Classification of 't Hooft instantons over multi-centered gravitational instantons

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Abstract

This work presents a classification of all smooth 't Hooft–Jackiw–Nohl–Rebbi instantons over Gibbons–Hawking spaces. That is, we find all smooth $SU(2)$ Yang–Mills instantons over these spaces which arise by conformal rescalings of the metric with suitable functions.

Since the Gibbons–Hawking spaces are hyper-Kähler gravitational instantons, the rescaling functions must be positive harmonic. By using twistor methods we present integral formulae for the kernel of the Laplacian associated to these spaces. These integrals are generalizations of the classical Whittaker–Watson formula. By the aid of these we prove that all 't Hooft instantons have already been found in a recent paper [10].

This result also shows that actually all such smooth 't Hooft–Jackiw–Nohl–Rebbi instantons describe singular magnetic monopoles on the flat three-space with zero magnetic charge moreover the reducible ones generate the full L^2 cohomolgy of the Gibbons–Hawking spaces.

1 Introduction

The simplest non-flat geometries in four dimensions are provided by non-compact hyper-Kähler spaces. These spaces also appear naturally in recent investigations in high energy physics. For example, physicists call these spaces as "gravitational instantons" because hyper-Kähler metrics are Ricci flat hence solve Einstein's equatios and have self-dual curvature tensor [13].

Moreover, on the one hand, motivated by Sen's S-duality conjecture [27] from 1994, recently there have been some interest in understanding the L^2 cohomology of certain moduli spaces of magnetic monopoles carrying natural hyper-Kähler metrics. Probably the strongest evidence

in Sen's paper for his conjecture was an explicit construction of an L^2 harmonic 2-form on the universal double cover of the Atiyah–Hitchin manifold. Sen's conjecture also predicted an L^2 harmonic 2-form on the Euclidean Taub–NUT space. This was found later by Gibbons in 1996 [12]. He constructed it as the exterior derivative of the dual of the Killing field of a canonical $U(1)$ action. In a joint paper with T. Hausel we imitated Gibbons' construction yielding a classification of L^2 harmonic forms on the Euclidean Schwarzschild manifold [8]. Recently Hausel, Hunsicker and Mazzeo classified the L^2 harmonic forms over various spaces with prescribed infinity, including gravitational instantons [14].

On the other hand, it is natural to ask wether or not the ADHM construction, originally designed for finding Yang–Mills instantons over flat $\mathbb{R}^4$, is extendible for hyper-Kähler spaces. Kronheimer provided an *A-D-E*-type classification of the so-called ALE (Asymptotically Locally Euclidean) hyper-Kähler spaces in 1989 [20]. Later Cherkis and Kapustin went on and constructed *A-D* sequences of so-called ALF (Asymptotically Locally Flat) and five D_4 type ALG (this shortening is by induction) gravitational instantons [5][6]. These are also non-compact hyper-Kähler spaces with different asymptotical behaviour. The question naturally arises what about the ADHM construction over these new gravitational instantons.

Kronheimer and Nakajima succeeded to extend the ADHM construction to ALE gravitational instantons yielding a classification of Yang–Mills instantons over these spaces in 1990 [21]. In the above scheme the Gibbons–Hawking gravitational instantons appear as the A_k ALE and A_k ALF spaces and are distinguished because the metrics are known explicitly on them. In our papers [9][10] we could construct $SU(2)$ Yang–Mills instantons over these spaces via the conformal rescaling method of Jackiw–Nohl–Rebbi [18]. Moreover it turned out that finding the reducible instantons among our solutions we could describe all the L^2 harmonic forms on them in a natural geometric way yielding a kind of unification of the two above, apparently independent problems.

In this paper, as a completion of these investigations we prove that we have already found all instantons which can be constructed by the conformal rescaling method [10]. The proof is based on a standard twistor theoretic construction of the kernel of the Laplacian.

The paper is organized as follows. In Sec. 2 we recall the 't Hooft–Jackiw–Nohl–Rebbi construction in the form of the Atiyah–Hitchin–Singer theorem [1]. This enables us to conclude that the scaling functions we are seeking must be harmonic with respect to the Gibbons–Hawking geometries.

In Sec. 3 we construct all solutions of the Laplace equation over $\mathbb{C}^2 \setminus \{0\}/\mathbb{Z}_N$ via a classical harmonic expansion. This is possible because the metric collapsed to this space possesses a large $U(2)$ isometry. From here we can see that only those harmonic functions give rise to everywhere non-singular instantons which are positive everywhere with at most pointlike singularities.

In Sec. 4 we perturb the above singular solutions into regular ones by constructing integral formulae for harmonic functions in the non-singular case based on twistor theory. These integral formulae are generalizations of the classical Whittaker–Watson formula over flat $\mathbb{R}^3$ [30].

In light of these investigations we can conclude that all 't Hooft instantons have in fact been found in our earlier paper [10]. Since all these functions are invariant under the natural $U(1)$ action on the Gibbons–Hawking spaces the resulting smooth Yang–Mills instantons describe singular magnetic monopoles on the flat $\mathbb{R}^3$, as it was pointed out by Kronheimer in his diploma thesis [19]. We also can see that the reducible solutions generate the full L^2 cohomology of the spaces in question. All these theorems are collected in Sec. 5.

Finally, we take conclusion and outlook in Sec. 6 and argue that our methods work for the above mentioned "exotic" gravitational instantons, too.

2   The Atiyah–Hitchin–Singer theorem

First we recall the general theory from [1]. Let (M, g) be a four-dimensional Riemannian spin-manifold. Remember that via $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$ we have a Lie algebra isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}(2)^+ \oplus \mathfrak{su}(2)^-$. Consider the Levi–Civitá connection ∇ which is locally represented in a fixed gauge by an $\mathfrak{so}(4)$-valued 1-form ω on TU for $U \subset M$. Because M is spin and four-dimensional, we can consistently lift this connection to the spin connection ∇_S, locally given by ω_S, on the spin bundle SM (which is a complex bundle of rank four) and can project it to the $\mathfrak{su}(2)^\pm$ components denoted by $\nabla^\pm$. The projected connections loacally are given by $\mathfrak{su}(2)^\pm$-valued 1-forms $A^\pm$ and live on the chiral spinor bundles $S^\pm M$ where the decomposition $SM = S^+ M \oplus S^- M$ corresponds to the above splitting of $\mathrm{Spin}(4)$. One can raise the question what are the conditions on the metric g for either ∇^+ or ∇^- to be self-dual (seeking antiself-dual solutions is only a matter of reversing the orientation of M).

Consider the curvature 2-form $R \in C^\infty(\Lambda^2 M \otimes \mathfrak{so}(4))$ of the metric. There is a standard linear isomorphism $\mathfrak{so}(4) \cong \Lambda^2 \mathbb{R}^4$ given by $A \mapsto \alpha$ with $xAy = \alpha(x, y)$ for all $x, y \in \mathbb{R}^4$. Therefore we may regard R as a *2-form-valued 2-form* in $C^\infty(\Lambda^2 M \otimes \Lambda^2 M)$ i.e. for vector fields X, Y over M we have $R(X, Y) \in C^\infty(\Lambda^2 M)$. Since the space of four dimensional curvature tensors, acted on by $SO(4)$, is 20 dimensional, one gets a 20 dimensional reducible representation of $SO(4)$ (and of $\mathrm{Spin}(4)$, being M spin). The decomposition into irreducible components is (see [3], pp. 45-52)

$$R = \frac{1}{12}\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} + \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} + \begin{pmatrix} W^+ & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & W^- \end{pmatrix}, \tag{1}$$

where s is the scalar curvature, B is the traceless Ricci tensor, $W^\pm$ are the Weyl tensors. The splitting of the Weyl tensor is a special four-dimensional phenomenon and is related with the above splitting of the Lie algebra $\mathfrak{so}(4)$. There are two Hodge operations which can operate on R. One (denoted by $*$) acts on the *2-form part of R* while the other one (denoted by $\star$) acts on the *values* of R (which are also 2-forms). In a local coordinate system, these actions are given by

