A review of Yang–Mills theory over asymptotically locally flat spaces

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The rough concept of an ALF space

A connected orientable Riemannian 4-manifold (M, g) is called an asymptotically locally flat (ALF) space in the broad sense if

(i) *topologically* M decomposes as $M = K \cup W$ where K is the compact interior part and $W \cong N \times \mathbb{R}^+$ is the asymptotic part such that N is a connected closed 3-manifold admitting a fibration

$$\pi : N \xrightarrow{F} B_{+\infty}$$

where $F \cong S^1$ and $B_{+\infty}$ being a connected compact (not necessarily orientable!) surface;

(ii) *geometrically* g is a complete metric of asymptotic shape

$$g|_W \sim \mathrm{d}r^2 + r^2 g_{B_{+\infty}} + g_F \quad \text{as } r \to +\infty$$

on $W \cong N \times \mathbb{R}^+$ (with $r \in \mathbb{R}^+$) and $|R_g|_W|_g = O(r^{-3})$.

Remark

Further assumptions on the metric g (e.g. self-duality, Ricci flatness, hyper-Kählerity) also can be imposed. In these cases we can talk about an ALF gravitational instanton.

Originally these spaces come from physics:

(i) Euclidean quantum gravity;
(ii) Finite-temperature phenomena (black hole temperature, finite temperature Yang–Mills theory, etc.);
(iii) Low energy supersymmetric solutions of string theory (if they are hyper-Kähler);
(iv) S-duality tests in supersymmetric quantum Yang–Mills theory;
(v) Geometric models of matter (Atiyah–Manton–Schroers 2011, Franchetti–Manton 2013).

Mathematically they fit into the

$$\text{Compact} \to \text{ALE} \to \text{ALF} \to \text{ALG} \to \text{ALH}$$

hierarchy which is a natural relaxation of compactness.

Examples:

(i) $\mathbb{R}^3 \times S^1$ (a flat space);

(ii) multi-Taub–NUT (or A_k Gibbons–Hawking) spaces, D_k spaces (hyper-Kähler spaces);

(iii) Riemannian Schwarzschild, Kerr (Ricci flat spaces).

Yang–Mills theory is well-understood in the Compact and ALE cases. What about the ALF case? E.g. it might be used to classify ALF spaces *á la* Donaldson or for physics.

Classical Yang–Mills theory over an ALF space

Let E be an $\mathrm{SU}(2)$ complex vector bundle over M and ∇_A a smooth $\mathrm{SU}(2)$ connection with curvature F_A. Over (M, g) the gauge equivalence class $[\nabla_A]$ of ∇_A is called an $\mathrm{SU}(2)$ (anti)instanton if $*F_A = \pm F_A$ and has finite energy (or action):

$$e(\nabla_A) := \frac{1}{8\pi^2}\|F_A\|^2_{L^2(M)} = -\frac{1}{8\pi^2}\int\limits_M \mathrm{tr}(F_A \wedge *F_A) < +\infty.$$

In general $e(\nabla_A) \notin \mathbb{N}$ because M is not compact! Indeed:

(i) On the multi-Taub–NUT space: the Killing field of the metric gives a family of reducible antiinstantons with any $e \in \mathbb{R}^+$;

(ii) On the Riemannian Schwarzschild space: a deformation of the "metric instanton" gives a family of irreducible instantons such that $e \in [1, 2] \subset \mathbb{R}^+$ (Mosna–Tavares 2009).

Experienced with instanton theory over compact spaces we expect that instantons form nice moduli spaces if and only if their energies form a discrete set. Fortunately the above pathological solutions are excluded by a plausible "admissibility" condition on instantons.

Remark

Recall that $M = K \cup W$ with $W \cong N \times \mathbb{R}^+$, $r \in \mathbb{R}^+$. For $0 < R < +\infty$ we put

$$\overline{M}_R := K \cup \{x \in W \mid r(x) \leqq R\} \subsetneqq M.$$

This is a compact truncation of M.

