Gravity as a four dimensional algebraic quantum field theory

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Introduction

Summary: while seeking unitary representations of the diffeomorphism group of a smooth oriented manifold, exactly in 4 dimensions a C^*-algebra has been identified naturally containing curvature tensors.
Representation theory of this C^*-algebra is investigated. There exist quantum representations carrying unitary representations of the full diffeomorphism group, too ("quantum gravity") as well as classical representations in which the diffeomorphism group symmetry is spontaneously broken down to a finite dimensional subgroup ("general relativity").

The C^*-algebra of an oriented smooth 4-manifold

Consider a connected oriented 4-manifold M together with its group of orientation-preserving diffeomorphisms $\mathrm{Diff}^+(M)$. The space $\Omega^2_c(M;\mathbb{C})$ carries a unitary representation with respect to the non-degenerate indefinite scalar product

$$\langle \alpha, \beta \rangle_{L^2(M)} := \int\limits_M \overline{\alpha} \wedge \beta$$

of $\mathrm{Diff}^+(M)$ from the right by pullback: $\omega \mapsto f^*\omega$. The space $\Omega^2_c(M;\mathbb{C})$ cannot be completed to a Hilbert space but at least this scalar product gives rise to an adjoint operation

$$\circledast : \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \longrightarrow \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$$

$\langle A^{\circledast}\alpha, \beta \rangle_{L^2(M)} := \langle \alpha, A\beta \rangle_{L^2(M)}$ on its space of all linear maps.

Lemma

Consider the indefinite unitary reprsentation of $\mathrm{Diff}^+(M)$ *from the right on the incomplete space* $(\Omega^2_c(M;\mathbb{C}), \langle\,\cdot\,,\,\cdot\,\rangle_{L^2(M)})$.

(i) *A vector* $\omega\in\Omega^2_c(M;\mathbb{C})$ *satisfies* $f^*\omega=\omega$ *for all* $f\in\mathrm{Diff}^+(M)$ *if and only if* $\omega=0$ *("no vacuum");*

(ii) *The subspaces* $B(M)\subsetneqq Z(M)\subset\Omega^2_c(M;\mathbb{C})$ *of* d*-exact or -closed* 2*-forms respectively, are invariant under* $\mathrm{Diff}^+(M)$. $\diamond$

Remark

(i) $\Omega^2_c(M;\mathbb{C})$ as a semi-normed space is incomplete but admits non-canonical splittings $\Omega^2_c(M;\mathbb{C})=\Omega^+_c(M;\mathbb{C})\oplus\Omega^-_c(M;\mathbb{C})$ into orthogonal maximal definite subspaces with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(M)}$ hence admit non-canonical split completions $\mathscr{H}^+(M)\oplus\mathscr{H}^-(M)$ not respected by diffemorphisms.

(ii) $\Omega^2_c(M;\mathbb{C})$ as a representation space is reducible.

Consider $(\Omega^2_c(M;\mathbb{C}), \langle \cdot\,,\,\cdot \rangle_{L^2(M)})$ together with the adjoint operator $\circledast : \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \to \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$.

Lemma

Take the $\circledast$-closed vector space $V := \left\{A \in \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \mid r(A^\circledast A) < +\infty\right\}$ defined by the spectral radius

$$r(B) := \sup_{\lambda\in\mathbb{C}} \left\{|\lambda| \;\middle|\; B - \lambda\cdot \mathrm{Id}_{\Omega^2_c(M;\mathbb{C})} \text{ is not invertible}\right\} \quad .$$

Then $\sqrt{r}$ is a norm and the corresponding completion of V renders $(V, \circledast)$ a unital C^*-algebra containing $\mathrm{Diff}^+(M)$. This C^*-algebra will be denoted by $\mathfrak{B}(M)$.

Idea of the proof. Use any particular split Hilbert space completion $\mathscr{H}^+(M) \oplus \mathscr{H}^-(M)$ of $\Omega^2_c(M;\mathbb{C})$ to check that the spectral radius gives rise to a norm. $\diamond$

Remark

(i) If M is compact then $\mathfrak{B}(M)$ can be regarded as the enhancement of the commutative unital C^*-algebra $C^0(M;\mathbb{C})$ of continuous $\mathbb{C}$-valued functions on the topological space M by taking into account the orientation and smooth structure on M as well. Indeed,

$$f \longmapsto f \cdot \mathrm{Id}_{\Omega^2_c(M;\mathbb{C})}$$

gives rise to a continuous embedding $C^0(M;\mathbb{C}) \subsetneqq \mathfrak{B}(M)$ of unital C^*-algebras;

(ii) So far the construction works in any $\dim_{\mathbb{R}} M = 4k$ dimensions (if $\dim_{\mathbb{R}} M = 4k+2$ then $(\Omega^2_c(M;\mathbb{C}), \langle \cdot\, , \,\cdot \rangle_{L^2(M)})$ is a symplectic space).