$$(\star R)_{ijkl} = \frac{1}{2}\sqrt{\det g}\, \varepsilon_{ijmn} R^{mn}{}_{kl},$$

$$(*R)_{ijkl} = \frac{1}{2}\sqrt{\det g}\, R_{ij}{}^{mn}\varepsilon_{mnkl}.$$

It is not difficult to see that the projections $p^\pm : \mathfrak{so}(4) \to \mathfrak{su}(2)^\pm$ are given by $R \mapsto F^\pm := \frac{1}{2}(1 \pm \star R)$, and $F^\pm$ are self-dual with respect to g if and only if $*(1 \pm \star R) = (1 \pm \star R)$. Using the previous representation for the decomposition of R suppose $\star$ acts on the left while $*$ on the right, both of them via

$$\begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix}.$$

In this case the previous self-duality condition looks like ($\overline{W}^\pm := W^\pm + \frac{1}{12}s$)

$$\begin{pmatrix} \overline{W}^+ \pm \overline{W}^+ & -(B \pm B) \\ B^T \mp B^T & -\left(\overline{W}^- \mp \overline{W}^-\right) \end{pmatrix} = \begin{pmatrix} \overline{W}^+ \pm \overline{W}^+ & B \pm B \\ B^T \mp B^T & \overline{W}^- \mp \overline{W}^- \end{pmatrix}.$$

From here we can immediately conclude that F^+ is self-dual if and only if $B = 0$ i.e. g is *Einstein* while F^- is self-dual if and only if $\overline{W}^- = 0$ i.e. g is *half-conformally flat (i.e. self-dual) with vanishing scalar curvature*. Hence we have proved [1]:

Theorem 2.1 (Atiyah–Hitchin–Singer) *Let (M, g) be a four-dimensional Riemannian spin manifold. Then*
(i) F^+ is the curvature of an self-dual $SU(2)$-connection on S^+M if and only if g is Einstein, or
(ii) F^- is the curvature of a self-dual $SU(2)$-connection on S^-M if and only if g is half conformally flat (i.e. self-dual) with vanishing scalar curvature. $\diamondsuit$

Remember that both the (anti)self-duality equations

$$*F = \pm F$$

and the action

$$\|F\|^2 = \frac{1}{8\pi^2} \int_M |F|_g^2 = -\frac{1}{8\pi^2} \int_M \mathrm{tr}(F \wedge *F)$$

are conformally invariant in four dimensions; consequently if we can rescale g with a suitable function f producing a metric $\tilde{g}$ which satisfies one of the properties of the previous theorem then we can construct instantons over the original manifold (M, g). This idea was used by Jackiw, Nohl and Rebbi to construct instantons over the flat $\mathbb{R}^4$ [18]. Consequently we find it convenient to make the following

Definition 2.1 *Let (M, g) be a four dimensional Riemannian spin manifold and $\nabla^\pm$ be a smooth self-dual $SU(2)$ connection of finite action on the chiral spinor bundle $S^\pm M$. If there is a smooth function $f : M \to \mathbb{R}$ such that the projected Levi–Cívitá connection of $f^2 g$ is gauge equivalent to $\nabla^\pm$ then this connection is called an 't Hooft instanton over (M, g).*

First consider the case of F^+, i.e. part (i) of the above theorem [9]. Let (M, g) be a Riemannian manifold of dimension $n > 2$. Remember that $\psi : M \to M$ is a *conformal isometry* of (M, g) if there is a function $f : M \to \mathbb{R}$ such that $\psi^* g = f^2 g$. Notice that being ψ a diffeomorphism, f cannot be zero anywhere i.e. we may assume that it is positive, $f > 0$. Ordinary isometries are the special cases with $f = 1$. The vector field X on M, induced by the conformal isometry, is called a *conformal Killing field*. It satisfies the *conformal Killing equation* ([29], pp. 443-444)

$$L_X g - \frac{2\mathrm{div}X}{n} g = 0$$

where L is the Lie derivative while div is the divergence of a vector field. If $\xi = \langle X, \cdot \rangle$ denotes the dual 1-form to X with respect to the metric, then consider the following *conformal Killing data*:

$$(\xi, \quad \mathrm{d}\xi, \quad \mathrm{div}X, \quad \mathrm{ddiv}X). \tag{2}$$

These satisfy the following equations (see [11]):

$$\nabla \xi = (1/2)\mathrm{d}\xi + (1/n)(\mathrm{div}X)g,$$

$$\nabla(\mathrm{d}\xi) = (1/n)(g \otimes \mathrm{ddiv}X - (\mathrm{ddiv}X) \otimes g) + 2R(\,\cdot\,,\,\cdot\,,\,\cdot\,,\xi),$$

$$\nabla(\mathrm{div}X) = \mathrm{ddiv}X, \tag{3}$$

$$\nabla(\mathrm{ddiv}X) = -(n/2)\nabla_X P - (\mathrm{div}X)P - (n/2)Q.$$

Here R is understood as the $(3,1)$-curvature tensor while

$$P = r - \frac{s}{(n-1)(n-2)}g, \quad Q = \operatorname{tr}(P \otimes \mathrm{d}\xi + \mathrm{d}\xi \otimes P)$$

with r being the Ricci-tensor. For clarity we remark that $Q_{ij} = (P_i^k \mathrm{d}\xi_{jk} + P_j^k \mathrm{d}\xi_{ik})$ i.e., tr is the only non-trivial contraction. If γ is a smooth curve in M then fixing conformal Killing data in a point $p = \gamma(t)$ we can integrate (3) to get all the values of X along γ. Actually if X is a conformal Killing field then by fixing the above data in one point $p \in M$ we can determine the values of X over the *whole* M, provided that M is connected. Consequently, if these data vanish in one point, then X vanishes over all the M.

Furthermore a Riemannian manifold (M, g) is called *irreducible* if the holonomy group, induced by the metric, acts irreducibly on each tangent space of M.

Now we can state [9]:

Proposition 2.2 *Let (M, g) be a connected, irreducible, Ricci-flat Riemannian manifold of dimension $n > 2$. Then $(M, \tilde{g})$ with $\tilde{g} = \varphi^{-2}g$ is Einstein if and only if φ is a non-zero constant function on M.*

Remark. Notice that the above proposition is not true for reducible manifolds: the already mentioned Jackiw–Nohl–Rebbi construction [18] provides us with non-trivial Einstein metrics, conformally equivalent to the flat $\mathbb{R}^4$.

Proof. If η is a 1-form on (M, g) with dual vector $Y = \langle \eta, \cdot \rangle$, then the $(0, 2)$-tensor $\nabla\eta$ can be decomposed into antisymmetric, trace and traceless symmetric parts respectively as follows (e.g. [24], p. 200, Ex. 5.):

$$\nabla\eta = \frac{1}{2}\mathrm{d}\eta - \frac{\delta\eta}{n}g + \frac{1}{2}\left(L_Y g + \frac{2\delta\eta}{n}g\right) \tag{4}$$

where δ is the exterior codifferentiation on (M, g) satisfying $\delta\eta = -\operatorname{div}Y$. Being g an Einstein metric, it has identically zero traceless Ricci tensor i.e. $B = 0$ from decomposition (1). We rescale g with the function $\varphi : M \to \mathbb{R}^+$ as $\tilde{g} := \varphi^{-2}g$. Then the traceless Ricci part of the new curvature is (see [3], p. 59)

$$\widetilde{B} = \frac{n-2}{\varphi}\left(\nabla^2\varphi + \frac{\triangle\varphi}{n}g\right).$$