Definition

An arbitrary finite energy $\mathrm{SU}(2)$-connection ∇_A on a rank 2 complex $\mathrm{SU}(2)$ vector bundle E over M is said to be admissible if it satisfies two conditions:

(i) The first is called the weak holonomy condition and says that to ∇_A there exist constants $0 < R < +\infty$ and $0 < c(g) < +\infty$, this latter being independent of R, and a smooth flat $\mathrm{SU}(2)$-connection $\nabla_\Gamma|_W$ on $E|_W$ along the end $W \subset M$ such that there exists a gauge on $M \setminus \overline{M}_R \subset W$ satisfying

$$\|A - \Gamma\|_{L^2_{1,\Gamma}(M\setminus\overline{M}_R)} \leqq c\|F_A\|_{L^2(M\setminus\overline{M}_R)};$$

(ii) The second condition requires ∇_A to decay rapidly at infinity:

$$\lim_{R\to+\infty} \sqrt{R}\, \|F_A\|_{L^2(M\setminus\overline{M}_R)} = 0.$$

Admissibility is a natural condition on instantons:

Theorem

Let (M, g) be an ALF manifold with $M = K \cup W$, $W \cong N \times \mathbb{R}^+$ and $\pi : N \xrightarrow{F} B_{+\infty}$. Assume that

(i) *either N is an arbitrary circle bundle over $B_{+\infty} \ncong S^2, \mathbb{R}P^2$;*

(ii) *or N is the trivial circle bundle over $B_{+\infty} \cong S^2, \mathbb{R}P^2$.*

Then the weak holonomy condition is satisfied for any finite energy $\mathrm{SU}(2)$ connection ∇_A on E over M.

Proof. Given an ALF space (M, g) let X denote its compactification by shrinking all the fibers of $\pi : N \xrightarrow{F} B_{+\infty}$ into points. Then X is a connected orientable closed 4-manifold, the Hausel–Hunsicker–Mazzeo compactification of (M, g). Note that $M = X \setminus B_{+\infty}$. Hence we can work over $X \setminus B_{+\infty}$ and refer to a codimension 2 singularity removal theorem of Sibner–Sibner (1992) and Råde (1994). $\diamond$

Remark

(i) Consequently the weak holonomy condition part of admissibility is in fact just a mild topological condition. This condition rules out the multi-Taub–NUT-type continuous energy antiinstantons;

(ii) But the rapid decay condition part of admissibility is indeed a non-trivial analytical assumption. This condition rules out the Riemannian Schwarzschild-type continuous energy instantons.

Remark

If $\nabla_\Gamma|_W = \mathrm{d} + \Gamma$ is an asymptotic flat connection on $E|_W$ then write $\Gamma_{+\infty} := \lim_{r\to+\infty} \Gamma|_{\partial\overline{M}_r}$ for the limit of the restricted connection.

Admissible instantons have discrete energy spectrum:

Theorem

Let (M, g) be an ALF space with an end $W \cong N \times \mathbb{R}^+$. Let E be an $\mathrm{SU}(2)$ vector bundle over M with an admissible self-dual connection ∇_A. Then

$$e(\nabla_A) \equiv \tau_N(\Gamma_{+\infty}) \mod \mathbb{Z}$$

that is, its energy is congruent to a Chern–Simons invariant of the boundary given by the flat connection $\nabla_\Gamma|_W$ on $E|_W$ in the weak holonomy condition part of the Definition. $\diamond$

An even stronger result holds as follows.

Let (M,g) be an ALF space with $M = K \cup W$, $W \cong N \times \mathbb{R}^+$ and $\pi : N \xrightarrow{F} B_{+\infty}$. Fix a contractible open subset $U \subset B_{+\infty}$. Then for

$$U_R^\times := \pi^{-1}(U)\times(R,+\infty) \cong U\times S^1\times(R,+\infty) \subset N\times\mathbb{R}^+ \cong W \subset M$$

we find $U_R^\times \cong B^2 \times (B^2)^\times$ consequently $\pi_1(U_R^\times) \cong \mathbb{Z}$. It is generated by a fiber $F \cong S^1$. Let $\tau \in [0,2\pi)$ be the corresponding cyclic coordinate on $U_R^\times$. Then any $\nabla_\Gamma|_W$ on $E|_W$ locally can be gauge transformed into the shape $\nabla_\Gamma|_{U_R^\times} = \mathrm{d} + \Gamma_m$ where

$$\Gamma_m = \begin{pmatrix} \mathbf{i}m & 0 \\ 0 & -\mathbf{i}m \end{pmatrix} \mathrm{d}\tau, \quad m \in [0,1).$$

Let ∇_A be an admissible $\mathrm{SU}(2)$ connection and $\nabla_\Gamma|_W$ be its associated asymptotic flat connection. The real number $m \in [0,1)$ is called the local holonomy of ∇_A at infinity.