Interesting (but well-known) observation: if $\dim_{\mathbb{R}} M = 4$ then $\mathfrak{B}(M)$ contains curvature tensors of pseudo-Riemannian 4-manifolds! More precisely: Let (M, g) be a 4-dimensional Riemannian manifold. The Riemannian metric together with the orientation gives $* : \wedge^2 M \to \wedge^2 M$ with $*^2 = \mathrm{Id}_{\wedge^2 M}$ inducing a decomposition into (anti)self-dual 2-forms:

$$\wedge^2 M = \wedge^+ M \oplus \wedge^- M \quad .$$

Exactly in 4 dimensions the full Riemannian curvature tensor R_g can be regarded as a real linear map $R_g : \wedge^2 M \to \wedge^2 M$ which decomposes with respect to the splitting as

$$R_g = \begin{pmatrix} W_g^+ + \frac{s_g}{12} & B_g \\ B_g^* & W_g^- + \frac{s_g}{12} \end{pmatrix} \quad .$$

Then $R_g \in C^\infty(M; \mathrm{End}(\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C})) \subset \mathrm{End}(\Omega^2_c(M; \mathbb{C})) \subset \mathfrak{B}(M)$ provided M is compact.

The Einstein equation $r_g - \frac{1}{2}s_g g = 8\pi T - \Lambda_M g$ says

$$\begin{cases} B_g &= 8\pi T_0 \\ s_g &= 4\Lambda_M - 8\pi \operatorname{tr}_g T \end{cases}$$

where T_0 is the traceless part of the energy-momentum tensor and the vacuum $T = 0$ is equivalently characterized by the condition $B_g = 0$ i.e., R_g obeys the splitting. By the bundle splitting above

$$\Omega^2_c(M;\mathbb{C}) = \Omega^+_c(M;\mathbb{C}) \oplus \Omega^-_c(M;\mathbb{C})$$

inducing a canonical split Hilbert space completion

$$\overline{\Omega^2_c(M;\mathbb{C})} = \mathscr{H}^+(M) \oplus \mathscr{H}^-(M) \quad .$$

Hence the classical vacuum i.e., an Einstein 4-manifold (M, g) means that $R_g \in \mathfrak{B}(M)$ is a bounded linear operator acting on the canonically split Hilbert space induced by (M, g).

Gravity as algebraic quantum field theory

Take an oriented smooth 4-manifold M and consider the scalar product space $(\Omega^2_c(M;\mathbb{C})\,,\,\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(M)})$ with the adjoint operator $\divideontimes : \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \to \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$. Put

$$\begin{aligned} B(R) &:= \mathrm{e}^{R}, \quad R \in C^\infty_c(M; \mathrm{End}(\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C})); \\ B(X) &:= \mathrm{e}^{L_X}, \quad X \in C^\infty_c(M; TM). \end{aligned}$$

Pick a relatively compact open $\emptyset \subsetneqq U \subsetneqq M$ and write $\mathrm{Diff}^+_U(M) \subsetneqq \mathrm{Diff}^+(M)$ for the subgroup leaving U fixed (not pointwise). Let $\mathfrak{A}(U) \subset \mathfrak{B}(M)$ be the C^*-algebra generated by the $B(R)$'s and $B(X)$'s, more precisely

$$\mathfrak{A}(U) := \Big\langle B(R), B(X) \in \mathfrak{B}(M) \,\Big|\, \Big[B(R)|_{\Omega^2_c(M\setminus U;\mathbb{C})}, \mathrm{Diff}^+_U(M)\Big] = 0,$$

$$\Big[B(X)|_{\Omega^2_c(M\setminus U;\mathbb{C})}\,,\,\mathrm{Diff}^+_U(M)\Big] = 0\Big\rangle\,.$$

Definition

Let M be an oriented smooth 4-manifold, $\emptyset \subsetneqq U \subsetneqq M$ a relatively compact open subset. Then $\mathfrak{A}(U) \subset \mathfrak{B}(M)$ is the local generalized CCR algebra of M. Moreover $\mathfrak{A}(M)$ satisfies $\mathfrak{A}(M) = \varinjlim_{U \subset M} \mathfrak{A}(U)$ and is called the generalized CCR algebra of M.

Then our AQFT is defined by the assignment

$$\{U \longmapsto \mathfrak{A}(U)\}_{U \subsetneqq M}$$

giving rise to a covariant functor from the category of relative compact open subsets of M into the category of unital C^*-algebras.