Here $\triangle$ denotes the Laplacian with respect to g. From here we can see that if $n > 2$, the condition for $\tilde{g}$ to be again Einstein is

$$\nabla^2\varphi + \frac{\triangle\varphi}{n}g = 0.$$

However, if $X := \langle \mathrm{d}\varphi, \cdot \rangle$ is the dual vector field then we can write by (4) that

$$\nabla^2\varphi = \frac{1}{2}\mathrm{d}^2\varphi - \frac{\delta(\mathrm{d}\varphi)}{n}g + \frac{1}{2}\left(L_X g + \frac{2\delta(\mathrm{d}\varphi)}{n}g\right) = -\frac{\triangle\varphi}{n}g + \frac{1}{2}\left(L_X g + \frac{2\triangle\varphi}{n}g\right).$$

We have used $\mathrm{d}^2 = 0$ and $\delta\mathrm{d} = \triangle$ for functions. Therefore we can conclude that $\varphi^{-2}g$ is Einstein if and only if

$$L_X g + \frac{2\triangle\varphi}{n}g = L_X g - \frac{2\operatorname{div}X}{n}g = 0$$

i.e. X is a conformal Killing field on (M, g) obeying $X = \langle \mathrm{d}\varphi, \cdot \rangle$. The conformal Killing data (2) for this X are the following:

$$(\mathrm{d}\varphi, \quad \mathrm{d}^2\varphi = 0, \quad -\triangle\varphi, \quad -\mathrm{d}(\triangle\varphi)). \tag{5}$$

Now we may argue as follows: the last equation of (3) implies that

$$\nabla(\mathrm{d}(\triangle\varphi)) = 0$$

over the Ricci-flat (M, g). By virtue of the irreducibility of (M, g) this means that actually $\mathrm{d}(\triangle\varphi) = 0$ (cf. e.g. [3], p. 282, Th. 10.19) and hence $\triangle\varphi =$const. over the whole (M, g). Consequently, the second equation of (3) shows that for all Y, Z, V we have

$$R(Y, Z, V, \mathrm{d}\varphi) = 0.$$

Taking into account again that (M, g) is irreducible, there is a point where R_p is non-zero. Assume that the previous equality holds for all Y_p, Z_p, V_p but $\mathrm{d}\varphi_p \neq 0$. This is possible only if a subspace, spanned by $\mathrm{d}\varphi_p$ in T_p^*M, is invariant under the action of the holonomy group. But this contradicts the irreducibility assumption. Consequently $\mathrm{d}\varphi_p = 0$. Finally, the first equation of (3) yields that $\triangle\varphi(p) = 0$ i.e. $\triangle\varphi = 0$. Therefore we can conclude that in that point all the conformal data (5) vanish implying $X = 0$. In other words φ is a non-zero constant. $\diamond$

In light of this proposition, general Ricci-flat manifolds cannot be rescaled into Einstein manifolds in a non-trivial way. Notice that the Gibbons–Hawking spaces (see below) are irreducible Ricci-flat manifolds. If this was not the case, then, taking into account their simply connectedness and geodesic completeness, they would split into a Riemannian product $(M_1 \times M_2, g_1 \times g_2)$ by virtue of the de Rham theorem [26]. But it is easily checked that this is not the case. We just remark that the same is true for the Euclidean Schwarzschild manifold.

Consequently constructing instantons in this way is not very productive.

3 Construction of singular Laplace operators

Therefore we turn our attention to the condition on the F^- part of the metric curvature in the special case of the Gibbons–Hawking spaces.

First we give a brief description of the Gibbons–Hawking spaces denoted by M_V. These spaces can be understood topologically as follows (cf., e.g. [15]). Take an $N \in \mathbb{N}^+$ and consider the cyclic group $\mathbb{Z}_N \subset SU(2)$ acting on $\mathbb{C}^2$ induced by the $SU(2)$ action. The resulting quotient $\mathbb{C}^2/\mathbb{Z}_N$ is singular at the origin and looks like $\mathbb{R} \times (S^3/\mathbb{Z}_N)$ at infinity. If $(v, w) \in \mathbb{C}^2$ then the monomials v^N, w^N, vw are invariant under $\mathbb{Z}_N$. If we denote them by x, y, z then they satisfy an algebraic equation

$$xy - z^N = 0.$$

This gives an isomorphism between $\mathbb{C}^2/\mathbb{Z}_N$ and the complex surface $xy - z^N = 0$ in $\mathbb{C}^3$. We can look at deformations of this singularity. As it is well known, this surface is singular because the monomial z^N contains multiple roots i.e., its discriminant is zero. This can be removed by adding lower order terms in z such that the resulting polynomial in z is of nonzero discriminant ("blowing up"):

$$xy - (z^N + a_1 z^{N-1} + \ldots + a_N) = 0. \tag{6}$$

These complex surfaces still have the required topology at infinity but are nonsingular. We shall take M_V to be the underlying four-dimensional differentiable manifold of such a surface.

One also can construct M_V intuitively. Start with $\mathbb{R}^3 \times S^1$ acted on by $U(1)$ along the circles. Consider N distinct points $p_1, \ldots, p_N$ along the z axis of $\mathbb{R}^3$ for example and shrink the S^1 circles over them. The resulting space is $M_V = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}_N \cup E$ with E the exceptional divisor consisting of $N - 1$ copies of $\mathbb{C}P^1$'s, attached to each other according to the A_{N-1} Dynkin diagram. From here we can also see that there is a circle action on M_V with N fixed points $p_1, \ldots, p_N \in M_V$, called NUTs. The quotient is $\mathbb{R}^3$ and we denote the images of the fixed points also by $p_1, \ldots, p_N \in \mathbb{R}^3$. Then $U_V := M_V \setminus \{p_1, \ldots, p_N\}$ is fibered over $\mathbb{R}^3 \setminus \{p_1, \ldots, p_N\}$ with S^1 fibers. The degree of this circle bundle around each point p_i is one.

The metric g_V on U_V looks like (cf. e.g. [15] or p. 363 of [7])

$$ds^2 = V(dx^2 + dy^2 + dz^2) + \frac{1}{V}(d\tau + \alpha)^2, \tag{7}$$

where $\tau \in (0, 8\pi m]$ parametrizes the circles and $x = (x, y, z) \in \mathbb{R}^3$; the smooth function $V : \mathbb{R}^3 \setminus \{p_1, \ldots, p_N\} \to \mathbb{R}^+$ and the 1-form $\alpha \in C^\infty(\Lambda^1(\mathbb{R}^3 \setminus \{p_1, \ldots, p_N\}))$ are defined up to gauge transformations as follows:

$$V(x, \tau) = V(x) = c + \sum_{s=1}^{N} \frac{2m}{|x - p_s|}, \quad *_3 d\alpha = dV.$$

Here c is a parameter with values 0 or 1 and $*_3$ refers to the Hodge-operation with respect to the flat metric on $\mathbb{R}^3$. We can see that the metric is independent of τ hence we have a Killing field on $(U_V, g_V|_{U_V})$. This Killing field provides the above mentioned $U(1)$-action. Furthermore it is possible to show that, despite the apparent singularities in the NUTs, all things extend analytically over the whole M_V providing a real analytic space (M_V, g_V).

Notice that for a fixed manifold M_V we have *two* different metrics on it corresponding to the two possible values of c. Take $c = 0$ then for $N = 1$ we just find the flat metric on $\mathbb{R}^4$ while $N = 2$ gives rise to the Eguchi–Hanson space on $T^*\mathbb{C}P^1$. If $c = 1$ and $N = 1$ we recover the Taub–NUT metric on $\mathbb{R}^4$. All these spaces posses an $U(2)$ isometry. For higher N's the resulting spaces are the multi-Eguchi–Hanson and multi-Taub–NUT spaces respectively. These metrics admit only an $U(1)$ isometry coming from the circle action mentioned above.