In fact admissible instantons have integer energy spectrum and vanishing local holonomy at infinity:

Theorem

Let (M, g) be an ALF space with an end $W \cong N \times \mathbb{R}^+$. Let E be an $\mathrm{SU}(2)$ vector bundle over M with an admissible self-dual connection ∇_A on it. Then

$$e(\nabla_A) \in \mathbb{N}$$

that is, compared to the previous Theorem its energy is always integer.
Regarding the asymptotical shape of ∇_A if M is in addition simply connected then the associated flat connection $\nabla_\Gamma|_W$ in the Definition has trivial local holonomy at infinity i.e., $m = 0$ (in this case if $\nabla_\Gamma|_W$ is not the trivial flat connection then $\pi_1(B_{+\infty}) \neq 1$).
$\diamond$

Admissible instantons also form usual moduli spaces:

Theorem

Let (M, g) be an ALF space with an end $W \cong N \times \mathbb{R}^+$ as before. Assume furthermore that $\pi_1(M) \cong 1$ and g is Ricci flat. Consider a rank 2 complex $\mathrm{SU}(2)$ vector bundle E over M and denote by $\mathscr{M}(e, \Gamma)$ the framed moduli space of irreducible admissible $\mathrm{SU}(2)$ instantons on E with a fixed energy $e < +\infty$ and asymptotic flat connection $\nabla_\Gamma|_W$ on $E|_W$.
Then $\mathscr{M}(e, \Gamma)$ is either empty or a manifold of dimension

$$\dim \mathscr{M}(e, \Gamma) = 8e - 3b^-(X)$$

where X is the Hausel–Hunsicker–Mazzeo compactification of M with induced orientation.

Proof. The proof is based on a Gromov–Lawson relative index theorem for the pair $(X, M = X \setminus B_{+\infty})$. $\diamond$

Examples of moduli spaces:

1. The multi-Taub–NUT space (M_V, g_V) with $s > 0$ NUTs and orientation coming from the hyper-Kähler family. For its Hausel–Hunsicker–Mazzeo compactification one finds

$$X \cong \underbrace{\overline{\mathbb{C}P^2}\#\dots\#\overline{\mathbb{C}P^2}}_{s}.$$

Therefore the unframed moduli space of antiinstantons with unit energy is 5 dimensional and admits the following description. An antiinstanton $[\nabla_A]$ is described by the pair $(x, \lambda) \in M_V \times (0, +\infty]$ (λ is the "concentration parameter"). The moduli space of antiinstantons with $\lambda < +\infty$ forms a collar of M_V as in the compact case. It can be constructed via the conformal rescaling method as for S^4 for instance.

The global picture looks like this:

Theorem

Consider the multi-Taub–NUT space (M_V, g_V) *with* $s > 0$ *NUTs* $p_1, \dots, p_s \in M_V$*, equipped with the natural orientation by any complex structure. Then for (one connected component of) the unframed moduli space of unit energy* $\mathrm{SU}(2)^+$ *admissible anti-instantons decaying rapidly to the trivial flat connection* ∇_Θ *on* $E = \Sigma^+$ *(the positive chiral spinor bundle) we find*

$$\widehat{\mathscr{M}}(1, \Theta) \cong (M_V \times (0, +\infty])/\sim$$

where the equivalence relation $\sim$ *means that* $M_V \times \{+\infty\}$ *is pinched into* $\mathbb{R}^3$ *by collapsing the* S^1*-isometry orbits of* (M_V, g_V)*.*

(continued...)

(...continued)

Consequently there exists a singular fibration

$$\Phi : \widehat{\mathscr{M}}(1, \Theta) \longrightarrow \mathbb{R}^3$$

with generic fibers homeomorphic to the open 2-ball B^2 *and as many as* s *singular fibers homeomorphic to the semi-open 1-ball* $(0, +\infty]$*. Therefore (one connected component of) the moduli space is contractible and in particular is orientable.*
The images of the points $(p_i, +\infty)$ *in* $\widehat{\mathscr{M}}(1, \Theta)$ *with* $i = 1, 2, \dots, s$ *represent reducible antiinstantons and* $\widehat{\mathscr{M}}(1, \Theta)$ *around these points looks like a cone over* $\mathbb{C}P^2$ *or* $\overline{\mathbb{C}P^2}$ *(depending on the orientation).*

Proof. It is based on the algebraic geometry of the twistor space of the multi-Taub–NUT geometry. ◇

Explicit unit energy antiinstantons on the 1-Taub–NUT space:

For (M_V, g_V) we take $M_V := \mathbb{R}^4$ with $p_1 = 0 \in \mathbb{R}^4$ and g_V looks like

$$\mathrm{d}s^2 = \frac{r+m}{r-m}\mathrm{d}r^2 + (r^2 - m^2)\left(\sigma_x^2 + \sigma_y^2 + \left(\frac{2m}{r+m}\right)^2 \sigma_z^2\right).$$

With $\lambda \in (0, +\infty]$ the $(0, \lambda)$-antiinstanton $\nabla_{A_\lambda} = \mathrm{d} + A_\lambda$ looks like

$$A_\lambda = -\frac{\mathbf{i}}{2}\Psi\sigma_x + \frac{\mathbf{j}}{2}\Psi\sigma_y + \frac{\mathbf{k}}{2}\Phi\sigma_z$$

where we have introduced the notations

$$\Phi(r) := 1 - \frac{2m\lambda}{(r-m+\lambda)(r+m)}, \quad \Psi(r) := 1 - \frac{\lambda}{r-m+\lambda}.$$

The $\lambda = +\infty$ case is the aforementioned reducible solution:

$$A_{+\infty} = \frac{\mathbf{k}}{2}\frac{r-m}{r+m}\sigma_z.$$

2. The Riemannian Schwarzschild space (M, g) with any orientation. For its Hausel–Hunsicker–Mazzeo compactification one finds

$$X \cong S^2 \times S^2.$$

Therefore the unframed moduli space of instantons with unit energy is 2 dimensional. Consequently the classical "metric instanton" (Charap–Duff 1977) admits a 2-parameter deformation!

An explicit unit energy instanton on the Riemannian Schwarzschild space (Charap–Duff 1977):
For (M, g) we take $M \cong S^2 \times \mathbb{R}^2$ and g the Wick rotated Schwarzschild metric:

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\Theta^2 + \sin^2\Theta\mathrm{d}\phi^2).$$

Then in the convenient gauge we find $\nabla_A = \mathrm{d} + A$ with

$$A = \frac{1}{2}\sqrt{1 - \frac{2m}{r}}\mathrm{d}\Theta\mathbf{i} + \frac{1}{2}\sqrt{1 - \frac{2m}{r}}\sin\Theta\mathrm{d}\phi\mathbf{j} + \frac{1}{2}\left(\cos\Theta\mathrm{d}\phi - \frac{m}{r^2}\mathrm{d}\tau\right)\mathbf{k}.$$

What is its predicted energy-preserving 2-parameter deformation?
But ∇_A also admits another deformation giving the aforementioned family with energy in $[1, 2] \subset \mathbb{R}^+$ (Mosna–Tavares 2009).

3. What about instantons over the D_k ALF spaces? What is X in this case?

On the quantum Yang–Mills theory

Fix an ALF space (M, g). Assume that M is simply connected and if $M = X \setminus B_{+\infty}$ then $B_{+\infty}$ is orientable. Introducing a (supersymmetrized, twisted, etc.) $\mathrm{SU}(2)$ Yang–Mills theory over (M, g) with a θ-term

$$-\frac{\theta}{16\pi^2}\int\limits_M \mathrm{tr}(F_A \wedge F_A)$$

we can ask about the partition function $Z(M, g, \mathrm{SU}(2), \tau) \in \mathbb{C}$ of the underlying quantum gauge theory. Here $\tau = \frac{\theta}{2\pi} + \mathbf{i}\frac{4\pi}{e^2} \in \mathbb{C}^+$ is the complex coupling constant. This Z is obtained from a formal Feynman integral over the (gauge equivalence classes of) finite energy (i.e., finite action) connections. We may try to calculate Z over a bit more restricted affine space

$$\mathscr{A}(E) := \{\nabla_A \mid \nabla_A \text{ is admissible on } E \text{ in the sense of the Definition}\}.$$

Then from our considerations so far it follows that

(i) Any $\nabla_A \in \mathscr{A}(E)$ extends to $X = M \cup B_{+\infty}$;

(ii) If also ∇_A denotes this extension then $\nabla_A|_{B_{+\infty}}$ is a flat connection on the surface $B_{+\infty} \subset X$.

By (i) we are tempted to write

$$Z(M, g, \mathrm{SU}(2), \tau) = Z(X, \mathrm{SU}(2), \tau)$$

and calculate $Z(X, \mathrm{SU}(2), \tau)$ within the framework of axiomatic TQFT as follows. Recall that X is a smooth connected oriented closed 4-manifold hence we can regard Z as a linear map

$$Z(X, \mathrm{SU}(2), \tau) : \mathscr{H}_{-\infty}(\emptyset) \longrightarrow \mathscr{H}_{+\infty}(\emptyset)$$

where $\mathscr{H}_{\pm\infty}(\emptyset) \cong \mathbb{C}$ are the Hilbert spaces attached to the past and future boundaries of the closed space X now considered as a cobordism between two emptysets.

