An operator algebraic characterization of the Riemannian vacuum Einstein equation in four dimensions

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Abstract

In this paper, using connected compact oriented smooth 4-manifolds, some representations of the hyperfinite II_1-type factor von Neumann algebra are constructed. The Murray–von Neumann coupling constant of these representations gives rise to a new smooth 4-manifold invariant whose very first properties are investigated.

Moreover as a part of this construction, a connected oriented smooth 4-manifold admits an embedding into the hyperfinite II_1 factor. This embedding, on the one hand, induces a Riemannian metric on the manifold such that its Riemannian curvature tensor, if appropriately bounded, belongs to the von Neumann algebra; on the other hand the metric induces a periodic dynamics on the von Neumann algebra, what we call the Hodge dynamics on the hyperfinite II_1 factor. It is observed that the metric is Einstein i.e., satisfies the (Riemannian) vacuum Einstein equation with a possibly non-zero cosmological constant, if and only if its Riemannian curvature tensor belongs to the fixed-point-subalgebra of the Hodge dynamics.

Finally, we make a comprehensive enumeration of all representations of the hyperfinite II_1 factor constructed here, from the viewpoint of thermal equilibrium states and phase transitions in algebraic quantum field theory.

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1 Introduction and summary

The *hyperfinite factor von Neumann algebra of type* II_1 is distinguished among von Neumann algebras in many senses. Apparently this was von Neumann's favourite operator algebra and he was especially satisfied with its discovery. As it is known (cf. [23, pp. 22-32] for a possible reconstruction of the story) he attempted, but finally did not complete or abandoned, to use the hyperfinite II_1 factor to bring quantum mechanics to a not only mathematically, but even conceptionally sound basis, by interpreting quantum probabilities as relative frequencies of a particular statistical ensemble sorted from an absolute

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one *a priori* given by this von Neumann algebra due to its unique normalized trace. The idea looked appealing not only by the uniqueness of the hyperfinite II_1 factor among operator algebras, but also because of its unexpectedly rich representation theory among factors. Indeed, compared with others the hyperfinite II_1 factor has a proliferation of representations: the moduli space of its non-equivalent representations is isomorphic to $\mathbb{R}_+$ (and accordingly, the II_1 factor is the only one among factors whose K_0 group is non-trivial, namely isomorphic to $\mathbb{R}$). However the existence of inequivalent representations, or in other words the failure of the *Stone–von Neumann representation theorem* in this case, is an indicator that the hyperfinite II_1 factor is an operator algebra of a physical system possessing *infinitely many* degrees of freedom, like a (macroscopic) quantum statistical ensemble. As was mentioned by von Neumann, but with some uncertainty, in e.g. [23, Letters to P. Jordan, December 11, 1949 and January 12, 1950], because of this property the hyperfinite II_1 factor might even play a role in (relativistic) quantum field theory; although the recent conviction is that rather algebras of type III appear here, cf. [16, Section V.6].

The aim of this paper is to take two walks around representation theory of the hyperfinite II_1 factor: one *mathematical* and one *physical* (of course these are not unrelated). The mathematical trip is a substantial extension of our previous efforts in [11, 12] and concerns the following problem: despite the existence of many non-trivial inequivalent representations of the hyperfinite II_1 factor, only one of them appears as "reasonable", namely its *standard representation*; of course one can say that this is the most important while other representations, if cyclic, arise in principle via the *Gelfand–Naimark–Segal construction*, and the rest are direct sums of these. However one can also say that this general description of representations is not too informative. Our first result toward constructing new representations connects the general theory of the hyperfinite II_1 factor with four dimensional differential geometry:

Theorem 1.1. *Let M be a connected oriented smooth 4-manifold. Making use of its smooth structure only, out of M a von Neumann algebra $\mathfrak{R}$ can be constructed which is geometric in the sense that it is generated by local operators, including all bounded complexified algebraic (i.e., formal or stemming from a metric) curvature tensors of M. Moreover $\mathfrak{R}$ itself is a hyperfinite factor of type II_1 hence is unique up to abstract isomorphisms of von Neumann algebras.*

Furthermore M admits an embedding $M \subset \mathfrak{R}$ via projections. Two 4-manifolds M, N with corresponding embeddings have abstractly isomorphic von Neumann algebras however not canonically. Nevertheless different abstract isomorphisms between these von Neumann algebras induce orientation-preserving diffeomorphisms of M and N respectively i.e. leave their embeddings unchanged. Hence up to diffeomorphisms, all connected oriented smooth 4-manifolds embed into a commonly given abstract von Neumann algebra $\mathfrak{R}$ which is the hyperfinite II_1 factor.

The occurence of the hyperfinite II_1 factor in low dimensional differential topology is not only an immense source for new representations, but even brings a smooth 4-manifold invariant to life:

Theorem 1.2. *Assuming that M above is moreover compact, its von Neumann algebra $\mathfrak{R}$ admits a non-faithful representation on a certain complex separable Hilbert space, such that the unitary equivalence class of this representation is invariant under orientation-preserving diffeomorphisms of M. Consequently the Murray–von Neumann coupling constant of this representation gives rise to a smooth 4-manifold invariant γ. This invariant takes values in the semi-open real interval $[0,1)$ more precisely $\gamma(M) = 1 - \frac{1}{x}$ where $x \in \left\{4\cos^2\left(\frac{\pi}{n}\right) \,\middle|\, n = 3,4,\dots\right\} \cup [4,+\infty)$ is an element of the set of Jones' finite subfactor indices. Moreover γ behaves like*

$$\gamma(M\#N) = \gamma(M) + \gamma(N) - \gamma(M)\gamma(N)$$

under taking connected sum.

Some further very immediate properties of γ will be elaborated in Lemmata 2.4, 2.5 and 2.6 below, however let us make some general comments already here. The set of Jones indices hence the spectrum of γ within $[0,1)$ splits into a discrete and a continuous part. Subfactors belonging to the discrete part $\{4\cos^2\left(\frac{\pi}{n}\right) \mid n \geqq 3\}$ have been completely classified [25] and in turn they follow an *ADE* pattern (with the odditiy that no subfactors corresponding to D_{2k+1} and E_7 exist) [24]. The set of subfactors belonging to the continuous portion $[4,+\infty)$ is however very wild and only partial results are known mainly for the subinterval $[4,5] \subset [4,+\infty)$ or a bit more (cf. e.g. [21] for an excellent survey and recent results while for some further extension cf. [1]). Concerning the impact of this division on γ, on the one hand for the (standard) 4-sphere $\gamma(\mathbb{S}^4) = 0$ hence $\frac{1}{1-\gamma(\mathbb{S}^4)} = 1$ belongs to the discrete range of the Jones' index spectrum. On the other hand, the connected sum formula exhibited in Theorem 1.2 here and some further *ad hoc* computations carried out in Section 2 strongly indicate that if a compact 4-manifold M has non-zero γ-invariant then $\frac{1}{1-\gamma(M)} \in [4,+\infty)$ i.e., the corresponding Jones index already belongs to the continuous range. This observation is a hint that smooth 4-manifolds might provide a rich reservoir of subfactors in the wild i.e., continuous index range; moreover poses the question whether or not smooth 4-manifolds distinct from $\mathbb{S}^4$ having invariant in the tame i.e., discrete range (like for instance the hypothetic exotic or fake $\mathbb{S}^4$'s, see Lemma 2.6 below) exist at all.

Next turning toward physics: a longstanding problem of contemporary theoretical physics is how to unify the obviously successful and mathematically consistent *theory of general relativity* with the obviously successful but yet mathematically problematic *relativistic quantum field theory*. It has been generally believed that these two fundamental pillars of modern theoretical physics are in a clash not only because of the different mathematical tools they use but are in tension even at a deep conceptional level: for instance classical notions of general relativity such as a space-time event, the light cone or the event horizon of a black hole are “too sharp” objects and the theory itself is “too non-linear” from a quantum theoretic viewpoint; whereas relativistic quantum field theory is not background independent from the aspect of general relativity.

The demand by general relativity summarized as the *principle of general covariance* is perhaps one of the two main obstacles why general relativity has remained outside of the mainstream classical and quantum field theoretic expansion in the 20^{th} century. Indeed, an implementation of this inherent principle of general relativity forces that a robust group, namely the full diffeomorphism group of the underlying space-time manifold must belong to the symmetry group of a field theory compatible with general relativity. However an unwanted consequence of the vast diffeomorphism symmetry is that it even allows one to transform time itself away from the theory (known as the “problem of time” in general relativity, see e.g. [30, Appendix E] for a technical presentation as well as e.g. [6, Chapter 2] and [18, Subsection 2.1] for a broader philosophical survey on this problem) making it problematic to apply standard canonical quantization methods—based on Hamiltonian formulation hence on an essential explicit reference to an ”auxiliary time”—in case of general relativity. The other reason is the as well in-built core idea, the *equivalence principle* which renders general relativity a strongly self-interacting classical field theory in the sense that precisely in four dimensions the “free” and the “interaction-with-itself” modes of the gravitational field have energetically the same magnitudes, obfuscating perturbative considerations. In fact the equivalence principle says that there is no way to make a physical distinction between these two modes of gravity. Heisenberg and Pauli were still optimistic concerning canonical and perturbative quantization of gravity with respect to a fixed time or, more generally, a reference or ambient space-time in their 1929 paper [19]; however these initial hopes quickly evaporated already in the 1930’s by recognizing the *essential impossibility* of quantizing general relativity via canonical quantization and exhibit it as a perturbatively renormalizable quantum field theory in a coherent way. This was clearly observed by Bronstein [5] first; as he wrote in his 1936 paper: “[...] the elimination of the logical inconsistencies [requires] rejection of our ordinary concepts of space and time, modifying them by some much deeper and nonevident concepts." (also cited by Smolin [28, p. 85]).

Roughly the thinking about gravity has split into two main branches since the 1950-60's [18]. The first older and more accepted direction postulates that gravity should be quantized akin to other fundamental forces but with more advanced methods including (super)string theoretic [17], Feynman integral, loop quantum gravity or some further techniques—or at least one should construct it as a low energy effective field theory of an unknown high energy theory; the other newer and yet less-accepted attitude declares that gravity is an emergent macroscopic phenomenon in the sense that it always involves a huge amount of physical degrees-of-freedom (beyond the obvious astronomical evidences, also supported by various theoretical discoveries during the 1970-80's such as Hawking's area theorem, black hole radiation, all resembling thermodynamics) hence is not subject to quantization at all. Nevertheless, as a matter of fact in the 2020's, we have to admit that an overall accepted quantum theory of gravity does not exist yet and even general relativity as a classical field theory persists to keep its conceptionally isolated position within current theoretical physics [18]. Perhaps it is worth mentioning here that general relativity receives further challenges from low dimensional differential topology too by recent discoveries which were unforeseenable earlier, cf. e.g. [10, Section 1] for a brief summary.

Strongly motivated by these well-known general incompatibility comments, in the aforementioned second i.e., physical trip around the hyperfinite II_1 factor, an operator algebraic characterization of the vacuum Einstein equation is obtained, which can be summarized as follows:

Theorem 1.3. *Let M be a connected oriented smooth 4-manifold and consider its embedding $M \subset \mathfrak{R}$ as in Theorem 1.1. This embeding induces a Riemannian structure (M, g) whose Riemannian curvature, if bounded, satisfies $R_g \in \mathfrak{R}$. Morever if $*$ denotes the Hodge star operating on $\Omega^2_c(M;\mathbb{C})$ then $* \in \mathfrak{R}$. It is self-adjoint and satisfies $*^2 = 1$ hence is unitary thus generates a periodic inner $\ast$-automorphism of $\mathfrak{R}$ rendering $\mathfrak{R}$ a so-called* Hodge dynamical system $\left(\mathfrak{R}, \{\mathrm{Ad}_{*^t}\}_{t\in\mathbb{R}}\right)$. *Finally, $M \subset \mathfrak{R}$ is preserved by the Hodge dynamics, more precisely it is part of its fixed-point-subalgebra, and (M, g) is Einstein if and only if R_g belongs to this fixed-point-subalgebra too.*

This result can be regarded as a sort of "linearization via complex numbers" of the highly non-linear and inherently real Einstein equation.

The paper is organized as follows. Section 2 contains detailed proofs of Theorems 1.1 and 1.2 and some further results concerning smooth 4-manifolds in Lemmata 2.4, 2.5 and 2.6. Then Section 3 is devoted to the proof of Theorem 1.3 as well as placing representation theory of the hyperfinite II_1 factor into the context of algebraic quantum field theory.

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2 Emergence of the hyperfinite II_1 factor

In this section for completeness and the Reader's convenience we recall and partly extend further the mathematical exposition in [12] and give a detailed proof of Theorems 1.1 and 1.2. First we shall exhibit a simple self-contained two-step construction of a von Neumann algebra attached to any oriented smooth 4-manifold. Then the structure of this algebra will be explored in some detail. Finally we exhibit a new (i.e. not the standard) representation of this von Neumann algebra induced by the whole procedure leading to a new smooth 4-manifold invariant whose properties are also examined. For clarity we emphasize that the forthcoming constructions are rigorous in the sense that *no* physical ideas, considerations, steps, etc. are used.

Construction of an algebra. Take the isomorphism class of a connected oriented smooth 4-manifold (without boundary) and from now on let M be a once and for all fixed representative in it carrying the action of its own orientation-preserving group of diffeomorphisms $\mathrm{Diff}^+(M)$. Among all tensor bundles $T^{(p,q)}M$ over M the 2^{nd} exterior power $\wedge^2 T^*M \subset T^{(0,2)}M$ is the only one which can be endowed with a pairing in a natural way i.e., with a pairing extracted from the smooth structure (and the orientation) of M alone. Indeed, consider its associated vector space $\Omega^2_c(M) := C^\infty_c(M; \wedge^2 T^*M)$ of compactly supported smooth 2-forms on M. Define a pairing $\langle \cdot\,,\cdot\rangle_{L^2(M)} : \Omega^2_c(M)\times\Omega^2_c(M)\to\mathbb{R}$ via integration:

$$\langle\varphi,\psi\rangle_{L^2(M)} := \int\limits_M \varphi\wedge\psi \tag{1}$$

and observe that this pairing is non-degenerate however is *indefinite* in general thus can be regarded as an indefinite scalar product on $\Omega^2_c(M)$. It therefore induces an indefinite real quadratic form Q on $\Omega^2_c(M)$ given by $Q(\varphi) := \langle\varphi,\varphi\rangle_{L^2(M)}$. Let $C(M)$ denote the complexification of the infinite dimensional real Clifford algebra associated with $(\Omega^2_c(M), Q)$. Because Clifford algebras are usually constructed out of definite quadratic forms, we summarize this construction [22, Section I.§3] to make sure that the resulting object $C(M)$ is well-defined i.e. is not sensitive for the indefiniteness of (1). To begin with, let $V_m \subset \Omega^2_c(M)$ be an m dimensional real subspace and assume that $Q_{r,s} := Q|_{V_m}$ has signature (r,s) on V_m that is, the maximal positive definite subspace of V_m with respect to $Q_{r,s}$ has dimension r while the dimension of the maximal negative definite subspace is s such that $r+s=m$ by the non-degeneracy of $Q_{r,s}$. Then out of the input data $(V_m, Q_{r,s})$ one constructs in the standard way a finite dimensional real Clifford algebra $C_{r,s}(M)$ with unit $1\in C_{r,s}(M)$ and an embedding $V_m\subset C_{r,s}(M)$ with the property $\varphi^2 = Q_{r,s}(\varphi)1$ for every element $\varphi\in V_m$. This real algebra depends on the signature (r,s) however fortunately its complexification $C_m(M) := C_{r,s}(M)\otimes\mathbb{C}$ is already independent of it. In fact, if $\mathfrak{M}_k(\mathbb{C})$ denotes the algebra of $k\times k$ complex matrices, then it is well-known [22, Section I.§3] that $C_0(M)\cong\mathfrak{M}_1(\mathbb{C})$ while $C_1(M)\cong\mathfrak{M}_1(\mathbb{C})\oplus\mathfrak{M}_1(\mathbb{C})$ and the higher dimensional cases follow from the complex periodicity $C_{m+2}(M)\cong C_m(M)\otimes\mathfrak{M}_2(\mathbb{C})$. Consequently depending on the parity $C_m(M)$ is isomorphic to either $\mathfrak{M}_{2^{\frac{m}{2}}}(\mathbb{C})$ or $\mathfrak{M}_{2^{\frac{m-1}{2}}}(\mathbb{C})\oplus\mathfrak{M}_{2^{\frac{m-1}{2}}}(\mathbb{C})$. These imply that 2-step-chains of successive embeddings of real subspaces $V_m\subset V_{m+1}\subset V_{m+2}\subset\Omega^2_c(M)$ starting with $V_0=\{0\}$ and given by iterating $\omega\mapsto\binom{\omega}{0}$ provide us with injective algebra homomorphisms $\mathfrak{M}_{2^{\frac{m}{2}}}(\mathbb{C})\hookrightarrow\mathfrak{M}_{2^{\frac{m}{2}+1}}(\mathbb{C})$ having the shape $A\mapsto\begin{pmatrix}A&0\\0&A\end{pmatrix}$. Therefore $C(M)$ is isomorphic to the injective limit of this directed system, that is there exists a linear algebraic isomorphism

$$C(M)\cong\bigcup_{n=0}^{+\infty}\mathfrak{M}_{2^n}(\mathbb{C}) \tag{2}$$

or equivalently

$$C(M)\cong\mathfrak{M}_2(\mathbb{C})\otimes\mathfrak{M}_2(\mathbb{C})\otimes\dots$$

because this injective limit is also isomorphic to the infinite tensor product of $\mathfrak{M}_2(\mathbb{C})$'s. For clarity note that being (1) a non-local operation, $C(M)$ is a genuine global infinite dimensional object.