These two infinite sequences of metrics are hyper-Kähler spaces i.e. for their curvature $s = 0$ and $W^- = 0$ holds (using another terminology they are half conformally flat with vanishing scalar curvature). Consequently part (ii) of the Atiyah–Hitchin–Singer theorem can be applied for them. Our aim is to find metrics $\tilde{g}$ (as much as possible), conformally equivalent to a fixed Gibbons–Hawking metric g_V, such that $\tilde{g}$'s are self-dual and have vanishing scalar curvature: in this case the metric instantons in $\tilde{g}$'s provide self-dual connections on them, as we have seen. Taking into account that the $(3, 1)$-Weyl tensor W is invariant under conformal rescalings i.e. $\widetilde{W} = W$, the condition $\widetilde{W}^- + \tilde{s}/12 = 0$ for the $\tilde{g}$'s settles down for having vanishing scalar curvature $\tilde{s} = 0$. Consider the rescaling $g \mapsto \tilde{g} := f^2 g$ where $f : M_V \to \mathbb{R}$ is a function. In this case the scalar curvature transforms as $s \mapsto \tilde{s}$ where $\tilde{s}$ satisfies (see [3], pp. 58-59):

$$f^3 \tilde{s} = 6 \triangle f + f s.$$

Taking into account that the Gibbons–Hawking spaces are Ricci-flat, our condition for the scaling function amounts to the simple condition

$$\triangle f = 0 \tag{8}$$

i.e. it must be a harmonic function (with respect to the Gibbons–Hawking geometry). In other words we have to analyse the kernel of the Laplacian for our aim.

To analyse harmonic functions in these geometries, first we construct Laplacians on the collapsed spaces $\mathbb{C}^2 \setminus \{0\}/\mathbb{Z}_N$ because these operators and their associated harmonic functions can be constructed explicitly.

Assume therefore that all the NUTs are pulled together in (6) i.e. we return to the singular quotient $\mathbb{C}^2/\mathbb{Z}_N$. If we remove the singular origin the resulting space is topologically a lense space: $\mathbb{C}^2 \setminus \{0\}/\mathbb{Z}_N \cong \mathbb{R}^+ \times (S^3/\mathbb{Z}_N)$. We shall denote this space by M_{V_0}. Regarding it as a plane bundle over S^2 and denoting by (Θ, ϕ) the spherical coordinates while by (r, τ) polar coordinates on the fibres we simply get

$$V_0(r) = \frac{cr + 2mN}{r}, \quad \alpha_0 = 2m \cos \Theta \mathrm{d}(N\phi)$$

under this blowing down process. Consequently the Gibbons–Hawking metric (7) reduces to a metric g_0 of the local form

$$\mathrm{d}s_0^2 = \frac{cr + 2mN}{r}(\mathrm{d}r^2 + r^2(\mathrm{d}\Theta^2 + \sin^2 \Theta \mathrm{d}\phi^2)) + \frac{r}{cr + 2mN}(2mN)^2(\mathrm{d}\tau + \cos \Theta \mathrm{d}\phi)^2$$

on M_{V_0} where

$$0 < r < \infty, \quad 0 \leq \tau < \frac{4\pi}{N}, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \Theta < \pi.$$

We remark that for $N = 1$ this metric extends as the flat metric for $c = 0$ and as the Taub-NUT metric for $c = 1$ over $\mathbb{R}^4$. One easily calculates $\det g_0 = 4m^2N^2(cr + 2mN)^2r^2 \sin^2 \Theta$ and by using the local expression

$$\triangle = \sum_{i,j} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x^j} \right),$$

where g^{ij} are the components of the inverse matrix, the singular Laplacian looks like

$$\triangle_0 = \frac{(cr + 2mN)^2 \sin^2 \Theta + 4m^2N^2 \cos^2 \Theta}{4m^2N^2(cr^2 + 2mNr) \sin^2 \Theta} \frac{\partial^2}{\partial \tau^2} + \frac{r}{cr + 2mN} \frac{\partial^2}{\partial r^2} + \frac{2}{cr + 2mN} \frac{\partial}{\partial r}$$

$$+ \frac{1}{cr^2 + 2mNr} \left(\frac{\partial^2}{\partial \Theta^2} + \cot \Theta \frac{\partial}{\partial \Theta} \right) + \frac{1}{(cr^2 + 2mNr) \sin^2 \Theta} \left(\frac{\partial^2}{\partial \phi^2} - 2 \cos \Theta \frac{\partial^2}{\partial \tau \partial \phi} \right).$$

This apparently new Laplacian can be rewritten in old terms as follows. The lense space with its standard metric induced from the round S^3 has an associated Laplacian

$$\triangle_{S^3/\mathbb{Z}_N} = \frac{1}{\sin^2 \Theta} \left(\frac{\partial^2}{\partial \tau^2} - 2 \cos \Theta \frac{\partial^2}{\partial \tau \partial \varphi} + \frac{\partial^2}{\partial \varphi^2} \right) + \frac{\partial^2}{\partial \Theta^2} + \cot \Theta \frac{\partial}{\partial \Theta}$$

in Euler coordinates. By the aid of this we obtain

$$\triangle_0 = \frac{r}{cr + 2mN} \frac{\partial^2}{\partial r^2} + \frac{2}{cr + 2mN} \frac{\partial}{\partial r} + \frac{1}{cr^2 + 2mNr} \triangle_{S^3/\mathbb{Z}_N} + \frac{(cr + 2mN)^2 - 4m^2N^2}{4m^2N^2(cr^2 + 2mNr)} \frac{\partial^2}{\partial \tau^2}.$$

Now we wish to solve (8) in its full generality in this case. To this end we introduce an orthonormal system of bounded smooth functions on $S^3/\mathbb{Z}_N$ by the (three dimensional) spherical

harmonics Y_j^{kl} with $j = 0, 1, \ldots$ and $k, l = 0, \ldots, j$ (for a fixed j, there are $(j + 1)^2$ independent spherical harmonics on $S^3/\mathbb{Z}_N$). In Euler coordinates these take the shape

$$Y_j^{kl}(\tau, \phi, \Theta) = \frac{1}{4\pi} \sqrt{\frac{(2j+1)(j - |k - l|)!}{(j + |k - l|)!}} \, \mathrm{e}^{\mathbf{i}kN\tau} \, \mathrm{e}^{\mathbf{i}l\phi} P_j^{|k-l|}(\cos\Theta)$$

with $P_j^m(x)$, $-1 \leq x \leq 1$, being an associated Legendre polynomial defined by the following generating function:

$$(2m - 1)!! \frac{(1 - x^2)^{\frac{m}{2}} t^m}{(1 - 2tx + t^2)^{m + \frac{1}{2}}} = \sum_{j=m}^{\infty} t^j P_j^m(x).$$

Consequently, using the decomposed Laplacian the general solution can be written in the form

$$f_0(r, \tau, \phi, \Theta) = \sum_{j=0}^{\infty} \sum_{k,l=0}^{j} \lambda_j^{kl} \varrho_j^{kl}(r) Y_j^{kl}(\tau, \phi, \Theta) \tag{9}$$

where λ_j^{kl} are complex numbers and ϱ_j^{kl} are smooth radial functions. Notice that the spherical harmonics obey

$$\triangle_{S^3/\mathbb{Z}_N} Y_j^{kl} = -j(j+1) Y_j^{kl}, \quad \frac{\partial}{\partial\tau} Y_j^{kl} = \mathbf{i}kN Y_j^{kl}$$

yielding that the solution of (8) reduces to the ordinary differential equations

$$r^2 \frac{\mathrm{d}^2 \varrho_j^{kl}}{\mathrm{d}r^2} + 2r \frac{\mathrm{d}\varrho_j^{kl}}{\mathrm{d}r} - j(j+1) \varrho_j^{kl} = 0 \tag{10}$$

if $c = 0$ and

$$r^2 \frac{\mathrm{d}^2 \varrho_j^{kl}}{\mathrm{d}r^2} + 2r \frac{\mathrm{d}\varrho_j^{kl}}{\mathrm{d}r} - \left(j(j+1) + \frac{r^2 + 4mNr}{4m^2} k^2 \right) \varrho_j^{kl} = 0 \tag{11}$$

if $c = 1$, $j = 0, 1, \ldots$, $0 \leq k, l \leq j$. From here we can see that ϱ_j^{kl} hence f can have an isolated singularity only in the origin of $\mathbb{C}^2/\mathbb{Z}_N$.