Remark

(i) $\mathfrak{A}(U)$ contains (exponentiated) algebraic curvature tensors as well as $\mathrm{Diff}^+_U(M) \subsetneqq \mathrm{Diff}^+(M)$ by construction;

(ii) At least when $U \subsetneqq M$ is a coordinate ball, it also contains a usual CCR algebra generated by self-adjoint elements $B(R)$ and $B(X)$ such that all the R's and X's either commute with each other or form canonically conjugate pairs. R corresponds to **Q** while X to **P**.

(iii) This AQFT by construction possesses diffeomorphism group symmetry i.e.,

$$f^*(\mathfrak{A}(U))(f^{-1})^* = \mathfrak{A}(f(U))$$

for all $\emptyset \subsetneqq U \subsetneqq M$ and $f \in \mathrm{Diff}^+(M)$.

Definition

Take a local generalized CCR algebra $\mathfrak{A}(U)$. For a differentiable 1-parameter subgroup $\{A_t\}_{t\in\mathbb{R}} \subset \mathfrak{A}(U)$ with $A_0 = 1 \in \mathfrak{A}(U)$ a local observable of the infinitesimal form

$$Q := \left.\frac{\mathrm{d}A_t}{\mathrm{d}t}\right|_{t=0} \in T_1\mathfrak{A}(U)$$

is a called a local quantum gravitational field on $U \subsetneqq M$.
Take any split Hilbert space $\mathscr{H}^+(M) \oplus \mathscr{H}^-(M)$ (note that this breaks the diffeomorphism symmetry). The off-blockdiagonal part of Q is the material content of the local quantum gravitational field relative to the splitting. In particular Q is called a local quantum vacuum gravitational field relative to the splitting if its material content relative to the splitting vanishes i.e., $Q(\mathscr{H}^\pm(M) \cap D) \subsetneqq \mathscr{H}^\pm(M)$ on a dense $D \subsetneqq \mathscr{H}^+(M) \oplus \mathscr{H}^-(M)$.

Remark

Consequently the distinction between "matter" and "gravity" requires a choice $\mathscr{H}^+(M) \oplus \mathscr{H}^-(M)$ for $\overline{\Omega^2_c(M;\mathbb{C})}$ hence the diffeomorphism symmetry is lost!

The assignment $\{U \mapsto \mathfrak{A}(U)\}_{U \subseteqq M}$ satisfies the usual Haag–Kastler axioms of AQFT. However the causality axiom is satisfied only in a trivial sense as follows:

Lemma

Let M be an oriented smooth 4-manifold, $\emptyset \subseteqq U \subseteqq M$ open and $\{U \mapsto \mathfrak{A}(U)\}_{U \subseteqq M}$ any assignment of (non-commutative) C^*-algebras satisfying

$$f^*(\mathfrak{A}(U))(f^{-1})^* = \mathfrak{A}(f(U)) \qquad (1)$$

for all $f \in \mathrm{Diff}^+(M)$. Then $[\mathfrak{A}(U), \mathfrak{A}(V)] = 0$ if and only if $U \cap V = \emptyset$.

Proof. It follows from (1) that $\mathfrak{A}(U)$ must commute with those elements of $\mathrm{Diff}^+(M)$ which are the identity on U; hence if $A \in \mathfrak{A}(U) \subset \mathfrak{B}(M)$ then $A|_{\Omega^2_c(M\setminus U;\mathbb{C})} \in \mathfrak{Z}(\mathfrak{B}(M \setminus U)) = \mathbb{C}\,\mathrm{Id}_{\Omega^2_c(M\setminus U;\mathbb{C})}$. But $\Omega^2_c(V;\mathbb{C}) \subset \Omega^2_c(M \setminus U;\mathbb{C})$ if $U \cap V = \emptyset$. $\diamond$

Remark

Therefore the diffeomorphism symmetry kills causality hence dynamics. Consequently in this diffeomorphism-invariant quantum field theory a very elementary, pre-causal level of physical reality is exhibited. In order to obtain non-trivial causal structure the diffeomorphism symmetry must be broken (a distinction between "gravity" and "matter" does this job).

