g_L-orthogonal for the level sets which are spacelike. In other words: *The vector field v in (11) is an infinitesimal observer in the space-time $(X^\times, g_L)$. If v has the form (12) then v can be identified with a global classical observer in the sense that the level value $t \in (-\infty, 0]$ corresponds to its global classical proper time as moves along its future directed own timelike curves (i.e., the integral curves of $v = \mathrm{grad}f$) and the level sets $f^{-1}(t) \subset X^\times$ correspond to its global classical spacelike submanifolds*. However this picture is too naive because f may attain critical points i.e., $p \in X^\times$ where $\mathrm{grad}f(p) = 0$ as we know from Morse theory. Hence the nowhere-vanishing vector field (11) cannot globally look like (12).

A rapid course on Morse theory. The following things are well known [28, 44] but we summarize them here for completeness and convenience. Let N be a smooth n-manifold. The point $p \in N$ is a *critical point* of a smooth function $f : N \to \mathbb{R}$ iff in a local coordinate system $(U_p, x_1, \dots, x_n)$ centered at p all the partial derivatives vanish there i.e., $\partial_i f(p) = 0$ for all $i = 1, \dots, n$ and it is *non-degenerate* iff the matrix $(\partial^2_{ij} f(p))_{i,j=1,\dots,n}$ is not singular. Moreover $c \in \mathbb{R}$ is a *critical value* iff the level set $f^{-1}(c) \subset N$ contains a critical point. The smooth function $f : N \to \mathbb{R}$ is a *Morse function* along N iff it admits only non-degenerate critical points such that each critical value level set contains at most one critical point. (Being non-degenerate already implies that the critical points are isolated [44, Corollary 2.3]). We shall also assume below that the level set $f^{-1}(c) \subset N$ is compact as well, for all $c \in \mathbb{R}$.

We know the following things. If $c \in \mathbb{R}$ is non-critical then $f^{-1}(c) \subset N$ is a smooth $n-1$ dimensional submanifold. If $c \in \mathbb{R}$ critical with a single critical point $p \in f^{-1}(c) \subset N$ then (cf. [44, Lemma 2.2]) there exists a local coordinate system $(U_p, y_1, \dots, y_n)$ about p i.e., $y_1(p) = \dots = y_n(p) = 0$, in which

$$f|_{U_p}(y_1, \dots, y_n) = f(0, \dots, 0) - \sum_{i=1}^{k} y_i^2 + \sum_{i=k+1}^{n} y_i^2$$

and the number $0 \leqq k \leqq n$ is called the *index* of the critical point. Therefore a critical point of index $k = 0$ is a local minimum while with index $k = n$ is a local maximum of f. Take $c \in \mathbb{R}$, $\varepsilon > 0$ and suppose that $[c-\varepsilon, c+\varepsilon] \subset \mathbb{R}$ consists of non-critical values only. Then (cf. [44, Theorem 3.1]) $f^{-1}(c-\varepsilon)$ and $f^{-1}(c+\varepsilon)$ are diffeomorphic. If the only critical value in $[c-\varepsilon, c+\varepsilon]$ is c and its unique critical point $p \in f^{-1}(c)$ is of index k then (cf. [44, Theorem 3.2]) $f^{-1}(c+\varepsilon)$ is obtained from $f^{-1}(c-\varepsilon)$ by glueing to the boundary of $f^{-1}((-\infty, c-\varepsilon])$ a closed n-ball B^n in the form of a k-handle $B^k \times B^{n-k}$. More precisely take an embedding $\varphi_k : S^{k-1} \times B^{n-k} \to f^{-1}(c-\varepsilon)$ and glue B^n to $f^{-1}((-\infty, c-\varepsilon])$ by identifying

$$S^{k-1} \times B^{n-k} \subsetneqq \partial(B^k \times B^{n-k}) = (S^{k-1} \times B^{n-k}) \cup (B^k \times S^{n-k-1})$$

with the image $\varphi_k(S^{k-1} \times B^{n-k}) \subsetneqq \partial\left(f^{-1}((-\infty, c-\varepsilon])\right) = f^{-1}(c-\varepsilon)$. Then after "smoothing off the corners" we obtain an n dimensional manifold-with-boundary $f^{-1}((-\infty, c-\varepsilon]) \cup_{\varphi_k} B^n$ and $f^{-1}(c+\varepsilon)$ is diffeomorphic to $\partial\left(f^{-1}((-\infty, c-\varepsilon]) \cup_{\varphi_k} B^n\right)$. For instance if $k = 0$ then B^n is glued along $S^{-1} \times B^n$ where $S^{-1} = \emptyset$ i.e., it is not glued hence this critical point is a local minimum; while if $k = n$ then B^n is attached along $S^{n-1} \times B^0$ where B^0 is a point i.e., it is attached along its full boundary S^{n-1} hence this is a local maximum of f. Note that replacing the bottom-up function f with the top-down function $-f$ critical points with index k and $n-k$ interchange.