If (9) is real and converges, by projecting the corresponding Levi–Civitá connections of the rescaled metrics onto the $\mathfrak{su}(2)^-$ part we can produce self-dual connections via the Atiyah–Hitchin–Singer theorem. To get *regular* solutions however, we have to propose further regularity conditions on the rescaling harmonic functions. The Uhlenbeck theorem [28] guarantees that generally in a given gauge only *pointlike* singularities can be removed from self-dual connections. Therefore we have to study the singularities of the rescaled Levi–Civitá connections. The natural orthonormal system to g_V, associated to the gauge (7) is

$$\xi^0 = \frac{1}{\sqrt{V}}(\mathrm{d}\tau + \alpha), \quad \xi^1 = \sqrt{V}\mathrm{d}x, \quad \xi^2 = \sqrt{V}\mathrm{d}y, \quad \xi^3 = \sqrt{V}\mathrm{d}z. \tag{12}$$

Since $\tilde{g} = f^2 g$ we have $\tilde{\xi}^i = f\xi^i$ consequently in our gauge the Levi–Civitá connection $\tilde{\omega}$ with respect to the rescaled metric obey the following Cartan equation on U_V:

$$-\omega_j^i \wedge \xi^j + \mathrm{d}(\log f) \wedge \xi^i = -\tilde{\omega}_j^i \wedge \xi^j.$$

As it is proved in [10] in this gauge the original connection ω is self-dual hence cancels out if we project $\tilde{\omega}$ to the $\mathfrak{su}(2)^-$ component consequently all singularities of the self-dual connection ∇^-

must come from the $\mathrm{d}(\log f)$ term in our gauge. Hence if we can understand the singularities of the function $|\mathrm{d}(\log f)|_{g_V}$ then we can find the required regularity condition for the rescaling harmonic functions. Notice that if f is zero somewhere then $|\mathrm{d}(\log f)|_{g_V}$ diverges consequently *an acceptable harmonic function must be positive and bounded everywhere, except isolated points.*

Taking into account that $\overline{Y_j^{kl}} = Y_j^{-k,-l}$, a real basis for the solutions is given by

$$\frac{1}{2}(Y_j^{kl} + Y_j^{-k,-l}), \qquad \frac{1}{2\mathbf{i}}(Y_j^{kl} - Y_j^{-k,-l})$$

hence, via $\varrho_j^{kl} = \varrho_j^{-k,-l}$, if we have $\overline{\lambda_j^{kl}} = \lambda_j^{-k,-l}$ we get real solutions in (9).

For $j = 0$ (hence $k = l = 0$) both (10) and (11) yields $c_1 r^{-1} + c_2$. Moreover $Y_0^{00} = 1/4\pi$ hence separating the $j = 0$ term in (9) we can write a harmonic function f_0 over (M_{V_0}, g_0) as

$$f_0(r, \tau, \phi, \Theta) = c_1 + \frac{c_2}{r} + h_0(r, \tau, \phi, \Theta)$$

if $c = 0, 1$. The radial function $R^2 = |v|^2 + |w|^2 = r^2 + (\operatorname{Im} w)^2$ in $\mathbb{C}^2$ is invariant under $\mathbb{Z}_N$ hence we can talk about lense spaces of radius R centered at the origin of $\mathbb{C}^2/\mathbb{Z}_N$. Now take such a lense space. By the aid of the explicit expressions for the spherical harmonics it is not difficult to show that for $j > 0$

$$\int\limits_{S^3/\mathbb{Z}_N} \left(Y_j^{kl} + Y_j^{-k,-l} \right) \mathrm{d}V_g = \frac{1}{2\mathbf{i}} \int\limits_{S^3/\mathbb{Z}_N} \left(Y_j^{kl} - Y_j^{-k,-l} \right) \mathrm{d}V_g = 0$$

holds (integration is with respect to the induced metric on the lense space in question) hence these real functions must change sign somewhere on any lense space showing that in fact h_0 also changes sign i.e. vanishes along three dimensional subsets in the above decomposition.

Furthermore the coefficients of (10) and (11) are real analytic in r hence any solution of them must be also real analytic in r. In fact the general solution to (10) and (11) with $k = l = 0$ is $c_1 r^{-j-1} + c_2 r^j$ if $j \geq 0$. It is also not difficult to see that any non-trivial solution of (11) with $j > 1$, $k, l \geq 0$ behaves either as $\varrho_j^{kl}(r) \sim r^{-j-1}$ if $r \to 0$ or $\varrho_j^{kl}(r) \sim r^j$ if $r \to \infty$. This implies that in both cases h_0 dominates f_0 for small or large values of r hence the total solution f_0 is also plagued by small codimensional zero sets. For later convenience we formulate this in the following

Lemma 3.1 *Let B_ε^R be the intersection of a ball of radius R centered at origin with the complement of a small ball of radius ε also centered at the origin of $\mathbb{C}^2$. Let $U_\varepsilon^R := B_\varepsilon^R/\mathbb{Z}_N$ be its quotient. Then for a harmonic function*

$$\lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \inf_{p \in U_\varepsilon^R} f_0(p) = -\infty, \qquad \lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \sup_{p \in U_\varepsilon^R} f_0(p) = \infty$$

holds unless $h_0 = 0$ in its decomposition. $\diamond$

Consequently we must have

$$f_0(r) = \lambda_0 + \frac{\lambda_1}{r} \tag{13}$$

with $0 \leq \lambda_i \leq \infty$ for a generic acceptable harmonic function on (M_{V_0}, g_0) i.e., f_0 admits the full $U(2)$ symmetry. This solution was found in [9] and [10] for the Taub–NUT metric. We remark that the harmonic expansion on Eguchi–Hanson space is treated in e.g. [22].

Our aim is to find the appropriate generalization of the above classical expansion (9) in the general case by regarding the Gibbons–Hawking metric as a perturbation of the singular metric g_0. To achieve this, we have to construct the twistor spaces of the Gibbons–Hawking spaces.

4 Regularization

As it is well-known, Penrose' twistor theory, originally designed for flat Lorentzian $\mathbb{R}^4$ can be generalized to self-dual four-manifolds. From our viewpoint this is important because the Gibbons–Hawking spaces are self-dual (or half conformally flat) and by the aid of their twistor spaces various differential equations can be solved over them.

Let us recall the general theory [3]. Let (M, g) be a Riemannian spin four-manifold and consider the projective bundle $Z := P(S^-M)$ of the complex chiral spinor bundle S^-M over it. Clearly, Z admits a fiber bundle structure $p : Z \to M$ with $\mathbb{C}P^1$'s as fibers. By using the Levi–Civitá connection on (M, g) we can split the tangent space $T_z Z$ at any point $z \in Z$ and the horizontal and vertical subspaces can be endowed with natural complex structures hence Z, as a real six-manifold, carries an almost complex structure, too. A basic theorem of Penrose or Atiyah, Hitchin, Singer states that this complex structure is integrable, i.e. Z is a complex manifold if and only if (M, g) is half conformally flat ([3], p.380, Th. 13.46). Z is called the *twistor space* of (M, g). The fibers of $p : Z \to M$ are then holomorphic projective lines in Z with normal bundle $H \oplus H$ and belong to a family parameterized by a complex four-manifold $\mathbb{C}M$. (We denote by H^k ($k \in \mathbb{Z}$) the powers of the tautological line bundle H on $\mathbb{C}P^1$ while their induced sheaves of holomorphic sections are $\mathcal{O}(k)$ respectively.) If one endows Z with a real structure $\tau : Z \to Z$ with $\tau^2 =$Id such that the fibers are kept fixed by the real structure then $M \subset \mathbb{C}M$ is encoded in Z as the family of these "real lines". The basic example is $(M, g) = (S^4, \delta)$, the round sphere, $\mathbb{C}M = Q^4$ the Klein quadric and $Z = \mathbb{C}P^3$.