For a fixed $0 < R < +\infty$ let $B_{+\infty} \subset V_R \subset X$ be a tubular neighbourhood of $B_{+\infty} \subset X$. Assume that X is cut up as follows:

$$X = \overline{M}_R \cup_{\partial \overline{M}_R} V_R.$$

Let $\mathscr{H}_R(N)$ be the Hilbert space attached to $\partial \overline{M}_R \cong N \times \{R\}$. By usual axioms we expect to find $v_R \in \mathscr{H}_R(N)$, $w_R \in \mathscr{H}_R(N)^*$ such that

$$Z(X, \mathrm{SU}(2), \tau) = (v_R, w_R)$$

and $Z(X, \mathrm{SU}(2), \tau)$ to be independent of R. Letting $R \to +\infty$ we formally obtain

$$Z(X, \mathrm{SU}(2), \tau) = (v_{+\infty}, w_{+\infty})$$

where $v_{+\infty} \in \mathscr{H}_{+\infty}(N) \cong \mathscr{H}(B_{+\infty})$ and $w_{+\infty} \in \mathscr{H}(B_{+\infty})^*$ since when $R \to +\infty$ the space $\partial \overline{M}_R \cong N \times \{R\}$ cuts down to $B_{+\infty}$.

What sort of space is $\mathscr{H}(B_{+\infty})$ here?
By (ii) we know that admissible connections on $M \subset X$ decay to flat connections on $B_{+\infty} \subset X$ hence by the principles of geometric quantization we expect that

$$\mathscr{H}(B_{+\infty}) = \bigoplus_{k \in \mathbb{N}} H^0\left(\mathscr{M}_{B_{+\infty}}; \mathscr{O}(L^k)\right)$$

where $\mathscr{M}_{B_{+\infty}}$ is the moduli space of flat $\mathrm{SU}(2)$ connections on $B_{+\infty}$ and L is the usual holomorphic quantizing line bundle over $\mathscr{M}_{B_{+\infty}}$, $k = e \in \mathbb{N}$ is the energy of admissible instantons.

We use holomorphic polarization to make sense of $H^0\left(\mathscr{M}_{B_{+\infty}}; \mathscr{O}(L^k)\right)$ hence need a complex structure on $\mathscr{M}_{B_{+\infty}}$ coming from a complex structure on $B_{+\infty}$.

But: the whole construction must be independent of the particular complex structure on $B_{+\infty}$. Hence in fact we obtain a holomorphic vector bundle

$$P : \mathscr{E}_{g,n,k} \xrightarrow{H^0\left(\mathscr{M}_{B_{+\infty}};\mathscr{O}(L^k)\right)} \mathfrak{M}_{g,n}$$

over the moduli space of complex structures $\mathfrak{M}_{g,n}$ on $B_{+\infty} \setminus \{p_1, \dots, p_n\}$. (Here g is the genus of $B_{+\infty}$ and was $n = 0$ so far but the modification for $n > 0$ is obvious.) Moreover there is a (projectively) flat connection (Kniznik–Zamolodchikov connection) on this bundle which identifies the fibers. Consequently *á la* Segal for all $k \in \mathbb{N}$ the spaces $H^0\left(\mathscr{M}_{B_{+\infty}}; \mathscr{O}(L^k)\right)$ are conformal blocks of some conformal field theory at level k attached to $B_{+\infty} \setminus \{p_1, \dots, p_n\}$!

In summary given an $\mathrm{SU}(2)$ quantum gauge theory over an ALF space (M, g) with $M \cup B_{+\infty} = X$ then we would like to have:

$$Z(M, g, \mathrm{SU}(2), \tau) = (v_{+\infty}, w_{+\infty})$$

where $v_{+\infty}$, $w_{+\infty}$ are correlation functions of some CFT on $B_{+\infty} \setminus \{p_1, \dots, p_n\}$. Note that in this description the corresponding mapping class group acts on Z.

A benefit of this description: an S-duality test for Yang–Mills theory in the ALF scenario. If $B_{+\infty} \cong T^2$ then the mapping class group is $\mathrm{SL}(2, \mathbb{Z})$.

For further details please check:

`http://www.math.bme.hu/~etesi/publ.html`

Thank you!