Critical points necessarily occur. If N is compact then a fundamental result of Morse theory (cf. [44, Theorem 5.2]) states that if $m_k(N)$ denotes the number of critical points of index k and $b_k(N)$ the k^{th} Betti number of N then $b_k(N) \leqq m_k(N) < +\infty$. If N is not compact then in general no such lower bounds exist but some $m_k(N)$'s can be even infinite. For further details cf. [28, Chapter 4] or [44].

Returning to our problem, we therefore correct (12) as follows. Although critical points of f are unavoidable, they are at least isolated i.e., for all $p, q \in X^\times$ pairs of critical points there exist small surrounding open neighbourhoods $U_p, U_q \subset X^\times$ such that $U_p \cap U_q = \emptyset$. Then taking the union

$$C_f := \bigcup_{p \text{ is a critical point of } f} U_p$$

which is therefore disjoint and supposing that this set is sharply concentrated around the critical points of f in $X^\times$, let us correct (12) to

$$v := \operatorname{grad} f + w$$

where w is a smooth vector field (of the form $w = \mathrm{div}T$) on $X^\times$ such that $w(p) \neq 0$ in the critical point p but $\mathrm{supp}\, w \subset C_f$ i.e., w vanishes outside of $C_f \subset X^\times$. Fortunately this changes our physical picture on $(X^\times, g_L)$ only locally (i.e. close to a critical point only). More precisely, the classical observer picture of v breaks down only in the vicinity of critical points of its Morse function part. Therefore from now on we suppose: *if $v = \mathrm{grad} f + w$ is a non-vanishing vector field on $X^\times$ then the infinitesimal observer provided by v in the original space-time $(X^\times, g_L)$ gives rise to a global classical observer in the sense above at least on the open domain*

$$(X^\times \setminus \overline{C}_f\, ,\, g_L|_{X^\times \setminus \overline{C}_f}\,) \subsetneqq (X^\times, g_L) \qquad (13)$$

because $v = \mathrm{grad} f$ along this restriction as before.

Let us ask ourselves now about the "experiences" of this partial global classical observer, constructed from a Morse function, as it moves in $(X^\times, g_L)$. That is, consider a Morse function f on $X^\times$ as above (see Figure 6) with an associated global classical observer on the restricted domain $X^\times \setminus \overline{C}_f$. This observer has a global proper time $t \in (-\infty, 0]$ measured by f with the infinite past $t = -\infty$ being the creased end of $X^\times$ and also has corresponding global spacelike $\Sigma_t := f^{-1}(t) \subset X^\times \setminus \overline{C}_f$ for appropriate t's which are closed 3-manifolds. First, fix $-\infty < K < 0$ such that Σ_K is a submanifold and consider the compact part $f^{-1}([K,0]) \subsetneqq X^\times$. As the observer moves forwards in time i.e., from $t = K$ to $t = 0$ along the integral curves of $\mathrm{grad} f$ then only finitely many critical points occur. As we have seen, around these points the spacelike Σ_t's change topology by picking up a k-handle according to the index of the critical point.

Now consider the much more interesting non-compact $f^{-1}((-\infty, K]) \subset X^\times$ regime, the downward "neck" part in Figure 6. If $K < 0$ is sufficiently small (we mean $|K| > 0$ is sufficiently large) we can suppose that $f^{-1}((-\infty, K])$ is fully contained in the exotic but topologically trivial summand R^4 of $X^\times$ in its decomposition $X^\times = M \# R^4$. Therefore if $-\infty < t \leqq K$ then Σ_t is fully contained in the R^4 summand. We can without loss of generality suppose that Σ_K surrounds the attaching region of M and R^4 hence Σ_K is diffeomorphic to S^3. Now take an observer in $(X^\times, g_L)$ moving backwards in time along the integral curves of $\mathrm{grad} f$ i.e. from $t = K$ downwards $t = -\infty$. A generic value of t is not critical for f consequently the corresponding spacelike submanifold Σ_t exists. Consider a fixed time $-\infty < t_0 < K$ which is a critical value of f. How the corresponding transition between the Σ_t's then looks like? As we have seen, in this moment always a single 4-ball B^4, attached through its boundary S^3 in various ways to Σ_t depending on the index k of the critical point, is going to be removed from the latter space-time portion $f^{-1}([t_0, K])$. Therefore, as we move backwards in time provided by f (or move forwards in time provided by $-f$) and pass through the moment t_0 the space $\Sigma_{t_0+\varepsilon}$ undergoes one of the following topological transitions:

* If $k = 1$ then at t_0 an S^3, attached along two disjoint B^3's to $\Sigma_{t_0+\varepsilon}$, is annihilated (or equivalently, attached along a thickened S^2, is created);

* If $k = 2$ then at t_0 an S^3, attached along a thickened knot to $\Sigma_{t_0+\varepsilon}$, is annihilated (or equivalently, attached along a thickened knot, is created);

* If $k = 3$ then at t_0 an S^3, attached along a thickened S^2 to $\Sigma_{t_0+\varepsilon}$, is annihilated (or equivalently, attached along two disjoint B^3's, is created)

and in this way the latter space $\Sigma_{t_0+\varepsilon}$ evolves into to the earlier $\Sigma_{t_0-\varepsilon}$ as we move backwards in time. Strictly mathematically speaking this k-handle attachment is to be performed "instantaneosly" somewhere along the singular level surface Σ_{t_0} carrying a unique critical point p at the moment t_0; however from a physical viewpoint we can rather suppose that it occurs within the "non-classical" (with respect to the observer provided by grad$f$) region $\Sigma_t \cap U_p \subset C_f$ at some unspecified time $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $\Sigma_{t_0 \pm \varepsilon} \cap U_p$ are still not empty (see Figure 7). Beside the f Morse function picture, we have formulated all processes in the dual picture of the reversed Morse function $-f$ as well in order to gain full symmetry in the formulation. Moreover we note that applying diffeomorphisms on $X^\times$ (or equivalently, modifying f) we can assume that along $f^{-1}((-\infty, K])$ with $K < 0$ the $k = 0, 4$ handle attachment steps corresponding to local minima and maxima do not occur.

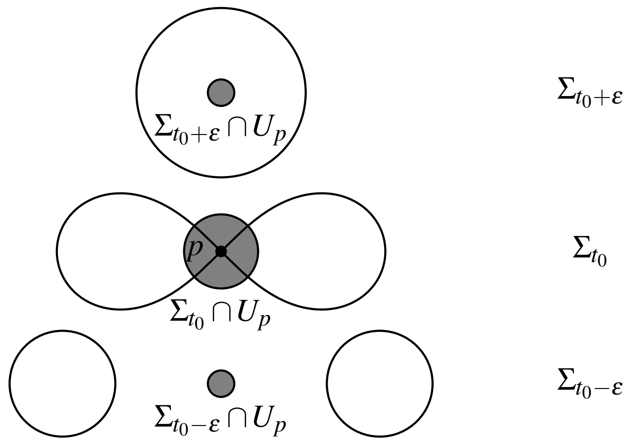

Figure 7. Topology change about the critical point $p \in \Sigma_{t_0} \cap U_p \subset X^\times$ about the moment $t_0 \in (-\infty, 0]$.

Taking $-\infty \leftarrow t$ i.e., as moving backwards in time till the creased end of $X^\times$ in Figure 6, in this process the collection $\{\Sigma_t\}_{-\infty < t \leqq K}$ of spacelike submanifolds looks like an evolution (in reversed time) from $\Sigma_K = S^3$ into a three dimensional "boiling foam" limit $\Sigma_{-\infty}$ or something like that. That is, these spacelike submanifolds unboundedly continue to switch their topology; or in other words the spatial oscillation between these states never stops and it is reasonable to expect that all closed orientable 3-manifolds occur as $-\infty \leftarrow t$. Indeed, as we noted in the Introduction, large exotic R^4's always require countably infinitely many handles in their handle decomposition therefore moving backwards in time the Σ_t's permanently continue changing their topological type. Moreover soon or later Σ_t very likely can be arbitrary since the $k = 2$ processes above are nothing but surgeries along knots and all connected, closed, orientable 3-manifolds arise this way from $S^3 = \Sigma_K$ by the Lickorish–Wallace theorem [40, 55]. This "boiling foam" picture therefore seems to be very weird and dynamical and the sole "driving force" behind this dynamics is the non-standard smooth structure along the end of $X^\times$. (Exactly the same thing is responsible for the role of these spaces in $\overline{\textbf{SCCC}}$, too.) The existence of topologically different Cauchy surfaces in $\mathbb{R}^4$ is already known to physicists, too [46].