If g is Ricci flat then S^-M with its induced connection is a flat bundle. Hence if M is moreover simply connected then Z can be contracted onto a particular fibre $\mathbb{C}P^1$ via parallel transport. Consequently in this case we have another fibration $\pi : Z \to \mathbb{C}P^1$ with fibers the copies of M. Hence if a holomorphic line bundle H^k is given on $\mathbb{C}P^1$ then it can be pulled back to line bundles $\pi^* H^k$ over Z. Consider the sheaf cohomology group $H^1(Z, \mathcal{O}(-2))$ of the line bundle $\pi^* H^{-2}$. Take an element $[\omega] \in H^1(Z, \mathcal{O}(-2))$ and consider for a representative $\omega \in [\omega]$ the complex valued real analytic function $f : M \to \mathbb{C}$ given by the restriction $\omega|_{Y_p}$ where $\mathbb{C}P^1 \cong Y_p \subset Z$ ($p \in M$) is a real line. Then we have $\triangle f = 0$ (with respect to g) and there is a natural bijection

$$T : H^1(Z, \mathcal{O}(-2)) \cong \ker \triangle,$$

onto real analytic soutions, called the *Penrose transform*. In fact if the original metric g is real analytic then this isomorphism gives rise to *all* solutions to the Laplace equation in question. For a proof and the more general statement see e.g. [16]. Taking into account the isomorphism $H^1(Y_p, \mathcal{O}(-2)) \cong H^{(1,1)}(\mathbb{C}P^1)$ and that $(1, 1)$-forms can be integrated over $Y_p \cong \mathbb{C}P^1$ we can see that the above transform is nothing but integration of forms over the real lines, that is,

$$f(p) = \int_{Y_p} \omega|_{Y_p}.$$

In what follows we construct the above objects explicitly in case of the Gibbons–Hawking spaces (M_V, g_V) based on Hitchin's works.

First consider the multi-Eguchi–Hanson spaces i.e. the $c = 0$ cases [15]. Since we have topologically $Z \cong M_V \times \mathbb{C}P^1$ the twistor space Z can be described analogouosly to (6) but instead of a single copy of M_V we need a collection of them, parameterized by a projective line. Therefore we have to replace locally the constants a_i with functions $a_i(u)$ depending holomorphically on a

complex variable $u \in \mathbb{C}P^1$. Hence, as a first approximation of Z we define $\tilde{Z}$ by the equation

$$xy - (z^N + a_1(u)z^{N-1} + \ldots + a_N(u)) = 0.$$

Since there are no nontrivial holomorphic functions on $\mathbb{C}P^1$ we have to assume that a_i is a holomorphic section of a bundle which implies that the independent variables are also to be interpreted as elements of various line bundles over $\mathbb{C}P^1$. The natural real structure $\tau : Z \to Z$

$$\tau(x, y, z, u) = ((-1)^N \sigma(y), \sigma(x), -\sigma(z), \sigma(u))$$

is induced by the $\mathbb{Z}_N$-invariant extension to $H^m \oplus H^n \oplus H^p$ of the natural antipodal map σ on $\mathbb{C}P^1$ with $\sigma(u) = -1/\overline{u}$. This map to be well-defined and the requirement that the real lines have normal bundle $H \oplus H$ implies that $m = n = N$ and $p = 2$ and finally $\tilde{Z}$ is defined as a three dimensional complex hypersurface $q(x, y, z, u) = 0$ in the complex four-manifold $H^N \oplus H^N \oplus H^2$ where $x, y \in H^N$, $z \in H^2$ and a_i is an holomorphic section of H^{2i} obeying the algebraic equation

$$q : H^N \oplus H^N \oplus H^2 \longrightarrow H^{2N}$$

given by

$$q(x, y, z, u) = xy - \prod_{s=1}^{N}(z - p_s(u)). \tag{14}$$

Here we use the factorization

$$\sum_{i=0}^{N} a_i(u)z^{N-i} = \prod_{s=1}^{N}(z - p_s(u))$$

and p_s' are interpreted as holomorphic sections of H^2 in this case. Unfortunately $\tilde{Z}$ is not a manifold because generically there are points $u_{ij}^{\pm} \in \mathbb{C}P^1$ for which $p_i - p_j = 0$ hence the discriminant of $\sum_{i=0}^{N} a_i(u)z^{N-i}$ vanishes. Consequently the fiber over this point is singular. More precisely, there is a singular point $z^* := (x = 0, y = 0, z = p_i(u_{ij}^{\pm}), u_{ij}^{\pm})$ on the fiber over $u_{ij}^{\pm}$.

These singularities can be removed by deforming $\tilde{Z}$ within a family of non-singular surfaces. During the course of this deformation typically we have to take a branched cover of $\mathbb{C}P^1$ over the singular points $u_{ij}^{\pm}$ hence the resolved Z would by first look fibered over a higher genus Riemann surface. However in our special case we know *a priori* that the twistor manifold Z exists hence the resolved model is still fibered over $\mathbb{C}P^1$. For our porposes it is enough to work with the singular model $\tilde{Z}$.

Now we construct the real lines explicitly, following [15]. Remember that these are invariant sections of the holomorphic bundle $\pi : \tilde{Z} \to \mathbb{C}P^1$ with respect to the real structure τ. If we take $z_p(u) = au^2 + 2bu + c$ and $p_s(u) = a_s u^2 + 2b_s u + c_s$ then the holomorphic sections $s_p : \mathbb{C}P^1 \to \tilde{Z}$ are given by polynomials $x_p(u)$, $y_p(u)$ and $z_p(u)$ obeying (14). Consequently we ought to solve it. However this equation can be solved by a simple factorization yielding

$$x_p(u) = A\prod_{s=1}^{N}(u - \alpha_{p_s}), \quad y_p(u) = B\prod_{s=1}^{N}(u - \beta_{p_s}), \quad z_p(u) = au^2 + 2bu + c \tag{15}$$

where $AB = \prod_{s=1}^{N}(a - a_s)$ and $\alpha_{p_s}, \beta_{p_s}$ are roots to the equations $z_p(u) - p_s(u) = 0$. The reality condition provided by the τ-invariance gives $a = -\bar{c}$, $b = \bar{b}$ and $a_s = -\bar{a}_s$, $b_s = \bar{b}_s$ hence the roots explicitly look like

$$\alpha_{p_s} = \frac{-(b - b_s) - \sqrt{(b - b_s)^2 + |a - a_s|^2}}{a - a_s}, \quad \beta_{p_s} = \frac{-(b - b_s) + \sqrt{(b - b_s)^2 + |a - a_s|^2}}{a - a_s}. \quad (16)$$

The reality condition also yields

$$A = \mathrm{e}^{\mathrm{i}\phi}\sqrt{\prod_{s=1}^{N}\left(b - b_s + \sqrt{(b - b_s)^2 + |a - a_s|^2}\right)}. \quad (17)$$

(In these equations the square root is understood as the *positive* square root to avoid ambiguity.) From here we can see again that any real section s_p i.e. a point $p \in M_V$ away from the NUTs is parameterized by $(\mathrm{Re}\, a, b, \mathrm{Im}\, a) \in \mathbb{R}^3$ and the angular variable $\arg A =: \phi \in S^1$.