It is well-known (cf. [8, Section I.3]) that *any* complexified infinite Clifford algebra like $C(M)$ above generates *the* II_1 type hyperfinite factor von Neumann algebra. Let us summarize this procedure too (cf. [2, Section 1.1.6]). It readily follows that $C(M)$ possesses a unit $1 \in C(M)$ and its center comprises the scalar multiples of the unit only. Moreover $C(M)$ continues to admit a canonical embedding $\Omega^2_c(M;\mathbb{C}) \subset C(M)$ satisfying $\omega^2 = Q(\omega)1$ where now Q denotes the quadratic form induced by the complex-bilinear extension of (1). We also see via (2) already that $C(M)$ is a complex $\circledast$-algebra whose $\circledast$-operation (provided by taking Hermitian matrix transpose, a non-local operation) is written as $A \mapsto A^{\circledast}$. The isomorphism (2) also shows that if $A \in C(M)$ then one can pick the smallest $n \in \mathbb{N}$ such that $A \in \mathfrak{M}_{2^n}(\mathbb{C})$ consequently A has a finite trace defined by $\tau(A) := 2^{-n}\mathrm{Trace}(A)$ i.e., taking the usual normalized trace of the corresponding $2^n \times 2^n$ complex matrix. It is straightforward that $\tau(A) \in \mathbb{C}$ does not depend on n. We can then define a sesquilinear inner product on $C(M)$ by $(A,B) := \tau(AB^{\circledast})$ which is non-negative and non-degenerate thus the completion of $C(M)$ with respect to the norm $\|\cdot\|$ induced by $(\,\cdot\,,\,\cdot\,)$ renders $C(M)$ a complex Hilbert space what we shall write as $\mathscr{H}$ and its Banach algebra of all bounded linear operators as $\mathfrak{B}(\mathscr{H})$. Multiplication in $C(M)$ from the left onitself is continuous hence gives rise to a representation $\pi : C(M) \to \mathfrak{B}(\mathscr{H})$. Finally our central object effortlessly emerges as the weak closure of the image of $C(M)$ under π within $\mathfrak{B}(\mathscr{H})$ or equivalently, by referring to von Neumann's bicommutant theorem [2, Theorem 2.1.3] we put

$$\mathfrak{R} := (\pi(C(M)))'' \subset \mathfrak{B}(\mathscr{H}) \ .$$

This von Neumann algebra of course admits a unit $1 \in \mathfrak{R}$ moreover continues to have trivial center i.e., is a factor. Moreover by construction it is hyperfinite. The trace τ as defined extends from $C(M)$ to $\mathfrak{R}$ and satisfies $\tau(1) = 1$. Moreover [2, Proposition 4.1.4] this trace is unique on $\mathfrak{R}$. Likewise we obtain by extension a representation $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$. Observe that here we have constructed $\mathscr{H}$ as a completion of $(C(M),\tau)$ however the same $\mathscr{H}$ arises if taking the completion of $(\mathfrak{R},\tau)$. Hence the two kinds of completions $\mathfrak{R}$ and $\mathscr{H}$ of one and the same object $C(M)$ in fact form an increasing chain $C(M) \subset \mathfrak{R} \subset \mathscr{H}$ as complex (complete) vector spaces. Thus the canonical inclusion $\Omega^2_c(M;\mathbb{C}) \subset C(M)$ recorded above automatically extends to inclusions $\Omega^2_c(M;\mathbb{C}) \subset \mathfrak{R} \subset \mathscr{H}$ too. Given $A, B \in \mathfrak{R}$ we shall write $A \in \mathfrak{R}$ but $\hat{B} \in \mathscr{H}$ from now on as usual. This is necessary since $\mathfrak{R}$ and $\mathscr{H}$ are very different for example as $\mathrm{U}(\mathscr{H})$-modules: given a unitary operator $V \in \mathrm{U}(\mathscr{H})$ then $A \in \mathfrak{R}$ is acted upon as $A \mapsto VAV^{-1}$ but $\hat{B} \in \mathscr{H}$ transforms as $\hat{B} \mapsto V\hat{B}$. Using this notation the trace always can be written as a scalar product with the image of the unit in $\mathscr{H}$ that is, for every $A \in \mathfrak{R}$ we have

$$\tau(A) = (\hat{A}, \hat{1})$$

yielding a general and geometric expression for the trace.

Exploring the algebra $\mathfrak{R}$. Before proceeding further let us make a digression here to gain a better picture. This is desirable because taking the weak closure like $\mathfrak{R}$ of some explicitly known structure like $C(M)$ often involves a sort of loosing control over the latter. Nevertheless we already know promisingly that $\mathfrak{R}$ is a hyperfinite factor von Neumann algebra of II_1 type. Let us now exhibit some of its elements.

1. Our first examples are the 2-forms themselves as it follows from the already mentioned canonical embedding $\Omega^2_c(M;\mathbb{C}) \subset C(M)$ combined with $C(M) \subset \mathfrak{R}$. This also implies that in fact $\mathfrak{R}$ is weakly generated by $1 \in C(M)$ and all finite products of 2-forms $\omega_1\omega_2\dots\omega_n$ within the associative algebra $C(M)$(and likewise, $\mathscr{H}$ is the closure of the unit and all finite products too). We might call this as the *first picture* on $\mathfrak{R}$ provided by the embedding $\Omega^2_c(M;\mathbb{C}) \subset C(M)$ however this description is not very informative.

2. To see more examples, let us return to the Clifford algebra in (2) for a moment. We already know that there exists a canonical embedding $\Omega^2_c(M;\mathbb{C}) \subset C(M)$. In addition to this let us find a Clifford module for $C(M)$. Consider again any finite *even* dimensional approximation $C_m(M) = C_{r,s}(M) \otimes \mathbb{C}$ constructed from $(V_m, Q_{r,s})$ where now $V_m \subset \Omega^2_c(M)$ is a real even $m = r+s$ dimensional subspace.

Choose any $2^{\frac{m}{2}}$ dimensional complex vector subspace S_m within $\Omega^2_c(M;\mathbb{C})$. If $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ denotes the associative algebra of *all* $\mathbb{C}$-linear transformations of $\Omega^2_c(M;\mathbb{C})$ then $S_m \subset \Omega^2_c(M;\mathbb{C})$ induces an embedding $\mathrm{End}\, S_m \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ moreover we know that $\mathrm{End}\, S_m \cong \mathfrak{M}_{2^{\frac{m}{2}}}(\mathbb{C}) \cong C_m(M)$. Therefore we obtain a non-canonical inclusion $C_m(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ for every fixed $m \in 2\mathbb{N}$. Furthermore $S_m \subset S_{m+2} \subset \Omega^2_c(M;\mathbb{C})$ given by $\omega \mapsto \binom{\omega}{0}$ induces a sequence $C_m(M) \subset C_{m+2}(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ for Clifford algebras which is compatible with the previous ascending chain of their matrix algebra realizations. Consequently taking the limit $m \to +\infty$ we come up with a non-canonical injective linear-algebraic homomorphism

$$C(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \tag{3}$$

and this embedding gives rise to the *second picture* on $\mathfrak{R}$. Of course, unlike the first picture above, this second one does not exist in the finite dimensional case.

Although the $\circledast$-algebra $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ is yet too huge, we can at least exhibit some of its elements. The simplest ones are the 2-forms themselves because $\Omega^2_c(M;\mathbb{C}) \subset C(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ holds as we already know. Furthermore orientation-preserving diffeomorphisms act $\mathbb{C}$-linearly on $\Omega^2_c(M;\mathbb{C})$ via pullbacks thus we conclude that $\mathrm{Diff}^+(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$. Likewise $\mathrm{Lie}(\mathrm{Diff}^+(M)) \cong C^\infty_c(M;TM)$ consisting of compactly supported real vector fields acts $\mathbb{C}$-linearly on $\Omega^2_c(M;\mathbb{C})$ through Lie derivatives hence we also find that $\mathrm{Lie}(\mathrm{Diff}^+(M)) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$.

Moreover $C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes \mathbb{C})) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ i.e. the bundle or in other words pointwisely defined endomorphisms are also included. As an introductory observation note that in case of such elements all operations in $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ stem from the corresponding pointwise operations i.e. if $A,B \in C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes \mathbb{C}))$ and $x \in M$ therefore $A_x, A^*_x, B_x \in \mathrm{End}(\wedge^2 T^*_x M \otimes \mathbb{C})$ then $(AB)_x = A_x B_x$ and $A^{\circledast}_x = A^*_x$. Operators of this kind are important because they allow to make a contact with local *four dimensional* differential geometry.[1] A peculiarity of four dimensions is that the space $C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes \mathbb{C}))$ contains curvature tensors (more precisely their complex linear extensions) on M. If (M,g) is an oriented Riemannian 4-manifold then its Riemannian curvature tensor R_g is indeed a member of this subalgebra: with respect to the splitting of complexified 2-forms into their (anti)self-dual parts its complex linear extension looks like (cf. [27])

$$R_g = \begin{pmatrix} \frac{1}{12}\mathrm{Scal} + \mathrm{Weyl}^+ & \mathrm{Ric}_0 \\ \mathrm{Ric}^*_0 & \frac{1}{12}\mathrm{Scal} + \mathrm{Weyl}^- \end{pmatrix} \quad : \quad \begin{matrix} \Omega^+_c(M;\mathbb{C}) \\ \bigoplus \\ \Omega^-_c(M;\mathbb{C}) \end{matrix} \longrightarrow \begin{matrix} \Omega^+_c(M;\mathbb{C}) \\ \bigoplus \\ \Omega^-_c(M;\mathbb{C}) \end{matrix} \tag{4}$$

hence is a self-adjoint operator $R^{\circledast}_g = R_g$. More generally, $C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes \mathbb{C}))$ contains the complex linear extensions of all algebraic (i.e. formal only, not stemming from a metric) curvature tensors R over M.

How to decide whether or not these elements of $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ belong to $\mathfrak{R}$? The key concept here is the trace. Compared with the above trace expression $\tau(A) = (\hat{A}, \hat{1})$ generally valid on $\mathfrak{R}$, more specific trace formulata are obtained if M is endowed with a normalized Riemannian metric g i.e., the corresponding volume form $\mu_g = *1$ satisfies $\int_M \mu_g = 1$. The unique sesquilinear extension of g induces a positive definite sesquilinear L^2-scalar product

$$(\varphi, \psi)_{L^2(M,g)} := \int\limits_{x \in M} g(\varphi_x, \psi_x)\mu_g(x) = \int\limits_M \varphi \wedge *\overline{\psi}$$

[1] In fact all the constructions so far work for an arbitrary oriented and smooth $4k$-manifold with $k = 0,1,2,\dots$ (note that in $4k+2$ dimensions the indefinite pairing (1) gives rise to a symplectic structure on $2k+1$-forms).

on $\Omega^2_c(M;\mathbb{C})$. If $\{\varphi_1,\dots,\varphi_{2^n}\}_{n=1,2,\dots}$ is an ascending orthonormal sub-base sequence in $\Omega^2_c(M;\mathbb{C})$ then it readily follows that the trace of any $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ formally looks like

$$\tau(B)=\lim_{n\to+\infty}\frac{1}{2^n}\sum_{i=1}^{2^n}(B\varphi_i\,,\,\varphi_i)_{L^2(M,g)} \tag{5}$$

and, if exists, is independent of the frame used. Obviously $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap C(M)$ if and only if the sum on the right hand side is constant after finitely many terms; and an inspection of this trace expression at finite stages shows that in general $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ if and only if $\tau(B)$ exists.[2] As a consequence note that $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ is already independent of the particular inclusion (3).

An example: $\Phi\in\mathrm{Diff}^+(M)$ acts on $\Omega^2_c(M;\mathbb{C})$ via pullback $(\Phi^{-1})^*$ but this action necessarily induces a $\mathbb{C}$-linear inner $*$-automorphism on $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ too consequently $(\Phi^{-1})^*$ must be unitary hence $|\tau((\Phi^{-1})^*)|=1$. Thus due to finiteness of its trace it extends to $\mathscr{H}$ (cf. Footnote 2) as a unitary operator hence $\mathrm{Diff}^+(M)\subset\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$. Another example: likewise if $X\in C^\infty_c(M;TM)$ and its corresponding Lie derivative L_X operating on 2-forms satisfies $\tau(L_X)<+\infty$ then it extends to $\mathscr{H}$ (cf. again Footnote 2) such that $L_X\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$.

Furthermore the curvature R_g of (M,g) as an operator in (4) acts on $\Omega^2_c(M;\mathbb{C})$. If in addition it is bounded which means that

$$\sup_{\|\omega\|_{L^2(M,g)}=1}\|R_g\omega\|_{L^2(M,g)}\leqq K<+\infty$$

then

$$0\leqq|\tau(R_g)|\leqq\lim_{n\to+\infty}\frac{1}{2^n}\sum_{i=1}^{2^n}|(R_g\varphi_i\,,\,\varphi_i)_{L^2(M,g)}|\leqq\lim_{n\to+\infty}\frac{1}{2^n}\sum_{i=1}^{2^n}\|R_g\varphi_i\|_{L^2(M,g)}\leqq K$$

thus extends over $\mathscr{H}$ (cf. again Footnote 2) and satisfies $R_g\in C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))\cap\mathfrak{R}$ and more generally any $R\in C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))\cap\mathfrak{R}$ if it is bounded.