All the things have described up to this point might seem as mere mathematical nonsense. However they get even physically interesting if we recognize that this vivid spatial topology oscillation in $(X^\times, g_L)$ appears as a cosmological redshift phenomenon to our observer moving in (13), as it looks back to the early creased end of $X^\times$ at late times. Let $E \in X^\times \setminus \overline{C}_f$ be a space-time event with a normalized future-directed timelike vector n_E where a photon is emitted; in the geometrical optics approximation this photon travels along a future-directed null geodesic γ in $(X^\times, g_L)$ till it is received in a later $R \in X^\times \setminus \overline{C}_f$ with corresponding receiver n_R. Taking any affine parameterization (i.e., $\nabla^L_{\gamma'}\gamma' = 0$) the emitted frequency measured by n_E is $\omega_E = -g_L(\gamma'_E, n_E)$ while $\omega_R = -g_L(\gamma'_R, n_R)$ is the frequency measured by the receiver. Then we define the *redshift factor* z in the standard way by the frequency ratio

$$1+z=\frac{\omega_E}{\omega_R}=\frac{g_L(\gamma'_E,n_E)}{g_L(\gamma'_R,n_R)}$$

and say that the photon is *redshifted* along γ if $z>0$. We adapt this general framework at least qualitatively to our setup as follows. Assume that the observer in the above process is given by $n=\frac{\mathrm{grad}f}{|\mathrm{grad}f|_{g_L}}$. Making use of the notation in the proof of Theorem 1.3, $\mathrm{grad}f$ is a section of $L\subset TX^\times$ hence $W_L\mathrm{grad}f=\sqrt{-1}\,\mathrm{grad}f$; moreover if $\gamma'=\gamma'_L+\gamma'_{L^\perp}$ is the unique decomposition according to $TX^\times=L\oplus L^\perp$ then $W_L\gamma'=W_L\gamma'_L+W_L\gamma'_{L^\perp}=\sqrt{-1}\gamma'_L+\gamma'_{L^\perp}\in\sqrt{-1}\,L\oplus L^\perp$. Consequently

$$g_L(\gamma'\,,n)=\frac{g_L(\gamma'\,,\mathrm{grad}f)}{|\mathrm{grad}f|_{g_L}}=\frac{g^{\mathbb{C}}(W_L\gamma'\,,W_L\mathrm{grad}f)}{|W_L\mathrm{grad}f|_{g^{\mathbb{C}}}}=\frac{-g(\gamma'_L\,,\mathrm{grad}f)}{-|\mathrm{grad}f|_g}=\frac{g(\gamma'\,,\mathrm{grad}f)}{|\mathrm{grad}f|_g}\;.$$

Moreover

$$\mathrm{d}g(\gamma'\,,\mathrm{grad}f)\gamma'=-\mathrm{d}g_L(\gamma'\,,\mathrm{grad}f)\gamma'=-g_L(\nabla^L_{\gamma'}\gamma'\,,\mathrm{grad}f)-g_L(\gamma'\,,\nabla^L_{\gamma'}\mathrm{grad}f)=-\mathrm{Hess}_f(\gamma',\gamma')$$

where $\mathrm{Hess}_f(x)=(\partial^2_{ij}f(x))_{i,j=1,\dots,4}$. Consider a non-critical point $q\in X^\times$ and its open neighbourhood $V_q\subset X^\times\setminus\overline{C}_f$ i.e. V_q surely does not contain any critical point of f. Then there exists a local coordinate system (V_q,t,x_1,x_2,x_3) centered at q i.e. $t(q)=x_i(q)=0$ such that $f|_{V_q}(t,x_1,x_2,x_3)=t$ implying $\mathrm{Hess}_f|_{V_q}=0$. Therefore $\mathrm{d}g(\gamma'\,,\mathrm{grad}f)\gamma'=\gamma'(g(\gamma'\,,\mathrm{grad}f))=0$ along V_q i.e. if the photon path γ does not intersect any critical point then $g(\gamma'\,,\mathrm{grad}f)$ is a non-zero constant along the whole γ. In this situation we end up with