Now we are ready to present integral formulae for the Laplacian over the multi-Eguchi–Hanson spaces. Pick up a particular fibre Y_p of Z, not identical to any of the NUTs Y_{p_s} and denote it by $\mathbb{C}P^1$. Take an ω with $[\omega] \in H^1(Z \setminus \cup_s Y_{p_s}, \mathcal{O}(-2))$. Since this is holomorphic we can Taylor expand it. Denote by ξ and υ the tautological sections of the two copies of $\pi^* H^N$ (as pulled back over H^N) and let ζ be the tautological section of $\pi^* H^2$ (as pulled back over H^2). Hence we can write

$$\omega = \sum_{m,n,i}(\pi^*\omega_{mni})\xi^m\upsilon^n\zeta^i$$

where for $j := Nm + Nn + 2i$

$$[\omega_{mni}] \in H^1(\mathbb{C}P^1, \mathcal{O}(-2 - j)), \quad \xi^m\upsilon^n\zeta^i \in H^0(H^j, \mathcal{O}(j)).$$

Since by definition the image of the tautological section is the zero section of the corresponding line bundle, we can write over $Z \setminus \cup_s Y_{p_s}$ that

$$\xi = \xi - \pi^* x_p + \pi^* x_p, \quad \upsilon = \upsilon - \pi^* y_p + \pi^* y_p, \quad \zeta = \zeta - \pi^* z_p + \pi^* z_p$$

and now $(\xi - \pi^* x_p)|_{Y_p} = 0$ etc., i.e. they vanish exactly on the image Y_p of the section $s_p = (x_p, y_p, z_p)$. Consequently, by using the isomorphism $s_p^* : H^0(Y_p, \mathcal{O}(j)) \cong H^0(\mathbb{C}P^1, \mathcal{O}(j))$ we can write

$$s_p^*(\omega|_{Y_p}) = \sum_{m,n,i}(s_p^*\pi^*\omega_{mni})s_p^*((\xi - \pi^* x_p)|_{Y_p} + \pi^* x_p)^m((\upsilon - \pi^* y_p)|_{Y_p} + \pi^* y_p)^n((\zeta - \pi^* z_p)|_{Y_p} + \pi^* z_p)^i$$

$$= \sum_{m,n,i}\omega_{mni}x_p^m y_p^n z_p^i.$$

Putting (15) into the above expression we find the following expansion for a harmonic function on the multi-Eguchi–Hanson space with possible singularities in the NUTs:

$$f(p) = f(\mathrm{Re}\, a, b, \mathrm{Im}\, a, \arg A) =$$

$$\sum_{m,n,i}\int_{\mathbb{C}P^1}\omega_{mni}(u, \bar{u})A^{m-n}\prod_{s=1}^{N}(a - a_s)^n (u - \alpha_{p_s})^m (u - \beta_{p_s})^n (au^2 + 2bu - \bar{a})^i \mathrm{d}u \wedge \mathrm{d}\bar{u}. \quad (18)$$

where α_{p_s}, β_{p_s} are given by (16) and A via (17).

Now we turn our attention to the case of multi-Taub–NUT spaces. The situation is similar [5][17]. In this case the twistor space topologically is the same as in the previous case but the complex structure on it is different. This difference is realized by twisting the previous line bundles with a new one. Consider the line bundle $L^\mu(m)$ over H^2 defined as follows. Let $U_0(u \neq \infty), U_1(u \neq 0)$ be the standard covers of $\mathbb{C}P^1$ and $\tilde{U}_0, \tilde{U}_1$ be the induced covers of H^2. The coordinates on $\tilde{U}_0$ are (u, z). Suppose H^m is the line bundle over H^2 pulled back from $\mathbb{C}P^1$. The line bundle $L^\mu(m)$ over H^2 is defined by the transition function (μ is a parameter)

$$g(z, u) := u^{-m} e^{-\mu z u^{-1}}$$

on $\tilde{U}_0 \cap \tilde{U}_1$. Write $z = au^2 + 2bu - \overline{u}$ and consider a section η^μ of $L^\mu(N)$ as follows:

$$\eta^\mu(z, u) := \begin{cases} e^{\mu(au+b)} & \text{on } \tilde{U}_0, \\ g(z, u)e^{\mu(au+b)} = u^{-N} e^{\mu(\overline{a}u^{-1} - b)} & \text{on } \tilde{U}_1. \end{cases}$$

This η^μ is well defined because it is regular at both $u = 0$ and $u = \infty$ i.e., on $\tilde{U}_0$ and $\tilde{U}_1$. By using this section the space $\tilde{Z}'$ is defined by

$$\tilde{Z}' := \left\{ (x', y', z') \in L^1(N) \oplus L^{-1}(N) \mid x'y' - \prod_{s=1}^N (z' - p_s(u)) \right\}.$$

where the real sections are of the form

$$x'_p(u) = \eta^1(z_p(u), u)x_p(u), \quad y'_p(u) = \eta^{-1}(z_p(u), u)y_p(u), \quad z'_p(u) = z_p(u) = au^2 + 2bu - \overline{a}.$$

Again, by the *a priori* knowledge of the existence of the multi-Taub–NUT metric, this space can be deformed into a complex analytic twistor space with holomorphic fibration $\pi' : Z' \to \mathbb{C}P^1$.

The corresponding integral formula, again with possible singularities over the NUTs looks like

$$f(p) = f(\mathrm{Re}\, a, b, \mathrm{Im}\, a, \arg A) =$$

$$\sum_{m,n,i} \int_{\mathbb{C}P^1} \omega_{mni}(u, \overline{u})(A\eta)^{m-n} \prod_{s=1}^N (a - a_s)^n (u - \alpha_{p_s})^m (u - \beta_{p_s})^n (au^2 + 2bu - \overline{a})^i \mathrm{d}u \wedge \mathrm{d}\overline{u} \quad (19)$$

together with (16) and (17). We may regard (18) and (19) as generalizations of the classical $U(2)$ symmetric harmonic expansion (9) to the general $U(1)$ symmetric case. Harmonic functions invariant under this $U(1)$ isometry arise by taking $m = n$. In some sense a complementary problem, i.e. the construction of the image of the Laplacian over Gibbons–Hawking spaces was considered by constructing Green's functions in [2] and [25].

After these preliminarities, we are ready to complete the analysis to find the acceptable harmonic functions. Fix an N and consider the space (M_V, g_V) with NUTs $p_1, \ldots, p_N \in \mathbb{R}^3$. If $t \in [0, 1]$ we can smoothly shrink the NUTs into the origin of $\mathbb{R}^3$ by taking $a_s(t) := ta_s$, $b_s(t) := tb_s$ yielding a one-parameter family (M_{V_t}, g_t) of Gibbons–Hawking spaces and associated Laplacians $\triangle_t$ which are regular for $t > 0$ and blow down to the singular space (M_{V_0}, g_0) with $\triangle_0$ if $t = 0$. Define the corresponding smooth functions $\alpha_{p_s}(t), \beta_{p_s}(t)$ and $A(t)$ by (16) and (17) respectively and consider the family of harmonic functions f_t constructed this way via (18) and (19). These satisfy $\triangle_t f_t = 0$ and the flow $t \mapsto f_t$ is *continuous* in the following sence. Note

that for a fixed N, the blow-up map, restricted to the complement of the exceptional divisor E, induces a biholomorphic mapping $\pi_t : M_{V_t} \setminus E \to M_{V_0}$. Hence write p for $\pi_t^{-1}(p)$ for all $t > 0$ and $p \in M_{V_0}$. Then there is a constant $C(p) > 0$, such that

$$|f_t(p) - f_0(p)| \leq C(p)t.$$

Moreover for fixed c and N this flow provides a 1:1 correspondence between the solutions for $t > 0$ and $t = 0$; however the latter case is explicitly known from the previous section. There we have seen that the only acceptable solutions are of the form (13) whose perturbation for $t > 0$ is (cf. [10])

$$f_t(x) = \lambda_0 + \sum_{s=1}^{N} \frac{\lambda_s}{|x - tp_s|}. \tag{20}$$

All other harmonic functions f_0 are zero along three dimensional subsets of M_{V_0} close to either infinity or the singularity. Consequently there is an $\varepsilon > 0$ such that if $0 < t < \varepsilon$ their continuous perturbation f_t has the same property because of Lemma 3.1.