Actually when $R\in C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))$ the previous trace formula can be further specified because one can compare the global trace $\tau(R)$ and the local trace function $x\mapsto\mathrm{tr}(R_x)$ given by the pointwise traces of the local operators $R_x:\wedge^2T^*_xM\otimes\mathbb{C}\to\wedge^2T^*_xM\otimes\mathbb{C}$ at every $x\in M$. Recall that $\mathfrak{R}$ has been constructed as the weak closure of the Clifford algebra (2). In fact [2, Section 1.1.6] the universality of $\mathfrak{R}$ permits to obtain it from other matrix algebras too, like for instance from $\bigcup\limits_{n=0}^{+\infty}\mathfrak{M}_{6^n}(\mathbb{C})$ whose weak closure therefore is again $\mathfrak{R}$. By the aid of this altered construction we can formally start with

$$\tau(R)=\lim_{n\to+\infty}\frac{1}{6^n}\sum_{i=1}^{6^n}(R\varphi_i\,,\,\varphi_i)_{L^2(M,g)}\ .$$

Fix $n\in\mathbb{N}$, write $M_n:=\bigcap\limits_{i=1}^{6^n}\mathrm{supp}\varphi_i\subseteqq M$ and take a point $x\in M_n$. Since $\dim_{\mathbb{C}}(\wedge^2T^*_xM\otimes\mathbb{C})=\binom{4}{2}=6$ the maximal number of completely disjoint linearly independent sub-6-tuples in $\{\varphi_{1,x},\varphi_{2,x},\dots,\varphi_{6^n,x}\}$

[2] By definition $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ contains those operators in $\mathfrak{R}\subset\mathfrak{B}(\mathscr{H})$ which are defined on the whole $\mathscr{H}$ but map $\Omega^2_c(M;\mathbb{C})\subset\mathscr{H}$ into itself; obviously such operators have finite trace. Conversely, given $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ we may try to extend it from $\Omega^2_c(M;\mathbb{C})$ to $\mathscr{H}$ step-by-step as follows. For a fixed n take the sub-basis $\{\varphi_1,\dots,\varphi_{2^n}\}$ in $\Omega^2_c(M;\mathbb{C})$ and consider the restriction B_n to the corresponding 2^n dimensional complex subspace; then iterating $B_n\mapsto\begin{pmatrix}B_n&0\\0&B_n\end{pmatrix}$ embed it into $C(M)$ and define the action of B_n on $\mathscr{H}$ by continuously extending over $\mathscr{H}$ the multiplication from the left on $C(M)\subset\mathscr{H}$ with its image; it is clear that $B_n\to B$ weakly as $n\to+\infty$ yielding a well-defined action of B on $\mathscr{H}$. This is indeed and extension of the action of B on $\Omega^2_c(M;\mathbb{C})$ to $\mathscr{H}$ simply because of (3); moreover $B\in\mathfrak{R}$ if $\tau(B)<+\infty$ i.e. in this case $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$.

is equal to $\frac{6^n}{6} = 6^{n-1}$. Moreover it follows from Sard's lemma that with a generic smooth choice for $\{\varphi_1, \varphi_2, \dots\}$ the subset of those points $y \in M_n$ where this number is less than 6^{n-1} has measure zero in M_n with respect to the measure μ_g. Consequently

$$\sum_{i=1}^{6^n} (R\varphi_i, \varphi_i)_{L^2(M_n,g)} = \int\limits_{x\in M_n} \sum_{i=1}^{6^n} g(R_x\varphi_{i,x}, \varphi_{i,x})\, \mu_g(x) = 6^{n-1} \int\limits_{x\in M_n} \mathrm{tr}(R_x)\mu_g(x) \ .$$

Since $\{\varphi_1, \varphi_2, \dots\}$ is a basis in $\Omega^2_c(M;\mathbb{C})$ therefore $M \setminus \bigcup\limits_{n=0}^{+\infty} M_n$ has measure zero as well we can let $n \to +\infty$ to end up with

$$\tau(R) = \frac{1}{6} \int\limits_M \mathrm{tr}(R)\mu_g \tag{6}$$

and observe that τ in this form is nothing else than the generalization of the total scalar curvature of a Riemannian manifold. Moreover if and only if (6) exists R extends to $\mathscr{H}$ (cf. Footnote (2) as usual) and gives $R \in C^\infty(M; \mathrm{End}(\wedge^2 T^*M \otimes \mathbb{C})) \cap \mathfrak{R}$. So if we start with (3) i.e. the *second picture* we can use several useful tracial criteria for checking whether or not an operator in $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ extends to an operator in $\mathfrak{R}$.

3. Now we are ready to exhibit an especially important class of elements in $\mathfrak{R}$ through a natural regular embedding of any connected oriented smooth 4-manifold M into its $\mathfrak{R}$ by the aid of the *first picture* as follows. To every sufficiently nice closed subset $\emptyset \subsetneqq X \subsetneqq M$ there exists an associated linear subspace $\Omega^2_c(M,X;\mathbb{C}) \subset \Omega^2_c(M;\mathbb{C}) \subset C(M) \subset \mathscr{H}$ consisting of compactly supported smooth 2-forms vanishing at least along X. In this way to every point $x \in M$ one can attach a closed subspace $\mathscr{V}_x \subset \mathscr{H}$ provided by the corresponding local Clifford algebra $C(M,x) \subset C(M)$ having the structure

$$C(M,x) = \mathbb{C}1 + \Omega^2_c(M,x;\mathbb{C}) + \Omega^2_c(M,x;\mathbb{C})\Omega^2_c(M,x;\mathbb{C}) + \dots \tag{7}$$

and taking its closure within $\mathscr{H}$. Let $P_x : \mathscr{H} \to \mathscr{V}_x$ be the corresponding orthogonal projection. Observe that *a priori* $P_x \in \mathfrak{B}(\mathscr{H})$ however in fact $P_x \in \mathfrak{R}$. Indeed, with respect to $\mathscr{H} = \mathscr{V}_x \oplus \mathscr{V}_x^\perp$ there is an induced decomposition $C(M) = \left(\begin{smallmatrix} C(M,x) & * \\ * & * \end{smallmatrix}\right)$ hence simply $P_x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in C(M) \subset \mathfrak{R}$. It readily follows that the resulting map

$$i_M : M \longrightarrow \mathfrak{R} \tag{8}$$

defined by $x \mapsto P_x$ is injective and continuous in the norm topology consequently gives rise to a continuous embedding of M into $\mathfrak{R}$ via projections.

Having understood $M \subset \mathfrak{R}$ for a given 4-manifold let us compare these embeddings for different spaces. So let M, N be two connected oriented smooth 4-manifolds and consider their corresponding embeddings into their von Neumann algebras via (8) respectively. Regardless what M or N are, their abstractly given algebras are both hyperfinite factors of II_1 type, therefore these latter objects are isomorphic [2, Theorem 11.2.2] however not in a canonical fashion. Indeed, if $F' : \mathfrak{R} \to \mathfrak{R}$ is an abstract isomorphism between the $\mathfrak{R}$'s for M and N respectively then any other abstract isomorphism between them has the form $F'' = \beta^{-1}F'\alpha$ where α and β are $\mathbb{C}$-linear $*$-automorphisms (in short from now on: automorphisms) of the abstractly given $\mathfrak{R}$ for M and the abstractly given $\mathfrak{R}$ for N respectively.

Therefore to understand the freedom how operator algebras for different 4-manifolds are identified we have to understand automorphisms of $\mathfrak{R}$. First, let us see how inner automorphisms of the weakly dense subalgebra $C(M)$ look like. Taking into account (3) all inner automorphisms of $C(M)$ are simply conjugations with appropriate elements of $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$ i.e., the group of all invertible $\mathbb{C}$-linear transformations of $\Omega^2_c(M;\mathbb{C})$. If $x \in M$ recall that $\Omega^2_c(M,x;\mathbb{C}) \subset \Omega^2_c(M;\mathbb{C})$ is a complex subspace and it is easy to show by invertability that subspaces of this kind are permuted by elements of $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$. Consequently any member of $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$ induces an orientation-preserving diffeomorphism Φ of M and a fiberwise $\mathbb{C}$-linear diffeomorphism f of $\wedge^2T^*M\otimes\mathbb{C}$ such that

$$\begin{array}{ccc} \wedge^2T^*M\otimes\mathbb{C} & \xrightarrow{\ f\ } & \wedge^2T^*M\otimes\mathbb{C} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ M & \xrightarrow{\ \Phi\ } & M \end{array}$$

is commutative i.e. yields a bundle isomorphism of $\wedge^2T^*M\otimes\mathbb{C}$. Conversely it is straightforward that every $\mathbb{C}$-linear bundle isomorphism of $\wedge^2T^*M\otimes\mathbb{C}$ gives rise to an element of $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$. Thus there exists an isomorphism of groups $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))\cong\mathrm{Iso}(\wedge^2T^*M\otimes\mathbb{C})$. Consider the short exact sequence $1\to\mathscr{G}(\wedge^2T^*M\otimes\mathbb{C})\to\mathrm{Iso}(\wedge^2T^*M\otimes\mathbb{C})\to\mathrm{Diff}^+(M)\to 1$ involving the fiberwise $\mathbb{C}$-linear automorphism group (the gauge group) $\mathscr{G}(\wedge^2T^*M\otimes\mathbb{C})\cong C^\infty(M;\mathrm{Aut}(\wedge^2T^*M\otimes\mathbb{C}))$ the global automorphism group $\mathrm{Iso}(\wedge^2T^*M\otimes\mathbb{C})\cong\mathrm{Aut}(C^\infty(M;\wedge^2T^*M\otimes\mathbb{C}))=\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$, both of the vector bundle $\wedge^2T^*M\otimes\mathbb{C}$, and the diffeomorphism group $\mathrm{Diff}^+(M)$ of the underlying space M respectively. This short exact sequence can therefore be re-written as

$$1\longrightarrow C^\infty(M;\mathrm{Aut}(\wedge^2T^*M\otimes\mathbb{C}))\longrightarrow\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))\longrightarrow\mathrm{Diff}^+(M)\longrightarrow 1\ . \tag{9}$$

In addition the map $\Phi\mapsto(\Phi^{-1})^*$ gives rise to a group injection $\mathrm{Diff}^+(M)\to\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$ whose composition with the projection $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))\to\mathrm{Diff}^+(M)$ is the identity consequently (9) can be supplemented to

$$1\longrightarrow C^\infty(M;\mathrm{Aut}(\wedge^2T^*M\otimes\mathbb{C}))\longrightarrow\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))\rightleftarrows\mathrm{Diff}^+(M)\longrightarrow 1$$

implying a splitting $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))=C^\infty(M;\mathrm{Aut}(\wedge^2T^*M\otimes\mathbb{C}))\rtimes\mathrm{Diff}^+(M)$ as a semi-direct product. Thus an inner automorphism of $C(M)$ as a conjugation with a suitable element of $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$ admits a unique decomposition $\mathrm{Ad}_\gamma\mathrm{Ad}_{(\Phi^{-1})^*}$ where γ is a $\mathbb{C}$-linear gauge transformation of the bundle $\wedge^2T^*M\otimes\mathbb{C}$ hence leaves M pointwise fixed, and Φ is a diffeomorphism of M which also preserves M as a whole. It then readily follows that the embedding $M\subset\mathfrak{R}$ given by (8) is preseved by this transformation. Next, consider the strong*-topology on $\mathfrak{B}(\mathscr{H})$. Taking $\mathfrak{R}\subset\mathfrak{B}(\mathscr{H})$ it is clear that $C(M)\subset\mathfrak{R}$ is a dense subalgebra; moreover, there exists an inclusion $\mathrm{Aut}\,\mathfrak{R}\subset\mathfrak{B}(\mathscr{H})$ for the full automorphism group too, and it is known [26, Theorem 4] that the subgroup of inner automorphisms is also dense within $\mathrm{Aut}\,\mathfrak{R}$. Consequently, any automorphism α of $\mathfrak{R}$ arises as

$$\alpha=\lim_i\left(\mathrm{Ad}_{\gamma_i}\mathrm{Ad}_{(\Phi_i^{-1})^*}\right) \tag{10}$$

where the limit is taken in the strong*-topology on $\mathfrak{B}(\mathscr{H})$, demonstrating that α also preserves $M\subset\mathfrak{R}$. Likewise, $\mathrm{Ad}_\delta\mathrm{Ad}_{(\Psi^{-1})^*}$ is the shape of an inner automorphism of $C(N)$ and a generic automorphism β arises as strong*-limit of them. These make sure that given an abstract isomorphism $F':\mathfrak{R}\to\mathfrak{R}$ between the von Neumann algebras constructed for M and N respectively then any other abstract isomorphism can be expressed as $F''=\lim\limits_i\left(\mathrm{Ad}_{\Psi_i^*}(\mathrm{Ad}_{\delta_i^{-1}}F'\mathrm{Ad}_{\gamma_i})\mathrm{Ad}_{(\Phi_i^{-1})^*}\right)$ between them. Consequently abstract isomorphisms between a pair of abstractly given von Neumann algebras differ only by automorphisms which preserve their underlying 4-manifolds as embedded within their algebras via (8) respectively; i.e. differences between identifications are inessential in this sense. Our overall conclusion therefore is that *up to diffeomorphisms every connected oriented smooth* 4*-manifold M admits an embedding into a commonly given abstract von Neumann algebra $\mathfrak{R}$ via (8).*

4. We close the partial comprehension of $\mathfrak{R}$ with an observation regarding its general structure. The shape of (9) at the Lie algebra level looks like

$$0 \longrightarrow C^\infty(M;\mathrm{End}(\wedge^2 T^*M\otimes\mathbb{C})) \longrightarrow \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \longrightarrow \mathrm{Lie}(\mathrm{Diff}^+(M)) \longrightarrow 0\ .$$

We already have an embedding (3). In addition to this there exists an isomorphism of Lie algebras $L : C^\infty_c(M;TM) \to \mathrm{Lie}(\mathrm{Diff}^+(M))$ such that $X \mapsto L_X$ is nothing but taking Lie derivative with respect to a compactly supported real vector field where the first-order $\mathbb{C}$-linear differential operator L_X is supposed to act on 2-forms hence $\mathrm{Lie}(\mathrm{Diff}^+(M)) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ as we know already too. Therefore the intersection of this sequence with the embedding $C(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ from (3) is meaningful and gives

$$0 \longrightarrow C^\infty(M;\mathrm{End}(\wedge^2 T^*M\otimes\mathbb{C}))\cap C(M) \longrightarrow C(M) \longrightarrow \mathrm{Lie}(\mathrm{Diff}^+(M))\cap C(M) \longrightarrow 0\ .$$

The second term consists of fiberwise algebraic hence local operators (including e.g. algebraic curvature tensors) having finite trace (6) and likewise the fourth term consists of Lie derivatives having finite trace via (5) thus belongs to the class of local operators too. Since the vector spaces underlying $C(M)$ considered either as an associative or a Lie algebra are isomorphic we conclude, as an important structural observation, that the overall construction here is geometric in the sense that *the algebra* $\mathfrak{R}$ *is generated by local operators*.