$$1+z=\frac{|\mathrm{grad}f(R)|_g}{|\mathrm{grad}f(E)|_g}\;.$$

As we emphasized throughout this note, the level surfaces $f^{-1}(t)\subset X^\times$ attain critical points more and more frequently as $-\infty\leftarrow t$. Consequently, the earlier space-time event $E\in f^{-1}(t_E)$ is "more likely" to be in the vicinity of a critical point $p_E\in f^{-1}(t_E)$ satisfying $\mathrm{grad}f(p_E)=0$ than the later event $R\in f^{-1}(t_R)$ with $t_R>t_E$. Therefore, acknowledging that a more careful statistical analysis is surely required, it is reasonable that "typically" $|\mathrm{grad}f(E)|_g\approx 0$ meanwhile $|\mathrm{grad}f(R)|_g\approx 1$ implying that the gradient ratio on the right hand side of $1+z$, when calculated for the "typical" early photon emitting event $E\in X^\times\setminus\overline{C}_f$ and late photon receiving event $R\in X^\times\setminus\overline{C}_f$, is large resulting in $z>0$. By the same reasoning this ratio even seems to be capable to be unbounded hence "typically" even $z>2$ seems reasonable which is exclusively characteristical for *cosmological* (i.e., not gravitational caused by a compact body, etc.) redshift. A cosmological context here is not surprising since our solutions $(X^\times,g_L)$ are smooth while it has been known for a long time that in general relativity the gravitational field of an isolated massive object cannot be regular everywhere [16, 17].

Finally, one may raise the question about the place or role or relevance of this topology changing phenomenon within the full theory of (classical or even quantum) general relativity. Regarding this it is worth calling attention again that the Riemannian solutions $(X^\times,g)$ underlying our smooth vacuum space-times $(X^\times,g_L)$ are not only Ricci-flat but even self-dual (see Theorem 1.2 here). Consequently they are gravitational instantons and their appearance here looks reasonable for they are expected to generate these topology changes as tunnelings at the semi-classical (i.e. the leading term of quantum corrections) level. At first sight the whole picture presented here strongly resembles the structure of the vacuum sector of a non-Abelian gauge theory in temporal gauge over Minkowski space: in analogy with the present situation instantons of the Euclidean Yang–Mills theory over the Euclidean flat space execute semi-classical tunnelings between topologically (hence classically) separated classical vacua along space-like submanifolds in the original Minkowskian Yang–Mills theory over the Minkowskian flat space.

However there is a subtle difference between the two tunneling processes which is probably worth recording here. In case of Yang–Mills theory all the aforementioned topologically different states connected by (anti)instanton effects are *vacua* hence the corresponding tunneling mechanism is time-symmetric which means that both instantons and antiinstantons occur and play a role. On the contrary in our gravitational situation the family $\{\Sigma_t\}_{-\infty<t\leqq 0}$ of topologically different spatial submanifolds with their corresponding Riemannian metrics inherited from their embeddings into $(X^\times, g_L)$ and connected by instanton effects are *not flat*; rather as $t \to 0$ this family looks like a sequence descending from quite complicated, topologically non-trivial highly curved compact 3-spaces (Σ_t's with $t \ll 0$, the bottom part of Figure 6) towards topologically trivial 3-spheres carrying metrics already close to the standard round metric (Σ_t's with $t \lessapprox 0$, the top of Figure 6). Therefore, as moving *forwards* in time the whole process seems to describe a sort of monotonic decay mechanism converting the gravitational degrees of freedom into other ones (like Yang–Mills fields, fermions, etc.) before reaching the gravitational vacuum (in our spatially compact situation the standard round S^3 plays the role of the flat geometry i.e. the gravitational vacuum). This process therefore seems to be not reversible and having a creased end introduces a sort of time direction along the cosmological space-time $(X^\times, g_L)$. Consequently the gravitational instantons provided by the spaces $(X^\times, g)$ are asymmetric unlike the gravitational instanton-antiinstanton pairs considered in [58, Section III].

Are then $(X^\times, g)$'s physically relevant? Based on cluster decomposition Witten argues that a non-perturbative field is still relevant in a quantum theory if it is continuously deformable to the vacuum in an appropriate configuration space [58, Section III]. Consider the case of traditional general relativity when space-time is topologically $\mathbb{R}^m$ and in particular the vacuum is the flat $\mathbb{R}^m$. Then by this argument gravitational instantons restricted to be exotic m-spheres if $m \neq 4$. However if $m = 4$ we cannot forget about exotic $\mathbb{R}^4$'s. In this exceptional situation we can follow Gompf [28, Chapter 9.4] and consider the configuration space $\mathscr{R}_\sim$ of compact equivalence classes of smooth structures on $\mathbb{R}^4$. The set $\mathscr{R}_\sim$ can be given the structure of a connected metrizable topological space with countable basis in which therefore the vacuum i.e. the standard $\mathbb{R}^4$ is represented by a point while our gravitational instanton R^4 by another point. Consequently within $\mathscr{R}_\sim$ the gravitational instanton considered here is deformable into the vacuum. However the relevance of this purely formal observation is not clear neither from a physical nor a mathematical viewpoint.

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