Representations

The generalized CCR algebra $\mathfrak{A}(M)$ admits several non-equivalent Hilbert space representations via the Gelfand–Najmark–Segal construction. These come in two main series:

(i) Classical representations π_ω on Hilbert spaces $\mathscr{H}_\omega$ provided by the functional $\Psi_\omega : \mathfrak{A}(M) \to \mathbb{C}$ of the form

$$\Psi_\omega(A) := \int_M \overline{\omega} \wedge A\omega$$

with $\omega \in \Omega^2(M;\mathbb{C})$ a non-degenerate 2-form along the whole M. The $\mathscr{H}_\omega$'s also carry unitary representations of finite dimensional subgroups $\mathrm{Iso}^+(M,g) \subset \mathrm{Diff}^+(M)$ interpreted as the isometry group of an emerging space-time (M,g) where g is a complex metric induced by ω;

(i) Quantum representations $\pi_{\Sigma,\omega}$ on Hilbert spaces $\mathscr{H}_{\Sigma,\omega}$ given by the functional $\Phi_{\Sigma,\omega} : \mathfrak{A}(M) \to \mathbb{C}$ of the form

$$\Phi_{\Sigma,\omega}(A) := \frac{1}{2\pi\mathbf{i}} \int\limits_{\Sigma} A\omega$$

where $\Sigma \subset M$ is a compact orientable surface and $\omega \in \Omega^2_c(M;\mathbb{C})$ is a non-degenerate 2-form along Σ. The $\mathscr{H}_{\Sigma,\omega}$'s also carry unitary representations of the full $\mathrm{Diff}^+(M)$.

Regarding the quantum representation more precisely we can state

Theorem

Take an oriented closed surface Σ. Let $(\Sigma, p_1, \dots, p_n)$ denote a generic smooth immersion $i : \Sigma \looparrowright M$ where the points $p_1, \dots, p_n \in \Sigma$ are the preimages of the double points of this immersion. Moreover take any d*-closed $\omega \in \Omega^2_c(M; \mathbb{C})$. Assume that*

(i) $\frac{1}{2\pi\mathbf{i}} \int_\Sigma \omega = 1$;

(ii) *ω is non-degenerate along Σ and for all complex structures $C = C(\Sigma)$ on Σ there exist positive definite unitary holomorphic vector bundle structures on the vector bundle $E := TM \otimes_{\mathbb{R}} \mathbb{C}|_C$ over $C \subset M$ compatible with ω such that* $\dim_{\mathbb{C}} H^0(C; \mathscr{O}(E)) = 4$.

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Then $(\Sigma, p_1, \dots, p_n, \omega)$ *gives rise to a so-called positive mass representation* $\pi_{\Sigma,\omega}$ *of* $\mathfrak{A}(M)$ *on a Hilbert space* $\mathscr{H}_{\Sigma,\omega}$ *as follows:*

(i) $\mathscr{H}_{\Sigma,\omega}$ *also carries a unitary representation* $U_{\Sigma,\omega}$ *of the group* $\mathrm{Diff}^+(M)$. *A vector* $v \in \mathscr{H}_{\Sigma,\omega}$ *satisfies* $U_{\Sigma,\omega}(f)v = v$ *for all* $f \in \mathrm{Diff}^+(M)$ *if and only if* $v = 0$ *("no vacuum");*

(ii) *On a dense subset of states* $0 \neq [A] \in \mathscr{H}_{\Sigma,\omega}$ *a complex 4-vector* $P_{C,\omega,A} \in H^0(C; \mathscr{O}(E))$ *can be defined together with its length* $m_{C,\omega,A} := \|P_{C,\omega,A}\|_{L^2(C)} \geqq 0$ *with respect to a natural Hermitian scalar product* $(\,\cdot\,,\,\cdot\,)_{L^2(C)}$ *on* $C^\infty(C;E)$. *It has the property that if* $[1] \in \mathscr{H}_{\Sigma,\omega}$ *is a state corresponding to vanishing algebraic curvature* $R = 0$ *then* $P_{C,\omega,1} = 0$ *hence* $m_{C,\omega,1} = 0$.

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The collection of all positive mass representations gives rise to a modular functor τ *which associates to every positive mass representation* $\pi_{\Sigma,\omega}$ *a finite dimensional complex vector space*

$$\tau(\Sigma, p_1, \ldots, p_n, \omega)$$

i.e., a conformal field theory in the sense of G. Segal. $\diamond$

Remark

Consequently 3 independent concepts meet up naturally in positive mass representations:

(i) Representations of the algebra of observables of a quantum field theory;

(ii) Quasilocal quantities of general relativity;

(iii) Conformal field theory *à la* G. Segal (modular functors).

Conclusion

We presented a diffeomorphism-invariant algebraic quantum field theory which exactly in 4 dimensions contains classical general relativity. In this theory the fundamental objects are the curvature tensors (regarded as bounded linear operators) instead of the metric. The presence of the giant diffeomorphism symmetry yields:

(i) "Matter" cannot be distinguished from "gravity";

(ii) Non-trivial causality hence dynamics cannot be introduced without breaking the diffeomorphism symmetry. In order to introduce these concepts the diffeomorphism symmetry must be cut down to some significantly smaller group.

For further details please check:

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http://www.math.bme.hu/~etesi/qgravity4.pdf

Thank you!