Summing up our findings so far. Given a connected oriented smooth 4-manifold M there exists a hyperfinite factor von Neumann algebra of II_1 type $\mathfrak{R}$ associated to M such that the solely input in its construction has been the pairing (1). Hence $\mathfrak{R}$ depends only on the orientation and the smooth structure of M. It contains, certainly among many other non-geometric operators, the space M itself as projections, its orientation-preseving diffeomorphisms as well as the space of bounded algebraic curvature tensors. Nevertheless $\mathfrak{R}$ is geometric in the sense that it is generated by M's local operators alone. It is remarkable that despite the *plethora* of smooth 4-manifolds detected since the early 1980's their associated von Neumann algebras here are unique offering a sort of justification terming $\mathfrak{R}$ as "universal". Moreover one is permitted to say that every connected oriented smooth 4-manifold M (perhaps together with its curvature tensor) embeds up to diffeomorphisms hence in a functorial way into a *common* $\mathfrak{R}$ and to look upon this von Neumann algebra as a natural common non-commutative space generalization of all oriented smooth 4-manifolds (or all 4-geometries). This universality also justifies the simple notation $\mathfrak{R}$ used throughout the text.

Proof of Theorem 1.1: Putting together all considerations so far the theorem is proved. □

Representations of $\mathfrak{R}$ *and a new smooth* 4*-manifold invariant.* The next lemmata closely follow [12, Lemmata 2.1-2.4] but with substantially improved constructions and extended contents.

Lemma 2.1. *Let M be a connected compact oriented smooth* 4*-manifold and* $\mathfrak{R}$ *its von Neumann algebra with trace* τ *as before. Then there exists a complex separable Hilbert space* $\mathscr{I}(M)^\perp$ *and a representation* $\rho_M : \mathfrak{R} \to \mathfrak{B}(\mathscr{I}(M)^\perp)$ *with the following properties. If* $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$ *is the representation constructed above then* $0 \subseteqq \mathscr{I}(M)^\perp \subsetneqq \mathscr{H}$ *and* $\rho_M = \pi|_{\mathscr{I}(M)^\perp}$ *holds.*

Moreover the unitary equivalence class of ρ_M *is invariant under orientation-preserving diffeomorphisms of M. Thus the Murray–von Neumann coupling constant*[3] *of* ρ_M *is invariant under orientation-preserving diffeomorphisms. Writing* $P_M : \mathscr{H} \to \mathscr{I}(M)^\perp$ *for the orthogonal projection and taking into account the characterization of* $\mathscr{I}(M)^\perp$ *the coupling constant is equal to* $\tau(P_M) \in [0,1)$ *hence* $\gamma(M) := \tau(P_M)$ *is a smooth* 4*-manifold invariant. Consequently* $\gamma(M) \in [0,1)$ *holds.*

[3] Also called the $\mathfrak{R}$-*dimension* of a left $\mathfrak{R}$-module hence denoted $\dim_{\mathfrak{R}}$, cf. [2, Chapter 8].

Proof. First let us exhibit a representation of $\mathfrak{R}$; this construction follows the Gelfand–Naimark–Segal technique however relative to the already existing standard representation. As we have seen every connected oriented smooth 4-manifold M admits an embedding into its von Neumann algebra $\mathfrak{R}$ via (8) and this map looks like $x \mapsto P_x$ i.e. to every point $x \in M$ the map $i_M : M \to \mathfrak{R}$ assigns a projection $P_x \in \mathfrak{R}$. Take any Riemannian metric g on M having volume form μ_g over M. Suppose furthermore that M is compact. Then consider the map $F_{M,g} : \mathfrak{R} \to \mathbb{C}$ given by

$$F_{M,g}(A) := \int\limits_{x\in M} i_M^* \tau\big(AP_x\big)\mu_g(x)$$

which is well-defined due to compactness of M. This map is obviously $\mathbb{C}$-linear, continuous and satisfies $F_{M,g}(A^*) = \overline{F_{M,g}(A)}$. Moreover we compute $F_{M,g}(A^*A) = \int_M i_M^* \big\|\widehat{AP_x}\big\|^2 \mu_g(x) \geqq 0$ consequently $F_{M,g}$ is a non-negative functional on $\mathfrak{R}$. In particular $F_{M,g}(1^*1) > 0$ hence an application of the standard inequality $|F_{M,g}(A^*B)|^2 \leqq F_{M,g}(A^*A)F_{M,g}(B^*B)$ implies that $0 \subseteqq I(M,g) \subseteqq \mathfrak{R}$ defined by the elements satisfying $F_{M,g}(B^*B) = 0$ is a multiplicative left-ideal in $\mathfrak{R}$ such that $\mathbb{C}1 \not\subset I(M,g)$ thus surely $0 \subseteqq I(M,g) \subsetneqq \mathfrak{R}$. In fact $I(M,g)$ is independent of the metric g involved in its definition: if h is another Riemannian metric on M then there exists a positive function $f : M \to \mathbb{R}$ such that

$$0 \leqq F_{M,h}(A^*A) = \int\limits_{x\in M} i_M^* \tau\big(A^*AP_x\big)\mu_h(x) = \int\limits_{x\in M} i_M^* \tau\big(A^*AP_x\big)f(x)\mu_g(x) \leqq F_{M,g}(A^*A)\|f\|_{L^\infty(M)}$$

hence $I(M,g) \subseteqq I(M,h)$ and likewise we see that $I(M,h) \subseteqq I(M,g)$ i.e. $I(M,h) = I(M,g)$. Therefore we shall denote this ideal simply as $I(M)$ from now on.

Recall that the standard representation $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$ arises via multiplication from the left in $\mathfrak{R}$ on itself. Since $0 \subseteqq I(M) \subsetneqq \mathfrak{R}$ is a left-ideal π restricts to a representation of $\mathfrak{R}$ on the Hilbert space completion $0 \subseteqq \mathscr{I}(M) \subseteqq \mathscr{H}$ of $I(M)$. Let us proceed further by exploting that the scalar product on $\mathfrak{R} \subset \mathscr{H}$ looks like $(\hat{A},\hat{B}) = \tau(AB^*)$ hence satisfies the identity $(\widehat{AB},\widehat{C}\,) = (\hat{B},\widehat{A^*C})$; consequently the standard representation restricts to the orthogonal complementum $0 \subseteqq \mathscr{I}(M)^\perp \subseteqq \mathscr{H}$ as well. Note that $\mathscr{I}(M)^\perp$ as a complete complex vector is isomorphic to $\mathscr{H}/\mathscr{I}(M)$. Thus for a given M we define $\rho_M : \mathfrak{R} \to \mathfrak{B}(\mathscr{I}(M)^\perp)$ to be simply the restricted representation $\pi|_{\mathscr{I}(M)^\perp}$. From the general theory [2, Chapter 8] we know that if $P_M : \mathscr{H} \to \mathscr{I}(M)^\perp$ is the orthogonal projection then simply $P_M \in \mathfrak{R}'$ because $0 \subseteqq \mathscr{I}(M)^\perp \subseteqq \mathscr{H}$ i.e. it lies in the standard $\mathfrak{R}$-module (and not in $\mathscr{H} \otimes \ell^2(\mathbb{N})$ as in general). The Murray–von Neumann coupling constant of ρ_M is therefore equal to $\tau(P_M) \in [0,1]$ and depends only on the unitary equivalence class of ρ_M. In particular it is preserved by orientation-preserving diffeomorphisms of M hence we conclude that $\gamma(M) := \tau(P_M) \in [0,1]$ is a smooth invariant of M.

Concerning an important restriction on the spectrum of γ, first note that $A \in I(M)$ if and only if $AP_x = 0$ for all $x \in M$. This demonstrates that being P_x surely not invertible, non-trivial solutions in principle are allowed and belong to $0 \subsetneqq \mathfrak{R}\,(1-P_x) \subsetneqq \mathfrak{R}$. These non-trivial subsets are weakly closed left-ideals and in fact

$$I(M) = \bigcap_{x\in M} \mathfrak{R}\,(1-P_x) \tag{11}$$

consequently $I(M)$ itself is a weakly closed left-ideal. Fix $x \in M$. The Hilbert space closure of $\mathfrak{R}(1-P_x)$ within $\mathscr{H}$ is also non-trivial and carries a non-trivial representation of $\mathfrak{R}$ in the usual way as above. Consequently there exists an invariant projection $0 \neq Q_x \in \mathfrak{R}'$ onto this invariant subspace having the property $\mathfrak{R}(1-P_x) = \mathfrak{R}Q_x = Q_x\mathfrak{R}$. Take $0 \neq A_x \in \mathfrak{R}(1-P_x)$ and write it in the form $B_xQ_x = Q_xB_x$. If $y \in M$ is another point with similar $0 \neq A_y = B_yQ_y = Q_yB_y$ where $Q_y \in \mathfrak{R}'$ too then $A_xA_y = (A_xB_y)Q_y$ and $A_xA_y = (B_xA_y)Q_x$ implies $A_xA_y \in \mathfrak{R}(1-P_x) \cap \mathfrak{R}(1-P_y)$. Let $\{x_1, x_2, \dots\} \subset M$ be an ordered countable everywhere dense subset of points in M (such subset exists because by the standard definition M as a topological space admits a countable basis) and in addition take an open covering $\{U_k\}_{k\in\mathbb{N}}$ of M. By the *compactness of* M this covering possesses a finite subcovering $\{U_1,\dots,U_l\}$. For every $1 \leqq k \leqq l$ take a point $x_{j_k} \in U_k$ and some $0 \neq A_{x_{j_k}} \in \mathfrak{R}(1-P_{x_{j_k}})$ and for $l \in \mathbb{N}$ define $A_l := c_l A_{x_{j_1}} A_{x_{j_2}} \dots A_{x_{j_l}}$ where the order is determined by demanding that $1 \leqq j_1 < j_2 < \dots < j_l$. Perturbing slightly $A_{x_{j_k}}$ within $\mathfrak{R}(1-P_{x_{j_k}})$ if necessary, we can assume that $0 \neq A_l$ and $c_l \in \mathbb{C}$ is choosen so that $[[A_l]] = 1$ where $[[\cdot]]$ denotes the operator norm on $\mathfrak{R}$. Consequently taking refinements of the covering and performing re-indexing if necessary, we obtain a sequence $\{A_m\}_{m\in\mathbb{N}}$ in the unit sphere of $\mathfrak{R}$. However being the closed unit ball of any von Neumann algebra hence of $\mathfrak{R}$ weakly compact [2, Corollary 2.3.2], there exists an element $A \in \mathfrak{R}$ satisfying $[[A]] = 1$ and a (sub)sequence $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n \to A$ weakly as $n \to +\infty$. This limit operator has the following property. Take a subsequence $\{x_1, x_2, \dots\}$ within the by construction everywhere dense subset $\cup_{n=1}^{+\infty} \cup_{k=1}^{n} \{x_{j_k}\} \subset M$ converging to an arbitrary fixed $x \in M$. Passing to the corresponding subsequence $\{A_i\}_{i\in\mathbb{N}}$ within the already weakly convergent one $\{A_n\}_{n\in\mathbb{N}}$ also by construction we know that $A_i P_{x_i} = 0$. Hence picking $\xi, \eta \in \mathscr{H}$ we can expand the scalar product

$$\begin{aligned}\big(AP_x\xi,\eta\big) &= \big((A-A_i)P_x\xi,\eta\big) + \big(A_i(P_x-P_{x_i})\xi,\eta\big) + \big(A_iP_{x_i}\xi,\eta\big)\\ &= \big((A-A_i)P_x\xi,\eta\big) + \big(A_i(P_x-P_{x_i})\xi,\eta\big)\end{aligned}$$

and let $i \to +\infty$ hence $x_i \to x$ to see that $\big(AP_x\xi,\eta\big) = 0$. Now putting $\eta := AP_x\xi$ gives $\|AP_x\xi\| = 0$ for every $\xi \in \mathscr{H}$ i.e. $AP_x = 0$. Since $x \in M$ was arbitrary via (11) we conclude that the limit operator satisfies

$$0 \neq A \in \overline{\bigcap_{n=1}^{+\infty}\bigcap_{k=1}^{n} \mathfrak{R}\big(1-P_{x_{j_k}}\big)}^{\,\mathrm{w}} = \bigcap_{x\in M} \mathfrak{R}(1-P_x) = I(M)$$

thus $0 \neq I(M)$ hence for its Hilbert space closure also $0 \neq \mathscr{I}(M)$. Thus for the corresponding projection $1-P_M \neq 0$ consequently $\gamma(M) < 1$. Therefore the compactness of M implies that $\gamma(M) \in [0,1)$. □

*Remark***.** Before proceeding further we note that, taking into account that the inclusion (8) assigning to a point on the geometric side a projection on the algebraic side is defined for every connected smooth oriented 4-manifold, (11) permits to extend the construction of γ as in Lemma 2.1 to non-compact M's too. If M admits a smooth compactification $\tilde{M}$ then $0 \leqq \gamma(M) \leqq \gamma(\tilde{M}) < 1$ consequently $\gamma(M) \in [0,1)$ continues to hold. However if a connected oriented smooth 4-manifod X fails to have this property then in principle $\gamma(X) = 1$ can occur. An immense class of such X's exists in four dimensions e.g. taking any smooth compact M and any R^4 (an exotic or fake $\mathbb{R}^4$) the connected sum $X := M\#R^4$ provides a typical example. Topologically M is nothing else than the one-point compactification of X however smoothly not cf. [14, Theorem 2.1].

Jones' subfactor theory (for a summary cf. e.g. [2, Section 9.4] or [8, Chapter V.10]) imposes an interesting further restriction on the possible spectrum of γ just introduced.