Therefore we can conclude that a general acceptable harmonic function on a Gibbons–Hawking space is of the form (20). In the next section we summarize our findings. For this we introduce notations, also used in [10]. Consider the quaternion-valued 1-form ξ assigned to the basis (12) and the imaginary quaternion valued function assigned to (20) with $t = 1$ (we denote it by f) as follows:

$$\xi := \xi^0 + \xi^1 \mathbf{i} + \xi^2 \mathbf{j} + \xi^3 \mathbf{k}, \quad \mathbf{d}(\log f) := \frac{\partial \log f}{\partial x}\mathbf{i} + \frac{\partial \log f}{\partial y}\mathbf{j} + \frac{\partial \log f}{\partial z}\mathbf{k}.$$

5 The theorems

By collecting all the results from [9], [10] and the present work we can state:

Theorem 5.1 *Let (M_V, g_V) be a Gibbons–Hawking space with $U_V := M_V \setminus \{p_1, \ldots, p_N\}$ where $p_1, \ldots, p_N$ denote the NUTs of it. Then any smooth, finite action $SU(2)$ 't Hooft instanton lives on the negative chiral bundle $S^- M_V$ and such an $\nabla^-_{\lambda_0, \ldots, \lambda_N}$ over this space is given by*

$$A^-_{\lambda_0, \ldots, \lambda_N} = \mathrm{Im}\, \frac{\mathbf{d}(\log f)\xi}{2\sqrt{V}}$$

with $\nabla^-_{\lambda_0, \ldots, \lambda_N} = \mathrm{d} + A^-_{\lambda_0, \ldots, \lambda_N}$ in the gauge (12) over U_V.

The action of this connection is zero if $\lambda_s = 0$ ($s > 0$) otherwise

$$\|F^-_{\lambda_0, \ldots, \lambda_N}\|^2 = \begin{cases} n - (1/N) & \text{if } \lambda_0 = 0, c = 0, \\ \\ n & \text{otherwise,} \end{cases}$$

where N refers to the number of NUTs while n is the number of non-zero λ_s' ($s > 0$). $\diamond$

All these instantons are $U(1)$ invariant hence dividing by this action we have

Theorem 5.2 *Furthermore such an instanton corresponds to a $SU(2)$ magnetic monopole (Φ, A) over flat $\mathbb{R}^3$ where*

$$\Phi = \frac{\mathbf{d}(\log f)}{2}, \quad A = \mathrm{Im}\, \frac{\mathbf{d}(\log f)\mathbf{i}}{2}\mathrm{d}x + \mathrm{Im}\, \frac{\mathbf{d}(\log f)\mathbf{j}}{2}\mathrm{d}y + \mathrm{Im}\, \frac{\mathbf{d}(\log f)\mathbf{k}}{2}\mathrm{d}z$$

with singularities in the NUTs. All such magnetic monopoles have zero magnetic charge. $\diamond$

In [10] we found all reducible connections and identified their curvatures with L^2 harmonic 2-forms. Via a result in [14] these generate the whole L^2 cohomology of (M_V, g_V) hence:

Theorem 5.3 *A 't Hooft instanton* $\nabla^-_{\lambda_0,\ldots,\lambda_N}$ *over* (M_V, g_V) *is reducible if and only if for an* $s = 0, \ldots, N$ *we have* $\lambda_s \neq 0$ *and* $\lambda_r = 0$ *for* $r = 0, 1, \ldots, s-1, s+1, \ldots, N$; *in this case it can be gauged into the form*

$$B_s = \left(-\frac{\mathrm{d}\tau + \alpha}{|x - p_s|V} + \alpha_s \right) \frac{\mathbf{k}}{2},$$

where $*_3\mathrm{d}\alpha_s = \mathrm{d}V_s$ *with* $V_s(x) = c + (2m/|x - p_s|)$. *The curvature* F_s *of these connections generate the full* L^2 *cohomology of the Gibbons–Hawking space* (M_V, g_V). $\diamond$

6 Concluding remarks

In this paper we have classified all $SU(2)$ Yang–Mills instantons over the Gibbons–Hawking spaces which arise by conformal rescalings of the the metric (we called these instantons as 't Hooft instantons). During the course of this we encountered the twistor manifolds of the Gibbons–Hawking spaces which enables us to take an outlook.

Firstly at this point we remark that (18) and (19) are generalizations of the classical formula of Whittaker and Watson for solutions of the three dimensional Laplacian to the Gibbons–Hawking spaces [30]. This is because dividing a Gibbons–Hawking space via its $U(1)$ isometry we recover the flat $\mathbb{R}^3$. Indeed, taking $m = n$ in (18) or (19) they cut down to

$$f(p) = \int_{\mathbb{C}P^1} \sum_{m,i} \omega_{mi}(u, \overline{u}) \prod_{s=1}^N (z_p(u) - p_s(u))^m z_p^i(u) \mathrm{d}u \wedge \mathrm{d}\overline{u}.$$

Pulling all NUTs into the origin of $\mathbb{R}^3$ we can write this as

$$f(\mathrm{Re}\, a, b, \mathrm{Im}\, a) = \int_{\mathbb{C}P^1} \sum_k \omega_k(u, \overline{u})(au^2 + 2bu - \overline{a})^k \mathrm{d}u \wedge \mathrm{d}\overline{u}.$$

Writing $u = re^{\mathrm{i}\phi}$ and performing the radial integration this eventually yields

$$f(x, y, z) = \int_0^{2\pi} g(x\mathbf{i}\cos\phi + y\mathbf{i}\sin\phi + z, \phi)\mathrm{d}\phi$$

for some function g which is the classical Whittaker–Watson formula (also cf. [23]).

Secondly, our method applies to more general gravitational instantons of Kronheimer [20] as well as of Cherkis and Kapustin [5][6]. Taking into account that $M_V \setminus \{p_1, \ldots, p_N\}$ is connected and simply connected, by a theorem of Buchdahl [4] we can see that

$$H^1(Z \setminus \cup_s Y_{p_s}, \pi^*\mathcal{O}(-2-j)) \cong H^1(\mathbb{C}P^1, \mathcal{O}(-2-j))$$

and via Serre duality

$$H^1(\mathbb{C}P^1, \mathcal{O}(-2-j)) \cong H^0(\mathbb{C}P^1, \mathcal{O}(j)) \cong \mathbb{C}^{1+j}.$$

Hence the $\pi^*(\omega_{mni})$ terms of our series indeed can be interpreted as "coefficients" of a Taylor expansion yielding a decomposition

$$H^1(Z \setminus \cup_s Y_{p_s}, \mathcal{O}(-2)) \cong \bigoplus_j \mathbb{C}^{1+j}.$$

Hence $H^1(Z \setminus \cup_s Y_{p_s}, \mathcal{O}(-2))$ has an obvious element, namely the pullback of one of the generators of $H^1(\mathbb{C}P^1, \mathcal{O}(-2)) \cong \mathbb{C}$ i.e. the term corresponding to $j = 0$. Exactly this element gives the solutions (20) [2][16]. This distinguished element exists because the twistor space Z is holomorphically fibered over $\mathbb{C}P^1$. However this is a consequence of the fact that our gravitational instantons are not only self-dual but Ricci-flat, too. Because the same is true for the new "exotic" ALE, ALF and ALG spaces, without knowing the metric explicitly on them we can be shure that 't Hooft instanton type solutions of the $SU(2)$ self-duality equations exist over them. Identifying the reducible solutions as above we can construct explicitly the L^2 cohomology of these spaces.

Finally we remark that we certainly cannot solve the self-dual $SU(2)$ Yang–Mills equations in their full generality with the conformal rescaling method even in the simplest Gibbons–Hawking space. This observations calls for the generalization of the ADHM construction to all gravitational instantons.

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