Lemma 2.2. *Let M be a connected compact oriented smooth 4-manifold and $\gamma(M) \in [0,1)$ its smooth invariant. Then $\gamma(M) = 1 - \frac{1}{x}$ where $x \in \{4\cos^2\left(\frac{\pi}{n}\right) \mid n \geqq 3\} \cup [4,+\infty)$ that is, an element from the set of all possible finite Jones' subfactor indices.*

Proof. We go ahead working with the weakly closed left-ideal (11) distillated out of M alone. Observe first that its adjoint $I(M)^*$ is therefore a weakly closed right-ideal hence $I(M) \cap I(M)^* \subseteqq I(M) \subsetneqq \mathfrak{R}$ is a two-sided ideal thus the simplicity of $\mathfrak{R}$ (cf. [2, Proposition 4.1.5]) forces $I(M) \cap I(M)^* = 0$. Consequently $I(M) + I(M)^* = I(M) \oplus I(M)^*$. Likewise $I(M)^* I(M) \subseteqq I(M) \cap I(M)^* = 0$ too; finally

$0 \subsetneqq I(M)I(M)^* \subsetneqq \mathfrak{R}$ is also a two-sided ideal hence again is either 0 or $\mathfrak{R}$ but leveraging the commutativity of the trace $\tau|_{I(M)I(M)^*} = \tau|_{I(M)^*I(M)} = \tau|_0 = 0$ i.e. in fact $I(M)I(M)^* = 0$. Consequently $\left(I(M)\oplus I(M)^*\right)^2 \subseteqq I(M)\oplus I(M)^*$ demonstrating that this self-adjoint set is already a subalgebra of $\mathfrak{R}$ which is even weakly closed however without unit. Hence taking bicommutant within $\mathfrak{B}(\mathscr{H})$ we know that $\left(I(M)\oplus I(M)^*\right)'' = \mathbb{C}1\oplus I(M)\oplus I(M)^*$ thus introducing

$$\mathfrak{I}(M) := \mathbb{C}1\oplus I(M)\oplus I(M)^* \subsetneqq \mathfrak{R} \tag{12}$$

is by construction a non-trivial von Neumann subalgebra of $\mathfrak{R}$ since $0\neq I(M)$. (For clarity we note that the displayed spitting of $\mathfrak{I}(M)$ valid only as a complex complete vector space and not as an algebra.) Using the projection $P_M : \mathscr{H}\to\mathscr{I}(M)^\perp$ from Lemma 2.1 consider the set $\mathfrak{R}(1-P_M)$. Knowing that $1-P_M\in\mathfrak{R}'$ we can write $\mathfrak{R}(1-P_M) = (1-P_M)\mathfrak{R}$ hence $\mathfrak{R}(1-P_M) = \mathfrak{R}(1-P_M)\mathfrak{R}(1-P_M)$ i.e. it is an algebra. Since $0\neq 1-P_M$ because $\gamma(M)<1$ it operates on $(1-P_M)\mathscr{H} = \mathscr{I}(M)$ such that $1-P_M$ acts as the identity and extends by zero to $\mathscr{I}(M)^\perp$. Therefore $\mathfrak{R}(1-P_M)$ is a von Neumann algebra on $\mathscr{H}$. Consequently its intersection with $\mathfrak{R}$ within $\mathfrak{B}(\mathscr{H})$ is a von Neumann subalgebra of $\mathfrak{R}$ and in fact is equal to $\mathfrak{R}(1-P_M)$. Struggling further pick $A\in I(M)\subset\mathfrak{R}$ then $\hat{A}\in\mathscr{I}(M)\subset\mathscr{H}$ thus $\hat{A} = (1-P_M)\hat{A}$ hence $A = A(1-P_M)$ yielding $A\in\mathfrak{R}(1-P_M)$; moreover taking adjoint of the previous equation we get $A^* = ((1-P_M)A)^* = A^*(1-P_M)$ consequently $\left(\mathbb{C}1\oplus I(M)\oplus I(M)^*\right)(1-P_M)\subseteqq\mathfrak{R}(1-P_M)$. Conversely, if $A\in\mathfrak{R}(1-P_M)$ then $A\in\mathfrak{R}$ has the form $A = B(1-P_M)$ with some $B\in\mathfrak{R}$ hence we obtain $\hat{A} = (1-P_M)\hat{B}$; we can assume that $\hat{B}\in\mathscr{I}(M)$ therefore $\hat{A}=\hat{B}$ yielding $A = B\in I(M)$; or equivalently we can write $\hat{A} = J(1-P_M)J\widehat{B^*}$ such that $\widehat{B^*}\in\mathscr{I}(M)^*$ where $\mathscr{I}(M)^*\subset\mathscr{H}$ is the closure of $I(M)^*$; but this means that $A = B^*\in I(M)^*$; therefore $\left(\mathbb{C}1\oplus I(M)\oplus I(M)^*\right)(1-P_M)\supseteqq\mathfrak{R}(1-P_M)$ too. Actually the map $A\mapsto A(1-P_M)$ is a homomorphism from $\mathfrak{I}(M)$ to $\mathfrak{I}(M)(1-P_M)$ which is readily both injective and surjective. In this way we come up with an alternative characterization as an abstract isomorphism

$$\mathfrak{I}(M)\cong\mathfrak{I}(M)(1-P_M) = \mathfrak{R}(1-P_M)\subsetneqq\mathfrak{R}\ .$$

The map $A\mapsto A(1-P_M)$ is also a homomorphism from $\mathfrak{R}$ to $\mathfrak{R}(1-P_M)$ which is obviously surjective; but it is injective as well because $\mathfrak{R}$ is simple hence $\mathfrak{R}$ and $\mathfrak{R}(1-P_M)$ are abstractly isomorphic consequently $\mathfrak{R}(1-P_M)$ is also a type II_1 factor. Thus $0\subsetneqq\mathfrak{I}(M)\subsetneqq\mathfrak{R}$ is a subfactor of $\mathfrak{R}$ admitting a Jones index $[\mathfrak{R}:\mathfrak{I}(M)]$ satisfying $[\mathfrak{R}:\mathfrak{I}(M)]\in\{4\cos^2\left(\frac{\pi}{n}\right)\,|\,n\geqq 3\}\cup[4,+\infty]$.

The representation of $\mathfrak{R}$ on $\mathscr{I}(M)$ worked out in Lemma 2.1 restricts to $\mathfrak{I}(M)$ rendering $\mathscr{I}(M)$ a left-$\mathfrak{I}(M)$-module too. Moreover also by Lemma 2.1 as left-$\mathfrak{R}$-modules $\mathscr{H} = \mathscr{I}(M)\oplus\mathscr{I}(M)^\perp$. Recalling now the basic properties of the dimension function of a left von Neumann algebra module over the von Neumann algebra itself (cf. e.g. [2, Chapter 8]) we collect:

$$\begin{aligned}
\dim_{\mathfrak{R}}\mathscr{H} &= 1 \quad \text{(the standard left-}\mathfrak{R}\text{-module)}\\
\dim_{\mathfrak{R}}\mathscr{H} &= \dim_{\mathfrak{R}}\mathscr{I}(M)+\dim_{\mathfrak{R}}\mathscr{I}(M)^\perp \quad \text{(additivity)}\\
\dim_{\mathfrak{R}}\mathscr{I}(M)^\perp &= \gamma(M) \quad \text{(by Lemma 2.1 and Footnote 3)}\\
\dim_{\mathfrak{I}(M)}\mathscr{I}(M) &= [\mathfrak{R}:\mathfrak{I}(M)]\dim_{\mathfrak{R}}\mathscr{I}(M) \quad \text{(dimension comparison)}\\
\dim_{\mathfrak{I}(M)(1-P_M)}\mathscr{I}(M) &= \dim_{\mathfrak{I}(M)(1-P_M)}\big((1-P_M)\mathscr{I}(M)\big) = \tau(1-P_M)\dim_{\mathfrak{I}(M)}\mathscr{I}(M)\\
\| \quad & \qquad \text{(projection formula)}\\
\dim_{\mathfrak{R}(1-P_M)}\mathscr{I}(M) &= \dim_{\mathfrak{R}(1-P_M)}\big((1-P_M)\mathscr{H}\big) = \tau(1-P_M)\dim_{\mathfrak{R}}\mathscr{H}\\
& \qquad \text{(projection formula)}
\end{aligned}$$

from which it follows that $0\leqq\gamma(M) = 1-\frac{1}{[\mathfrak{R}:\mathfrak{I}(M)]}$. But we know from Lemma 2.1 already that $\gamma(M)<1$ hence $[\mathfrak{R}:\mathfrak{I}(M)]<+\infty$ as stated. □

Next we collect some basic useful properties of the invariant.

Lemma 2.3. *(Reversing orientation.) If M is a connected compact oriented smooth 4-manifold and $\overline{M}$ is its orientation-reversed form then $\gamma(\overline{M}) = \gamma(M)$.*

(Gluing principle.) Let M and N be two connected compact oriented smooth 4-manifolds and let $M\#N$ be their connected sum. With induced orientation $M\#N$ is a connected compact oriented smooth 4-manifold. Then

$$\gamma(M\#N) = \gamma(M) + \gamma(N) - \gamma(M)\gamma(N)$$

and in particular if $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the standard n-sphere then $\gamma(\mathbb{S}^4) = 0$.

(Blow-up.) If M' is a smooth blow-up of M then

$$\gamma(M) \leqq \gamma(M') = \gamma(M) + \big(1 - \gamma(M)\big)t$$

where $t := \gamma(\mathbb{C}P^2)$.

Proof. The first assertion is obvious from γ's construction carried out in the proof of Lemma 2.1.

Concerning the second assertion as a general observation note that the γ-invariant is a well-defined map from (the category) Man^4 of all orientation-preserving diffeomorphism classes of connected compact oriented smooth 4-manifolds into the real interval $[0,1) \subset \mathbb{R}$. But Man^4 forms a commutative semigroup with (one possible) unit $\mathbb{S}^4$ under the connected sum operation $\#$. That is, if $X,Y,Z \in \mathsf{Man}^4$ and $\mathbb{S}^4 \in \mathsf{Man}^4$ then $X\#Y \cong Y\#X$ and $(X\#Y)\#Z \cong X\#(Y\#Z)$ and $X\#\mathbb{S}^4 \cong X$. Pick $M,N \in \mathsf{Man}^4$ with their connected sum $M\#N \in \mathsf{Man}^4$ and consider the corresponding $\gamma(M),\gamma(N),\gamma(M\#N) \in [0,1)$. Introduce $\bullet : [0,1) \times [0,1) \to [0,1)$ by setting $\gamma(M) \bullet \gamma(N) := \gamma(M\#N)$. The $\bullet$-operation is therefore well-defined and satisfies $\gamma(X) \bullet \gamma(Y) = \gamma(Y) \bullet \gamma(X)$ and $(\gamma(X) \bullet \gamma(Y)) \bullet \gamma(Z) = \gamma(X) \bullet (\gamma(Y) \bullet \gamma(Z))$ and $\gamma(X) \bullet \gamma(\mathbb{S}^4) = \gamma(X)$ i.e. $\gamma(\mathbb{S}^4)$ is a unit. These ensure us that $([0,1),\bullet)$ is a unital commutative semigroup and $\gamma : (\mathsf{Man}^4,\#) \to ([0,1),\bullet)$ is a unital semigroup homomorphism. As a specific observation note that γ has been defined in Lemma 2.1 as the (continuous) dimension of a closed subspace within the $\mathfrak{R}$-module Hilbert space $\mathscr{H}$; more precisely $\gamma(M) = \tau(P_M)$ arises through a chain of assignments $M \mapsto \mathscr{I}(M)^\perp \mapsto \dim_{\mathfrak{R}} \mathscr{I}(M)^\perp$. Lemma 2.1 and in particular (11) imply that $0 \neq I(M\#N) = I(M) \cap I(N)$ which demonstrates that $0 \neq \mathscr{I}(M\#N) = \mathscr{I}(M) \cap \mathscr{I}(N) \subseteqq \mathscr{H}$ for the Hilbert space completion too. Introducing the Abelian group $(L(\mathscr{H}),+)$ of closed subspaces with respect to taking sum and hence $0 \in \mathscr{H}$ playing the role of the unit we see that the assignment $M \mapsto \mathscr{I}(M)^\perp$ has the property $\mathscr{I}(M\#N)^\perp = (\mathscr{I}(M) \cap \mathscr{I}(N))^\perp = \mathscr{I}(M)^\perp + \mathscr{I}(N)^\perp \subsetneqq \mathscr{H}$ hence induces a unital semigroup homomorphism $L : (\mathsf{Man}^4,\#) \to (L(\mathscr{H}) \setminus \{\mathscr{H}\},+)$. Thus γ factorizes as

$$\gamma : (\mathsf{Man}^4,\#) \xrightarrow{L} (L(\mathscr{H}) \setminus \{\mathscr{H}\},+) \xrightarrow{\dim_{\mathfrak{R}}} ([0,1),\bullet)$$

consequently the unique plain dimension function on subspaces as it is, must in fact be an Abelian unital semigroup homomorphism too hence its properties impose constraints on its target structure. Recall the γ-invariant when evaluated on a connected sum has been written as $\gamma(M\#N) = \gamma(M) \bullet \gamma(N)$ expressing that it depends only on $\gamma(M)$ and $\gamma(N)$. Putting $\mathscr{V} = \mathscr{I}(M)^\perp$ and $\mathscr{W} = \mathscr{I}(N)^\perp$ and knowing that the dimension function behaves like $\dim_{\mathbb{R}}(\mathscr{V} + \mathscr{W}) = \dim_{\mathfrak{R}} \mathscr{V} + \dim_{\mathfrak{R}} \mathscr{W} - \dim_{\mathfrak{R}}(\mathscr{V} \cap \mathscr{W})$ forces to write $\gamma(M) \bullet \gamma(N) = \gamma(M) + \gamma(N) - f\big(\gamma(M),\gamma(N)\big)$ and the unknown function is strongly determined by the geometric properties of intersection of subspaces. Namely we know that $f : [0,1) \times [0,1) \to [0,1)$ such that $f(s,t) = f(t,s)$ and $s \mapsto f(s,t)$ is linear, $f(0,t) = 0$ and $\lim_{s\to 1} f(s,t) = t$. The function $f(s,t) = st$ solves these constraints. This forces the Abelian semigroup multiplication law on $[0,1)$ to look like $s \bullet t = s + t - st$ with $0 \in [0,1)$ being the unit hence yielding the shape for $\gamma(M\#N)$ moreover constraining $\gamma(\mathbb{S}^4) = 0$ as stated.

Finally, since the blow-up in the smooth category $(\mathsf{Man}^4, \#)$ is by definition given by $M' := M\#\overline{\mathbb{C}P^2}$ the result follows if we put $t := \gamma(\overline{\mathbb{C}P^2}) = \gamma(\mathbb{C}P^2)$ in the connected sum formula. □

Proof of Theorem 1.2. Putting together Lemmata 2.1, 2.2 and 2.3 the theorem follows. □

An excursus on the invariant γ. Before proceeding forward let us take a closer look on the smooth 4-manifold concerning its effective computability. The general experience is that the more sensitive an invariant is, the less computable it is. By techniques taken from 4-manifold theory and the gluing principle, the following probably non-trivial but quite insensitive behaviour of γ shows up in the closed simply connected case.

Lemma 2.4. *If M' and M'' are connected, simply connected, closed smooth 4-manifolds which are homeomorphic then $\gamma(M') = \gamma(M'')$.*

In fact for any connected, simply connected, closed smooth 4-manifold

$$\gamma(M) = 1 - (1-t)^{b_2(M)}$$

with $t = \gamma(\mathbb{C}P^2)$ as before.

Proof. Concerning the first assertion if M' and M'' are as required then there exists an integer $k \geqq 0$ such that $M'\#k(\mathbb{C}P^1 \times \mathbb{C}P^1) \cong M''\#k(\mathbb{C}P^1 \times \mathbb{C}P^1)$, cf. e.g. [15, Theorem 9.1.12]. Thus we know that we have the equality $\gamma(M'\#k(\mathbb{C}P^1 \times \mathbb{C}P^1)) = \gamma(M''\#k(\mathbb{C}P^1 \times \mathbb{C}P^1))$. Then introducing $s := \gamma(k(\mathbb{C}P^1 \times \mathbb{C}P^1))$ and applying the gluing principle we find

$$\gamma(M') + s - \gamma(M')s = \gamma(M'') + s - \gamma(M'')s$$

and leveraging the invertability of the map $r \mapsto r + s - rs$ because $s < 1$ we obtain $\gamma(M') = \gamma(M'')$.

Concerning the second assertion, with some $s \in [0,1)$ take the recursive sequence

$$R_0(s) := 0\,,\, R_1(s) := s\,,\, \dots\,,\, R_k(s) := s + R_{k-1}(s) - sR_{k-1}(s)\,,\, \dots$$

for all $k = 0,1,2,\dots$ representing the semigroup $\{0\} \cup \mathbb{N}$ inside $[0,1)$ and put $t := \gamma(\mathbb{C}P^2) = \gamma(\overline{\mathbb{C}P^2})$. Now if M_1 and M_2 are connected, closed, simply connected, smooth then there exist integers $k_1, l_1 \geqq 0$ and $k_2, l_2 \geqq 0$ such that $M_1\#k_1\mathbb{C}P^2\#l_1\overline{\mathbb{C}P^2} \cong M_2\#k_2\mathbb{C}P^2\#l_2\overline{\mathbb{C}P^2}$ (cf. e.g. [15, Theorem 9.1.14]) which gives again that $\gamma(M_1\#k_1\mathbb{C}P^2\#l_1\overline{\mathbb{C}P^2}) = \gamma(M_2\#k_2\mathbb{C}P^2\#l_2\overline{\mathbb{C}P^2})$. Then by the gluing principle

$$\gamma(M_1) + R_{k_1+l_1}(t) - \gamma(M_1)R_{k_1+l_1}(t) = \gamma(M_2) + R_{k_2+l_2}(t) - \gamma(M_2)R_{k_2+l_2}(t)\ .$$

Let $M_1 := M$ be arbitrary and $M_2 := \mathbb{S}^4$ hence $\gamma(M_2) = 0$. Then we can suppose that $k_1 + l_1 \leqq k_2 + l_2$ therefore

$$\gamma(M) + R_{k_1+l_1}(t) - \gamma(M)R_{k_1+l_1}(t) = R_{k_2+l_2}(t)$$

from which again, taking into account that $t < 1$, by invertability we find $\gamma(M) = R_{k_2+l_2-k_1-l_1}(t)$ hence setting $n := k_2 + l_2 - k_1 - l_1 \geqq 0$ we get $\gamma(M) = R_n(t)$. Moreover it is clear from the proof that in fact $n = b_2(M)$. It is easy to see that actually $R_k(s) = 1 - (1-s)^k$ and inserting $k := b_2(M)$ and $s := t$ we end up with the stated formula for $\gamma(M)$ in the simply connected case. □

As an important but unexpectedly subtle task next we compute the number $0 \leqq t < 1$ assigned to $\mathbb{C}P^2$. This number appears in both Lemmata 2.3 and 2.4 and $t \neq 0$ is crucial for rendering things not completely trivial in the simply connected realm. Nevertheless γ in the non-simply connected regime might be more interesting; since in the well-known construction of 4-manifolds with prescribed finitely presented fundamental group the space $\mathbb{S}^1 \times \mathbb{S}^3$ plays a key role (since $\pi_1(\mathbb{S}^1 \times \mathbb{S}^3) \cong \mathbb{Z}$ is the free group generated by a single element) we compute its invariant here too for future reference.

Lemma 2.5. *An equality $\gamma(\mathbb{C}P^2)=\frac{8}{9}$ holds hence $t=\frac{8}{9}$. Thus $\gamma(M)=1-\left(\frac{1}{9}\right)^{b_2(M)}$ in Lemma 2.4 and the 2^{nd} Betti number of a connected simply connected closed smooth 4-manifold M can be expressed in terms of its γ-invariant and looks like*

$$b_2(M)=\frac{\log\frac{1}{1-\gamma(M)}}{\log 9}$$

(we arranged the expression so that the logarithms are positive).

Moreover $\gamma(\mathbb{S}^1\times\mathbb{S}^3)=\frac{3}{4}$.

Proof. If (M,g) is a compact oriented Riemannian 4-manifold then remember that, as a special four dimensional phenomenon, $R_g\in\mathfrak{R}$. Concerning the (complexified) curvature as an operator we make a general observation: if (M,g) is a homogeneous 4-space then for its curvature $R_g\in\mathfrak{I}(M)\subsetneqq\mathfrak{R}$ if and only if $R_g=\mathbb{R}1$. Indeed, referring to the splitting of $\mathfrak{I}(M)$ in (12) the self-adjoint curvature admits a unique decomposition $R_g=a1+A+A^*$ where $a\in\mathbb{R}$ and $A\in I(M)$. We already know via (11) that the condition $A\in I(M)$ is equivalent to $AP_x=0$ hence $P_xA^*=0$ too implying that $AP_x+P_xA^*=0$ for every $x\in M$. Also recall that here $0\neq P_x\in\mathfrak{R}$ has been constructed in (8) as the projection $P_x:\mathscr{H}\to\mathscr{V}_x$ where $\mathscr{V}_x\subset\mathscr{H}$ is the closure of the local Clifford algebra $C(M,x)\subset C(M)$. Recalling its structure exhibited in (7) consider its subspace $\Omega^2_c(M,x;\mathbb{C})\subset C(M,x)$ consisting of complex 2-forms vanishing at $x\in M$. Take an arbitrary element $\xi_x\in\Omega^2_c(M,x;\mathbb{C})\subsetneqq\mathscr{V}_x$. Then

$$0=P_x(AP_x+P_xA^*)P_x\xi_x=P_x(A+A^*)P_x\xi_x=P_x(R_g-a1)P_x\xi_x=P_x(R_g-a1)\xi_x=(R_g-a1)\xi_x$$

where at the fourth step we leveraged that $P_x\xi_x=\xi_x$ and at the fifth step that $R_g-a1\in\mathfrak{R}$ is in fact an extension of a geometric i.e. pointwisely defined operator (at this point cf. Footnote 2 again) hence preserves the subspace $\Omega^2_c(M,x;\mathbb{C})\subsetneqq\mathscr{V}_x$ too. Consequently $(R_g-a1)|_{\Omega^2_c(M,x;\mathbb{C})}=0$ for every $x\in M$. However (M,g) is homogeneous by assumption therefore R_g-a1 is a constant operator along M hence in this case the stronger property $(R_g-a1)|_{\Omega^2_c(M;\mathbb{C})}=0$ holds too implying $(R_g-a1)|_{\mathscr{H}}=0$. That is R_g-a1 is the zero operator on $\mathscr{H}$ demonstrating $A=0$ in the unique decomposition of R_g or in other words $R_g\in\mathbb{R}1\subset\mathfrak{I}(M)$ as claimed.

Now specializing to the complex projective space consider the Fubini–Study metric g on $\mathbb{C}P^2$ with corresponding curvature tensor $R_g\in\mathfrak{R}$. It is a classical fact that the Fubini–Study metric is a homogeneous Riemannian 4-manifold satisfying the Einstein condition with non-zero cosmological constant and having non-vanishing Weyl curvature. Therefore referring back to (4) surely $0\neq R_g\notin\mathbb{R}1$ consequently $0\neq R_g\notin\mathfrak{I}(\mathbb{C}P^2)$ hence $\mathfrak{I}(\mathbb{C}P^2)$ is a non-trivial subfactor of $\mathfrak{R}$. Proceeding further it is clear that as a therefore non-trivial condition $\mathfrak{I}(\mathbb{C}P^2)$ must be invariant within $\mathfrak{R}$ under $\mathrm{Diff}^+(\mathbb{C}P^2)$ acting as inner automorphisms on $\mathfrak{R}$. The isometry group $\mathrm{Iso}(\mathbb{C}P^2,g)\subsetneqq\mathrm{Diff}^+(\mathbb{C}P^2)$ of the Fubini–Study metric is large enough to act transitively along $\mathbb{C}P^2$ thus taking into account the decomposition in (11) we can see that by picking any point $x_0\in\mathbb{C}P^2$

$$I(\mathbb{C}P^2)=\bigcap_{x\in\mathbb{C}P^2}\mathfrak{R}(1-P_x)=\bigcap_{\Phi\in\mathrm{Diff}^+(\mathbb{C}P^2)}\mathfrak{R}\big(1-P_{\Phi(x_0)}\big)=\bigcap_{\Psi\in\mathrm{Iso}(\mathbb{C}P^2,g)}\mathfrak{R}\big(1-P_{\Psi(x_0)}\big)$$

and this together with (12) sharpens the invariance condition. Indeed, being an intersection, $I(\mathbb{C}P^2)$ is pointwisely fixed by $\mathrm{Iso}(\mathbb{C}P^2,g)$ hence so is $\mathfrak{I}(\mathbb{C}P^2)$ therefore the effective form of the invariance condition is that $\mathfrak{I}(\mathbb{C}P^2)$ be pointwisely fixed by the finite dimensional group $\mathrm{Iso}(\mathbb{C}P^2,g)$ yet acting as inner automorphisms on $\mathfrak{R}$. Actually $\mathrm{Iso}(\mathbb{C}P^2,g)$ acts on $\Omega^2_c(\mathbb{C}P^2;\mathbb{C})$ by pullbacks hence taking into account (3) acts with corresponding conjugations on the Clifford subalgebra $C(\mathbb{C}P^2)$ which then extend to inner automorphisms on the weak closure $\mathfrak{R}$. An action characterized by these properties is unique on $\mathfrak{R}$. For if $\alpha \in \mathrm{Aut}\,\mathfrak{R}$ belongs to a family of automorphisms abstractly isomorphic to $\mathrm{Iso}(\mathbb{C}P^2, g)$ and which pointwisely fixes $\mathfrak{I}(\mathbb{C}P^2)$ then under the embedding $\mathbb{C}P^2 \subset \mathfrak{R}$ provided by (8) α factorizes on the corresponding $C(\mathbb{C}P^2)$ according to (10) as the product of a gauge transformation on $\wedge^2 T^*\mathbb{C}P^2 \otimes \mathbb{C}$ which can be supposed to be the identity, and an orientation-preserving diffeomorphism of $\mathbb{C}P^2$. Moreover these latter transformations act transitively nevertheless preserve $\mathbb{C}P^2 \subset \mathfrak{R}$. Thus the action of this family on $\mathfrak{R}$ looks like as described above. An example for such a unique action of $\mathrm{Iso}(\mathbb{C}P^2, g) \cong \mathrm{PU}(3)$ on $\mathfrak{R}$ arises as conjugation by 3×3 unitary matrices on $\mathfrak{R} \cong \mathfrak{M}_3(\mathfrak{I}(\mathbb{C}P^2))$ hence

$$[\mathfrak{R} : \mathfrak{I}(\mathbb{C}P^2)] = [\mathfrak{M}_3(\mathfrak{I}(\mathbb{C}P^2)) : \mathfrak{I}(\mathbb{C}P^2)] = 3^2 = 9$$

demonstrating that $\gamma(\mathbb{C}P^2) = 1 - \frac{1}{[\mathfrak{R}:\mathfrak{I}(\mathbb{C}P^2)]} = 1 - \frac{1}{9} = \frac{8}{9} = t$ as stated.

Concering $\mathbb{S}^1 \times \mathbb{S}^3$ we repeat the previous steps. First let us make sure that $0 \neq \mathfrak{I}(\mathbb{S}^1 \times \mathbb{S}^3) \subsetneqq \mathfrak{R}$. Choosing an isomorphism $\mathbb{S}^1 \times \mathbb{S}^3 \cong \mathrm{U}(2)$ and taking the corresponding biinvariant metric g we obtain a homogeneous Riemannian 4-space structure on $\mathbb{S}^1 \times \mathbb{S}^3$ acted upon transitively by $\mathrm{U}(2)$. Assume that this metric is Einstein; then referring back to the decomposition (4) of the Riemannian curvature in the four dimensional oriented case this means that $\mathrm{Ric}_0 = 0$ i.e. the traceless Ricci tensor vanishes but in addition the Gauß–Bonnet–Chern theorem says that

$$0 = \chi(\mathbb{S}^1 \times \mathbb{S}^3) = \frac{1}{8\pi^2} \int\limits_{\mathbb{S}^1 \times \mathbb{S}^3} \left(|\mathrm{Weyl}^+|_g^2 + |\mathrm{Weyl}^-|_g^2 + \frac{1}{24}\mathrm{Scal}^2 \right) \mu_g$$

implying $\mathrm{Weyl}^\pm = 0$ and $\mathrm{Scal} = 0$ too hence actually that $R_g = 0$ i.e. g is flat. However this is not possible because a compact flat 4-manifold is always universally covered by the flat $\mathbb{R}^4$ which cannot be the case here. Consequently g is neither Einstein nor flat hence again by (4) we know that $0 \neq R_g \notin \mathbb{R}1$ and this together with its homogeneity permits to conclude that $0 \neq R_g \notin \mathfrak{I}(\mathbb{S}^1 \times \mathbb{S}^3)$ hence this subfactor is not trivial. Secondly repeating the remaining steps for $\mathbb{C}P^2$ but replacing $\mathrm{U}(3)$ with $\mathrm{U}(2)$ we find that

$$[\mathfrak{R} : \mathfrak{I}(\mathbb{S}^1 \times \mathbb{S}^3)] = [\mathfrak{M}_2(\mathfrak{I}(\mathbb{S}^1 \times \mathbb{S}^3)) : \mathfrak{I}(\mathbb{S}^1 \times \mathbb{S}^3)] = 2^2 = 4$$

consequently $\gamma(\mathbb{S}^1 \times \mathbb{S}^3) = 1 - \frac{1}{[\mathfrak{R}:\mathfrak{I}(\mathbb{S}^1 \times \mathbb{S}^3)]} = 1 - \frac{1}{4} = \frac{3}{4}$ as stated. □

We close our detour on γ as well as this section with two general observations. The first is that by Lemma 2.2 the range of γ naturally splits into a *discrete part* $\left\{ 1 - \frac{1}{4\cos^2\left(\frac{\pi}{n}\right)} \right\}_{n=3,4,\dots} \subset [0,1)$ and a *continuous part* $\left[\frac{3}{4}, 1\right) \subset [0,1)$. On the one hand $\gamma(\mathbb{S}^4) = 0$ by Lemma 2.3 hence belongs to the discrete part. On the other hand $\gamma(\mathbb{C}P^2) = \frac{8}{9}$ is a member of the continuous part by Lemma 2.5 hence the same holds for all simply connected closed spaces having non-zero 2^{nd} Betti number by Lemma 2.4; and likewise $\gamma(M\#N)$ belongs to the continuous part whenever $\gamma(M) > 0$ and $\gamma(N) > 0$ via Lemma 2.3; finally $\gamma(\mathbb{S}^1 \times \mathbb{S}^3) = \frac{3}{4}$ belongs there too via Lemma 2.5. It is an interesing question whether or not exists any M distinct from $\mathbb{S}^4$ such that $\gamma(M)$ belongs to the discrete range?

As a second observation, therefore apparently $\mathbb{S}^4$ is sharply separated from the rest of 4-manifolds from the viewpoint of the γ-invariant and this gap phenomenon might be linked with the difficulty underlying the 4 dimensional smooth Poincaré conjecture as follows. Lemma 2.4 unfortunately makes sure that γ is not injective at its certain values in the continuous part because γ depends only on the 2^{nd} Betti number and in fact γ gets less-and-less distinctive as the 2^{nd} Betti number inreases. On the contrary, as the 2^{nd} Betti number approaches zero, γ has a sharp content: an application to $\mathbb{S}^4$ satisfying $\gamma(\mathbb{S}^4) = 0$ implies that if the smooth four dimensional Poincaré conjecture fails then γ is not injective at zero too. A stronger assertion is the following.

Lemma 2.6. *The restriction $\gamma|_{\mathsf{Man}^4_0}$ to (the category of) all connected, simply connected closed smooth 4-manifolds is injective at zero if and only if the smooth Poincaré conjecture holds i.e. the 4-sphere admits a unique smooth structure.*

Proof. If S^4 is a differentiable manifold homeomorphic to $\mathbb{S}^4$ then $S^4 \in \mathsf{Man}^4_0$ such that $b_2(S^4) = 0$ therefore Lemma 2.4 implies $\gamma(S^4) = 1-(1-t)^0 = 0$. Assume that $\gamma|_{\mathsf{Man}^4_0}$ is injective at zero. Then $S^4 \cong \mathbb{S}^4$ and the 4 dimensional smooth Poincaré conjecture follows.

Conversely, take $S^4 \in \mathsf{Man}^4_0$ such that $\gamma(S^4) = 0$. Then by Lemmata 2.4 and 2.5 we conclude that $\gamma(S^4) = 1-\left(\frac{1}{9}\right)^{b_2(S^4)} = 0$ hence $b_2(S^4) = 0$. The validity of the 4 dimensional topological Poincaré conjecture [13] makes sure that S^4 is homeomorphic to $\mathbb{S}^4$. Assume that the smooth 4 dimensional Poincaré conjecture is true. Then $S^4 \cong \mathbb{S}^4$ hence $\gamma|_{\mathsf{Man}^4_0}$ is injective at zero. □

3 The Einstein equation and comparison of representations

In the previous section making use of the smooth structure and orientation of a 4-manifold alone, we have constructed a subfactor $\mathfrak{I}(M) \subsetneqq \mathfrak{R}$ encoding some information about topology and probably smoothness. In this section using additional geometric data we shall construct a normal subalgebra $\mathfrak{F}(M,g) \subsetneqq \mathfrak{R}$ related with an operator algebraic characterization of the Riemannian vacuum Einstein equation $\mathrm{Ric} = \Lambda g$ precisely in 4 dimensions.

Construction of a canonical metric. As a first step, let us observe that the plain manifold embedding $M \subset \mathfrak{R}$ in (8) naturally enhances to a Riemannian embedding $(M,g) \subset (\mathfrak{R}, \mathrm{Re}\,\tau)$. Consider the setup in Theorem 1.1 again i.e., take a connected oriented smooth 4-manifold M and consider its embedding into $\mathfrak{R}$ via (8) mapping $x \in M$ into $P_x \in \mathfrak{R}$. Fix a point $x \in M$ and a tangent vector $X \in T_xM$. Take a 1-parameter family $\{\Phi_t\}_{t\in(-\varepsilon,+\varepsilon)}$ of diffeomorphisms such that the curve $t \mapsto \Phi_t(x)$ within M satisfies $\Phi_0(x) = x \in M$ and $\dot{\Phi}_0(x) = X \in T_xM$ and consider the corresponding image curve $P(t) := i_M\Phi_t(x)$ in $\mathfrak{R}$ hence $P_x = P(0)$. Then $P(0) = i_Mx \in \mathfrak{R}$ and formally $\dot{P}(0) = \frac{\mathrm{d}}{\mathrm{d}t} i_M\Phi_t(x)|_{t=0} = i_{M*}\dot{\Phi}_0(x) = i_{M*}X \in T_{P_x}\mathfrak{R}$. Orientation-preserving diffeomorphisms act transitively on M thus for any two points $x,y \in M$ there exists $\Lambda \in \mathrm{Diff}^+(M)$ with $\Lambda(x) = y$ implying $(\Lambda^{-1})^*\mathscr{V}_x = \mathscr{V}_y$ consequently for their corresponding projections $P_y = (\Lambda^{-1})^*P_x\Lambda^*$ holds. Thus they are unitary equivalent (yielding τ is constant along $i_MM \subset \mathfrak{R}$). Therefore $P(t) = \Phi^*_{-t}P_x\Phi^*_t$ and formally $\dot{P}(0) = [P_x, L_X]$ where $L_X \in \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ is the Lie derivative constructed from the infinitesimal generator, also denoted as $X \in C^\infty_c(M;TM)$, of $\{\Phi_t\}_{t\in(-\varepsilon,+\varepsilon)}$. Without loss of generality we can assume $\Phi_t = \mathrm{id}_M$ outside a small neighbourhood of $x \in M$ hence the support of X is also small. Hence we can suppose $\tau(L_X) < +\infty$ allowing a unique extension of the Lie derivative, also written as $L_X \in \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \cap \mathfrak{R}$, implying that the commutator exists. Therefore $\dot{P}(0) = [P_x, L_X] \in \mathfrak{R}$. If $\{\Psi_t\}_{t\in(-\varepsilon,+\varepsilon)}$ is another similar family with corresponding projector curve $Q(t) = i_M\Psi_t(x)$ then we can pick an intertwining diffeomorphism having the property $\Lambda(\Phi_t(x)) = \Psi_t(x)$ for every t hence $Q(t) = (\Lambda^{-1})^*P(t)\Lambda^*$ such that $(\Lambda^{-1})^*P(0)\Lambda^* = P(0)$ and $(\Lambda^{-1})^*\dot{P}(0)\Lambda^* = \dot{P}(0)$. Consequently $\dot{Q}(0) = \frac{\mathrm{d}}{\mathrm{d}t}((\Lambda^{-1})^*P(t)\Lambda^*)|_{t=0} = (\Lambda^{-1})^*\dot{P}(0)\Lambda_* = \dot{P}(0)$. Thus $i_{M*} : TM \to \mathfrak{R} \subsetneqq T\mathfrak{R}$ defined by $i_{M*}X := [P_x, L_X]$ for every $x \in M$ and $X \in T_xM$ is well-defined i.e. is independent of how $X \in T_xM$ has been extended to an $X \in C^\infty_c(M;TM)$. Finally assume that $X \neq 0$ hence $L_X \neq 0$ however $[P_x, L_X] = 0$. This would imply that the image $\mathscr{V}_x$ of P_x is invariant under L_X however this is not possible for if e.g. φ_x is any 2-form vanishing at $x \in M$ hence belongs to $\mathscr{V}_x$ then in general $L_X\varphi_x$ does not vanish there hence is not in $\mathscr{V}_x$. All of these allow us to use the non-degenerate scalar product $(A,B) := \tau(AB^*)$ on $\mathfrak{R}$ (completing it to $\mathscr{H}$ as above) to obtain a non-degenerate Riemannian metric g on M via pullback i.e. put $g(X,Y) := \mathrm{Re}(i_{M*}X, i_{M*}Y)$ yielding

$$g_x(X,Y) := \mathrm{Re}\,\tau\big([P_x, L_X][P_x, L_Y]^{\circledast}\big)$$

for $X,Y \in T_xM$ and $x \in M$. This metric therefore does not depend on the particular extensions to $X,Y \in C_c^\infty(M;TM)$ and is canonical in many senses.[4] Finally we remark that (8) in its improved version $(M,g) \subset (\mathfrak{R}, \mathrm{Re}\,\tau)$ is analogous to embedding Riemannian manifolds into Hilbert spaces via heat kernel techniques [3].

The Einstein condition. Having constructed a canonical Riemannian metric g on M, one can introduce more structures on $\mathfrak{R}$. Consider the complexified Hodge operator $* \in C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes \mathbb{C}))$ acting on 2-forms along (M,g). It induces a pointwise g-orthogonal splitting of 2-forms into self-dual and anti-self-dual parts and looks like

$$* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad : \quad \begin{matrix} \Omega_c^+(M;\mathbb{C}) \\ \bigoplus \\ \Omega_c^-(M;\mathbb{C}) \end{matrix} \quad \longrightarrow \quad \begin{matrix} \Omega_c^+(M;\mathbb{C}) \\ \bigoplus \\ \Omega_c^-(M;\mathbb{C}) \end{matrix} \tag{13}$$

with respect to this splitting. This simple algebraic operator gives rise to an element in $\mathfrak{R}$. Take an ascending orthonormal sub-base sequence $\{\varphi_1, \dots, \varphi_{2^n}\}_{n=1,2\dots}$ in $\Omega_c^2(M;\mathbb{C})$ having the property that its first 2^{n-1}-tuple forms a sub-basis of $\Omega_c^+(M;\mathbb{C})$ while the rest gives a sub-basis in $\Omega_c^-(M;\mathbb{C})$. With respect to this sequence there is a decomposition into 2×2 complex matrices

$$* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \dots$$

thus, recalling the infinite tensor product decomposition of the Clifford algebra $C(M)$ as well as its embedding (3), we obtain an element $* \in C(M)$ hence $* \in \mathfrak{R}$. Moreover $*$ is self-adjoint, continues to satisfy $*^2 = 1$ and (6) immediately yields $\tau(*) = 0$. This simple algebraic operator generates a dynamics on $\mathfrak{R}$.

Definition 3.1. *Let M be a connected oriented smooth 4-manifold with its induced Riemannian metric g and corresponding Hodge operator $* \in \mathfrak{R}$ as above. The condition that $* \neq 1$ is self-adjoint and satisfies $*^2 = 1$ implies that there exists a basis in $\mathfrak{R}$ in which $* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Therefore introducig the skew-Hermitian operator* $\log * := \left(\begin{smallmatrix} 0 & 0 \\ 0 & \sqrt{-1}\pi \end{smallmatrix}\right)$ *for every $t \in \mathbb{R}$ we can set $*^t := \mathrm{e}^{t \log *}$ which is a unitary in $\mathfrak{R}$. The corresponding 1-parameter family of $\mathbb{C}$-linear inner $\circledast$-automorphisms on $\mathfrak{R}$ given by*

$$A \longmapsto *^t A *^{-t}$$

for all $A \in \mathfrak{R}$ and $t \in \mathbb{R}$ introduces a non-trivial periodic dynamics on $\mathfrak{R}$ what we call the Hodge dynamics. *Accordingly* $\big(\mathfrak{R}, \{\mathrm{Ad}_{*^t}\}_{t\in\mathbb{R}}\big)$ *is a* Hodge dynamical system on the hyperfinite II_1 factor.

This naturally appearing dynamical system on $\mathfrak{R}$ can be used to characterize not only the plain embedding $M \subset \mathfrak{R}$ but even the geometric properties of the Riemannian embedding $(M,g) \subset (\mathfrak{R}, \mathrm{Re}\,\tau)$ using the terminology of dynamical systems. The Hodge star as an element $* \in \mathfrak{R}$ is self-adjoint, satisfies $1 \neq *$ but $1 = *^2$ hence generates an Abelian von Neumann subalgebra $\langle * \rangle \subset \mathfrak{R}$ isomorphic to $\mathbb{C}1 \oplus \mathbb{C}*$; consequently its relative commutant $\mathfrak{F}(M,g) := \langle * \rangle' \cap \mathfrak{R}$ extends $\langle * \rangle$ to a by construction normal subalgebra of $\mathfrak{R}$ (i.e. is equal to its double relative commutant). It can be identified with the *fixed-point-subalgebra* of the Hodge dynamics: $\mathfrak{F}(M,g) = \big\{A \in \mathfrak{R} \,|\, \mathrm{Ad}_{*^t} A = A \text{ for every } t \in \mathbb{R}\big\}$. Converesely, every normal subalgebra of $\mathfrak{R}$ arises as the fixed-point-subalgebra of a periodic inner $\mathbb{C}$-linear $\circledast$-automorphism of $\mathfrak{R}$, cf. [29, Theorem 3.1].

[4]For instance if M is compact then with this metric the functional $F_{M,g}$ in Lemma 2.1 is normalized i.e. $F_{M,g}(1) = 1$. Morever one can compute that $\tau(P_x) = \frac{1}{\mathrm{Vol}(M,g)}$ holds for the trace of the projections in (8) hence $\mathrm{Vol}(M,g) > 1$.

Lemma 3.1. *Let M be a connected oriented smooth 4-manifold and consider its embedding (8) into $\mathfrak{R}$. Take the induced Riemannian metric g on M and Hodge dynamics $\{\mathrm{Ad}_{*^t}\}_{t\in\mathbb{R}}$ on $\mathfrak{R}$ as well as the subfactor $\mathfrak{I}(M)\subsetneqq\mathfrak{R}$ as in (12) and the Riemannian curvature tensor R_g; assume that $R_g\in\mathfrak{R}$.*

The projections of the embedding satisfy $P_x\in\mathfrak{F}(M,g)$ for every $x\in M$ i.e., $M\subset\mathfrak{R}$ is pointwise preserved by the Hodge dynamics; moreover $\mathrm{Ad}_{*^t}(\mathfrak{I}(M))=\mathfrak{I}(M)$ for every $t\in\mathbb{R}$ i.e., the subfactor itself generated by M is also invariant under the Hodge dynamics however not pointwise. Finally $R_g\in\mathfrak{F}(M,g)$ if and only if (M,g) is Einstein.

Proof. The complexified Hodge star is a bundle morphism of $\wedge^2T^*M\otimes\mathbb{C}$ hence we know from the general theory summarized in (9) that it leaves M pointwise fixed. However let us exhibit another, more direct proof. Consider the embedding (8) given by $x\mapsto P_x$. An important observation is that $*$ commutes with the projection $P_x\in\mathfrak{R}$ for every $x\in M$. Indeed, take $\mathscr{H}=\mathscr{V}_x\oplus\mathscr{V}_x^{\perp}$ with respect to $P_x:\mathscr{H}\to\mathscr{V}_x$; then $*$ obviously preserves $\Omega^2_c(M,x;\mathbb{C})\subset\Omega^2_c(M;\mathbb{C})$ hence $C(M,x)$ consequently $*\mathscr{V}_x=\mathscr{V}_x$; likewise $0=(\mathscr{V}_x,\mathscr{V}_x^{\perp})=(*^2\mathscr{V}_x,\mathscr{V}_x^{\perp})=(*\mathscr{V}_x,*\mathscr{V}_x^{\perp})=(\mathscr{V}_x,*\mathscr{V}_x^{\perp})$ therefore $*\mathscr{V}_x^{\perp}=\mathscr{V}_x^{\perp}$ yielding $[P_x,*]=0$ hence $P_x\in\mathfrak{F}(M,g)$. Moreover using the decomposition (12) pick $A\in\mathfrak{I}(M)=\mathbb{C}1\oplus I(M)\oplus I(M)^{*}$; if $A\in\mathbb{C}1$ then obviously $A\in\mathfrak{F}(M,g)$; if $A\in I(M)$ then by (11) this condition is equivalent to $AP_x=0$ for every $x\in M$ but $(*^tA*^{-t})P_x=*^t(AP_x)*^{-t}=0$ for every $x\in M$ consequently $*^tA*^{-t}\in I(M)$ too and likewise for $A\in I(M)^{*}$. Therefore $*^t(\mathfrak{I}(M))*^{-t}=\mathfrak{I}(M)$. Finally the assertion on the Einstein condition is straightforward, as following [27] we notice at once comparing (4) and (13) that g is Einstein i.e. the traceless Ricci part of R_g vanishes if and only if $[R_g,*]=0$ hence $R_g\in\mathfrak{F}(M,g)$. □

Proof of Theorem 1.3. Taking into account the construction of the metric g above, Definition 3.1 and Lemma 3.1 the theorem follows. □

Comparison of representations and physics. Lemma 3.1 can be regarded as a sort of compatibility result between two pieces of data on the hyperfinite II_1 factor namely a subfactor $\mathfrak{I}(M)\subsetneqq\mathfrak{R}$ and a normal subalgebra $\mathfrak{F}(M,g)\subsetneqq\mathfrak{R}$. A different, somewhat more physical way of understanding compatibility between these two structures is as follows. A connected, compact, oriented, smooth 4-manifold M also gives rise to a representation ρ_M of $\mathfrak{R}$ as in Lemma 2.1 and a Hodge dynamical system $\left(\mathfrak{R},\{\mathrm{Ad}_{*^t}\}_{t\in\mathbb{R}}\right)$ introduced in Definition 3.1. The mathematical fact that $\mathfrak{R}$ admits many *inequivalent* representations (i.e. the failure of the Stone–von Neumann representation theorem in this case) can be interpreted in the framework of *algebraic quantum field theory* as saying that $\mathfrak{R}$ is an operator algebra of a quantum system possessing infinitely many degrees of freedom like a quantum statistical ensemble [4, 16]. In this context, as well as recalling [9], the temptation here to interprete $\mathfrak{R}$ as the operator algebra of a relativistic quantum field theory at non-zero temperature involving gravity, is supported by the following further observations (also cf. [7]). On the one hand $\mathfrak{R}$ contains curvature tensors, the key objects of general relativity. On the other hand the periodicity of the Hodge dynamics on this operator algebra i.e., the plain mathematical property $*^2=1$ of the Hodge star on 2-forms in 4 dimensions, can also be interpreted along these lines in the well-known way, as the presence of a temperature in a statistical ensemble (cf. e.g. [20]). This temperature is the inverse of the period hence is equal to $\frac{1}{2}T_{\mathrm{Planck}}$ in natural units. This permits one to analyse the interference between the aforementioned structures within the realm of the theory of thermal equilibium states in algebraic quantum field theory. For stationarity and stability are expected properties of physical thermal equilibrium states (see e.g. [16, Section V.3]), as a first step in this analysis we record here the following stationarity and stability property of the representations, more precisely their corresponding states on $\mathfrak{R}$, against their induced Hodge dynamics and their perturbations on $\mathfrak{R}$.

Lemma 3.2. *Assume that M is compact and consider its embedding $M\subset\mathfrak{R}$ together with the induced oriented Riemannian 4-manifold (M,g). Also consider the associated Hodge dynamical system $\big(\mathfrak{R},\{\mathrm{Ad}_{*^t}\}_{t\in\mathbb{R}}\big)$.*

Then the state $F_{M,g}:\mathfrak{R}\rightarrow\mathbb{C}$ in the proof of Lemma 2.1 provided by (M,g) is stationary under the Hodge dynamics i.e. $F_{M,g}(*^tA*^{-t})=F_{M,g}(A)$ for every $A\in\mathfrak{R}$ and $t\in\mathbb{R}$.

Moreover take a unitary element $*'\in\mathfrak{R}$ and let $\big(\mathfrak{R},\{\mathrm{Ad}_{(*')^t}\}_{t\in\mathbb{R}}\big)$ be a "nearby" dynamical system in the sense that it preserves $M\subset\mathfrak{R}$ and satisfies $(*')^p=1$ i.e. is periodic with $1\leqq p<+\infty$. Then there exists a corresponding "nearby" state $F_{M,g'}$ on $\mathfrak{R}$, yet inducing the same representation ρ_M of $\mathfrak{R}$, which is stationary under the "nearby" dynamics.

Proof. Recall that $F_{M,g}:\mathfrak{R}\rightarrow\mathbb{C}$ has been defined in Lemma 2.1 as $F_{M,g}(A)=\int_M i^*_M\tau(AP_x)\mu_g$. Taking into account that $*$ commutes with P_x as in Lemma 3.1 and using the cyclic property of the trace it readily follows at once that $F_{M,g}(*^tA*^{-t})=F_{M,g}(A)$ for every $A\in\mathfrak{R}$ and $t\in\mathbb{R}$.

The "perturbed" dynamics generated by $*'$ as a 1-parameter inner $\circledast$-automorphisms of $\mathfrak{R}$ preserves M by assumption hence admits a unique decomposition into a 1-parameter family of gauge transformations and diffeomorphisms according to (10) hence $\mathrm{Ad}_{(*')^t}=\mathrm{Ad}_{\gamma_t}\mathrm{Ad}_{\Phi^*_{-t}}$. Being the "perturbed" dynamics periodic along M its orbits are compact consequently there exists a $*'$-averaged metric g' on M whose volume form $\mu_{g'}$ is preserved by the perturbed dynamics. Thus

$$\begin{aligned}F_{M,g'}\big((*')^tA(*')^{-t}\big) &= \int\limits_{x\in M} i^*_M\tau\big((\mathrm{Ad}_{(*')^t}A)P_x\big)\mu_{g'}(x)=\int\limits_{x\in M} i^*_M\tau\big(A(\mathrm{Ad}_{(*')^{-t}}P_x)\big)\mu_{g'}(x)\\ &= \int\limits_{x\in M} i^*_M\tau\big(A(\mathrm{Ad}_{\gamma_{-t}}\mathrm{Ad}_{\Phi^*_t}P_x)\big)\mu_{g'}(x)=\int\limits_{x\in M} i^*_M\tau\big(AP_{\Phi_{-t}(x)}\big)\mu_{g'}(x)\\ &= \int\limits_{\Phi_{-t}(x)\in M} i^*_M\tau\big(AP_{\Phi_{-t}(x)}\big)\mu_{g'}(\Phi_{-t}(x))\\ &= F_{M,g'}(A)\end{aligned}$$

demonstrating that $F_{M,g'}$ is stationary under the "perturbed" dynamics. Finally we have observed already in the proof of Lemma 2.1 that the ideal $0\neq I(M)\subsetneqq\mathfrak{R}$ in (11) consisting of elements satisfying $F_{M,g'}(A^{\circledast}A)=0$ is independent of the metric g' hence the representation of $\mathfrak{R}$ induced by $F_{M,g'}$ coincides with that one induced by $F_{M,g}$ i.e. with the representation ρ_M of $\mathfrak{R}$ constructed in Lemma 2.1 hence the result. □

Following [2, Chapter 8] appropriately finite representations of the hyperfinite II_1 factor are classified by their Murray–von Neumann coupling constants or $\mathfrak{R}$-dimensions, taking all possible values in the real half-line $[0,+\infty)$, see [2, Proposition 8.6.1]. A representation, uniquely characterized by its $\mathfrak{R}$-dimension $y\in[0,+\infty)$ as its numerical invariant, naturally decomposes according to $y=[y]+\{y\}$ i.e. splits into its integer part with $\mathfrak{R}$-dimension $[y]\in\{0,1,2,\dots\}\subset[0,+\infty)$ containing copies of the representation having $\mathfrak{R}$-dimension precisely 1 and into its fractional part given by $\{y\}\in(0,1)\subset[0,+\infty)$ describing another representation whose $\mathfrak{R}$-dimension falls within the open unit interval. Consequently it is enough to understand those representations which belong to the closed unit interval $[0,1]\subset[0,+\infty)$ only. Of course the representation characterized by $0\in[0,1]$ is just the *trivial representation*. The representation having $\mathfrak{R}$-dimension precisely $1\in[0,1]$ is the *standard representation* π of $\mathfrak{R}$ on itself by (left-)multiplications (as above). This is the best-known non-trivial representation possessing the following remarkable properties:

(i) within the framework of the *Gelfand–Naimark–Segal* (GNS) construction, the unique standard representation π of $\mathfrak{R}$ on a Hilbert space $\mathscr{H}$ can be obtained from a distinguished faithful state, namely the unique finite trace τ on $\mathfrak{R}$;

(ii) the *Tomita–Takesaki* modular theory is applicable to π and the corresponding modular operator Δ renders $\mathfrak{R}$ a dynamical system, however this modular dynamics is trivial because τ is tracial;

(iii) the state τ is a *Kubo–Martin–Schwinger* (KMS) state on $\mathfrak{R}$ with respect to the modular dynamics, however in a trivial way and the formal KMS temperature of this state is infinite, both because τ is tracial.

In Section 2 using smooth 4-manifolds M we have constructed an immense class of geometric representations ρ_M whose $\mathfrak{R}$-dimensions $\gamma(M)$ fall into $[0,1)$. What about properties (i)-(iii) concerning these fractional representations? Interestingly, we can can exhibit a list of analogous properties:

(iv) within the GNS construction every connected compact oriented smooth 4-manifold M gives rise to a representation ρ_M of $\mathfrak{R}$ on a Hilbert space $\mathscr{I}(M)^{\perp}$ obtained from a non-faithful state $F_{M,g}$ on $\mathfrak{R}$ (cf. Lemma 2.1);

(v) to every ρ_M as above there exists a unitary operator $*\in\mathfrak{R}$ which renders $\mathfrak{R}$ a dynamical system such that this Hodge dynamics is already non-trivial nevertheless always satisfies $*^2=1$ (cf. Definition 3.1);

(vi) the state $F_{M,g}$ is invariant under, and the corresponding representation ρ_M is stable against small perturbations of, the Hodge dynamics hence $F_{M,g}$ describes a thermal equilibrium state with respect to this periodic dynamics at a uniform formal temperature $\frac{1}{2}T_{\text{Planck}}$ (cf. Lemma 3.2 and the discussion before it).

This comprehensive view of representations strongly motivates the following physical picture: there exists a unique physical system whose operator algebra is $\mathfrak{R}$ but this system possesses different physical phases corresponding to inequivalent representations of $\mathfrak{R}$. Therefore, as a working hypothesis, it is challenging to physically interprete the quite circular interaction between $\mathfrak{R}$ and M unfolded here, by saying that the unique abstract triple $(\mathfrak{R},\mathscr{H},\pi)$ describes the *quantum phase* of, while a highly non-unique triple $(\mathfrak{R},\mathscr{I}(M)^{\perp},\rho_M)$ giving rise to a 4 dimensional vacuum space-time (M,g), describes a paritulcar state from the usual *classical phase* of Riemannian vacuum general relativity precisely in 4 dimensions. (A usual choice for (M,g) is the FLRW solution with or without cosmological constant.) We can display symbolically this passage as

$$\begin{array}{ccc} (\mathfrak{R},\mathscr{H},\pi) & \Longrightarrow & \big(\{P_x\}_{x\in M}\cup\{R_g\}\subset\mathfrak{R},\mathscr{I}(M)^{\perp}\subset\mathscr{H},\rho_M=\pi|_{\mathscr{I}(M)^{\perp}}\big) \\ \Updownarrow & & \Updownarrow \\ \text{The ``quantum space-time''} & & \text{A particular 4 dimensional Riemannian vacuum space-time } (M,g) \\ \text{at infinite temperature} & & \text{at temperature } \frac{1}{2}T_{\text{Planck}} \end{array}$$

having in mind a sort of phase transition from the quantum to the classical phase of the theory via spontaneous symmetry breaking driven by cooling (or by a spontaneous jump from the unique Tomita–Takesaki to a particular Hodge dynamics on $\mathfrak{R}$). Note that this transition from the unique quantum regime $(\mathfrak{R},\mathscr{H},\pi)$ to a particular 4 dimensional classical vacuum regime (M,g) given by another representation $(\mathfrak{R},\mathscr{I}(M)^{\perp},\rho_M)$ has been captured in the framework of algebraic quantum field theory

[4, 16] as switching from the unique representation π to a different particular representation ρ_M of the same algebra $\mathfrak{R}$. One can also formally label the transition with $\frac{1}{2}T_{\rm Planck} \approx 7.06 \times 10^{31}$ K which is the formal temperature associated with ρ_M; this high temperature is reasonable if we keep in mind that π corresponds to infinite temperature. Finally observe that during this spontaneous symmetry breaking procedure the original gauge group $\mathrm{U}(\mathscr{H}) \cap \mathfrak{R} \subset \mathrm{Aut}\,\mathfrak{R}$ breaks down to its subgroup $\mathrm{Diff}^+(M)$ justifying the terminology.

References

[1] Afzaly, N., Morrison, S., Penneys, D.: *Classification of subfactors with index at most* $5\frac{1}{4}$, preprint, 70 pp. (2015), arXiv: `1509.00038 [math.OA]`;

[2] Anantharaman, C., Popa, S.T.: *An introduction to* II_1 *factors*, perprint, 331 pp. (no year), available at `http://www.math.ucla.edu/~popa/books.html`;

[3] Bérard, P., Besson, G., Gallot, S.: *Embedding Riemannian manifolds by their heat kernel*, Geom. Functional Anal. **4**, 374-398 (1994);

[4] Bratteli, O., Robinson, D.W.: *Operator algebras and quantum statistical mechanics I-II*, Springer, New York (1979, 1981);

[5] Бронштейн, М.П.: *Квантование гравитационных волн*, ЖЭТФ **6**, 195-236 (1936);

[6] Callender, C.: *What makes time special?*, Oxford Univ. Press, Oxford (2017);

[7] Chandrasekaran, V., Longo, R., Penington, G., Witten, E.: *An algebra of observables for de Sitter space*, preprint, 56 pp. (2022), arXiv: `2206.10780 [hep-th]`;

[8] Connes, A.: *Noncommutative geometry*, Academic Press, New York (1994);

[9] Connes, A., Rovelli, C.: *Von Neumann algebra automorphisms and time-thermodynamics relation in general covariant quantum theories*, Class. Quant. Grav. **11**, 2899-2918 (1994);

[10] Etesi, G.: *Exotica and the status of the strong cosmic censor conjecture in four dimensions*, Class. Quantum Grav. **34**, No. 24, 245010, 26 pp. (2017);

[11] Etesi, G.: *The universal von Neumann algebra of smooth four-manifolds*, Adv. Theor. Math. Phys. **25**, 995-1024 (2021);

[12] Etesi, G.: *The universal von Neumann algebra of smooth four-manifolds revisited*, in: *Advances in functional analysis and operator theory* (eds.: Markin, M.V., Nikolaev, I.V., Trunk, C.), Contemporary Mathematics **798**, 125-151 (2024);

[13] Freedman, M.: *The topology of four-dimensional manifolds*, Journ. Diff. Geom. **17**, 357-454 (1982);

[14] Gompf, R.E.: *An exotic menagerie*, Journ. Diff. Geom. **37**, 199-223 (1993);

[15] Gompf, R.E., Stipsicz, A.I.: *4-manifolds and Kirby calculus*, GSM **20**, Amer. Math. Soc. Providence, Rhode Island (1999);

[16] Haag, R.: *Local quantum physics*, Springer–Verlag, Berlin (1993);

[17] Hedrich, R.: *String theory—nomological unification and the epicycles of the quantum field theory paradigm*, preprint, 23pp., arXiv: `1101.0690 [physics.gen-ph]` (2011);

[18] Hedrich, R.: *Raumzeitkonzeptionen in der Quantengravitation*, preprint, 270 pp., arXiv: `1101.1835 [physics.gen-ph]` (2011);

[19] Heisenberg, W., Pauli, W.: *Zur quantendynamik der Wellenfelder*, Zeit. für Physik **56**, 1-61 (1929);

[20] Itzykson, C., Zuber, J-B.: *Quantum field theory*, McGraw–Hill Inc., New York (1980);

[21] Jones, F.R.V., Morrison, S. Snyder, N.: *Classification of subfactors of index at most* 5, Bull. Amer. Math. Soc. **51**, 277-327 (2014);

[22] Lawson Jr., H.B., Michelsohn, M-L.: *Spin geometry*, Princeton Univ. Press, Princeton, New Jersey (1989);

[23] *John von Neumann: selected letters* (ed.: M. Rédei), History of Mathematics **27**, AMS, Providence, Rhode Island, 59-64 (2005);

[24] Ocneanu, A.: *Quantized groups, string algebras and Galois theory for algebras*, in: *Operator algebras and applications*, Vol. 2, London Math. Soc. Lecture Note Ser. **136**, Cambridge Univ. Press, Cambridge, 119-172 (1988);

[25] Popa, S.T.: *Classification of subfactors: the reduction to commuting squares*, Invent. Math. **101**, 19-43 (1990);

[26] Sakai, S.: *On automorphism groups of* II_1*-factors*, Tôhoku Math. Journ. **26**, 423-430 (1974);

[27] Singer, I.M., Thorpe, J.A.: *The curvature of* 4*-dimensional Einstein spaces*, in: *Global analysis*, Papers in honour of K. Kodaira, 355-365, Princeton Univ. Press, Princeton (1969).

[28] Smolin, L.: *The trouble with physics*, A Mariner Book, New York (2006);

[29] Størmer, E.: *Inner automorphisms of von Neumann algebras*, Comm. Math. Phys. **36**, 115-122 (1974);

[30] Wald, R.M.: *General relativity*, Chicago Univ. Press, Chicago (1984).