A rigidity theorem for non-vacuum initial data

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Abstract

In this paper we prove a theorem on initial data for general relativity. The result presents a "rigidity phenomenon" caused by the sign of the scalar curvature.

More precisely, we claim that for a non-vacuum, asymptotically flat initial data set if the spatial metric has everywhere non-positive scalar curvature then the extrinsic curvature cannot be compactly supported.

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1 Introduction

According to our experiences there are several different gravitational configurations in our physical world. Therefore if general relativity is a correct theory of gravitational phenomena (at least at low energies) then it is important to know whether or not these various patterns can be modeled in general relativity i.e., Einstein equations provide enough solutions for describing many different gravitational fields. Unfortunately or fortunately Einstein equations form an extraordinary difficult system of nonlinear partial differential equations for the four dimensional Lorentzian metric mainly because of the rich self-interactions of the gravitational field; hence in general it is a hard job to write down explicit solutions in this theory. Therefore all methods which prove at least the existence of solutions are very important. From this viewpoint, the Cauchy problem or initial value formulation of general relativity is maybe the most powerful method to generate plenty of solutions.

As it is well-known, the initial value formulation gives rise to a correspondence between globally hyperbolic space-times and gravitational initial data. Maybe we can say without an exaggeration that the class of globally hyperbolic space-times is the most important class of space-times from the physical point of view. Consequently the initial data formulation provides not only many but also physically relevant solutions. The constraint equations between initial data are in the focal point of the initial data formulation. The question is whether or not these constraint equations are easier to solve than the original Einstein equation itself making the method effective. Of course, the answer is typically yes.

This motivates the serious efforts made in order to understand the structure and provide solutions of constraint equations. Far from being complete we just mention the early works of Lichnerowicz [16], Choquet-Bruhat et al. [2][3][6] , Fisher-Marsden [8], Christodoulou–Kleinerman[7]. These works mainly deal with the analytical properties of the solutions. Witt proved the existence of solutions on a general three-manifold [20]. More recently, in a sequence of papers

Isenberg, Moncrief, Choquet-Bruhat and York proved the existence of solutions under milder and milder assumptions, cf. e.g. [12][4][14][5].

The constraint equations involve the scalar curvature of the metric on the underlying Cauchy surface which is a three dimensional smooth manifold. Various properties of the solutions depend crucially on the scalar curvature, especially on its sign. But we know that in the problem of describing the sign of the scalar curvature, especially on a compact manifold, one encounters the topology of the space. Parallelly to the investigations of solutions of the constraint equations by physicist and mathematicians, mathematicians proved remarkable results on the properties of the scalar curvature of Riemannian manifolds. By an early general result of Kazdan and Warner [15] we know that for compact manifolds of dimension greater than two there is no constraint on the scalar curvature *if there is at least one point where it is negative.* This shows that it is easy to construct manifolds with negative scalar curvature. If we wish to construct manifolds with non-negative scalar curvature however, we have to face various obstacles coming from the topology of the underlying manifold. We just mention two basic examples. By results of Lichnerowicz and Hitchin, on spin manifolds it is often impossible to construct metrics of positive scalar curvature because of a subtle topological invariant, the so-called $\hat{A}$-genus [11][17]. Moreover, in three dimensions, the size of the fundamental group provides another obstruction for positive scalar curvature by results of Gromow–Lawson [9] and Schoen–Yau [18]. An excellent survey on this branch of differential geometry is [1].

These observations make it not surprising that the topology of the Cauchy surface has a strong influence on the properties of initial data on it. The goal of this paper is to understand this link a bit better.

Our paper is organized as follows. The motivation is a result of Witt [20]: he proved that every connected, oriented three-maifold with an end admits asymptotically flat initial data (M, g, k). These asymptotically flat data are typically non-vacuum and are not maximal slices i.e., the resulting Cauchy surfaces are generally *not* space-like submanifolds of identically zero extrinsic curvature k. Hence, two natural questions arise. First, what are the conditions on the complete Riemannian manifold (M, g) which can carry a *vacuum*, asymptotically flat initial data set? Recently it has been proved by Isenberg, Mazzeo and Pollack that every punctured n-manifold can carry an asymptotically Euclidean vacuum solution of the Einstein constraint equations [13].

Secondly, if (M, g, k) is a non-vacuum, asymptotically flat initial data set with M being a non-maximal slice, in what extent is it non-maximal? In other words what are the conditions on a Riemannian manifold (M, g) for its extrinsic curvature in the Cauchy development to be compactly supported at least? This problem is the subject of our paper. We will see that if (M, g) is a complete Riemannian-manifold whose scalar curvature is everywhere non-positive, then the extrinsic curvature k is never compactly supported.

2 Background material

First let us introduce some notations. Let W be a smooth manifold. We will call a tensor field T of type (m, n) over W if it is a smooth section of the bundle

$$T^{(m,n)}W := \underbrace{TW \otimes \ldots \otimes TW}_{m} \otimes \underbrace{T^*W \otimes \ldots \otimes T^*W}_{n}.$$

Remember that an *initial data set for general relativity* is a triple (M, g, k), where M is a (not necessarily compact) connected, oriented, smooth three-manifold, $g = (g_{ij})$ is a smooth, complete Riemannian metric on M i.e. a non-degenerate smooth symmetric tensor field of type $(0,2)$ on M while $k=(k_{ij})$ is a smooth, symmetric tensor field on M also of $(0,2)$-type. These fields must satisfy the following constraint equations [10][19]:

$$\begin{cases} s_g - |k|_g^2 + \mathrm{tr}^2 k = 16\pi\rho, \\ \\ \mathrm{div}(k - (\mathrm{tr}k)g) = 8\pi J, \\ \\ \rho \geq |J|_g \geq 0. \end{cases} \tag{1}$$

Here s_g is the scalar curvature of the metric g and $|\cdot|_g$ denotes various norms given by the induced scalar product on $T^{(m,n)}M$, e.g. $|k|_g^2 = \langle k,k\rangle = k_{ij}k^{ij}$. The operator $\mathrm{tr}: T^{(m,n)}M \to T^{(m-1,n-1)}M$ is the *trace* or *contraction* with respect to the metric, e.g. $\mathrm{tr}k = k^i_{\ i}$. For the sake of simplicity in the second equation we also denote by g and k the $(1,1)$-tensors with respect to the metric g (i.e. $g=(g^i_{\ j})$, $k=(k^i_{\ j})$ in the second equation). The linear differential operator $\mathrm{div}: C^\infty(T^{(1,n)}M) \to C^\infty(T^{(0,n)}M)$ is the covariant divergence, defined by

$$\mathrm{div}T := \mathrm{tr}(\nabla T)$$

where T is a tensor field of $(1,n)$-type and ∇ is the Levi–Civitá covariant derivative of the metric g. The smooth function $\rho: M \to \mathbb{R}$ is the energy-density, and the smooth covector field $J \in C^\infty(T^*M)$ with $|J|_g^2 = \langle J,J\rangle = J_iJ^i$ is interpreted as the momentum-density of matter.

Supposing the energy- and momentum-densities correspond to classical non-dissipative matter sources or vacuum ($J=0$, $\rho=0$), the coupled Einstein-equations can be used to evolve the initial data set (M,g,k) into a (globally hyperbolic) smooth space-time (N,h) where $N \cong M\times\mathbb{R}$ and M is a Cauchy surface in N; furthermore $h|_M = g$ and k is the second fundamental form or extrinsic curvature of M in (N,h) [10] [19].

Also remember that an open, connected, oriented three-manifold M has an *end* $E\subset M$ if there is a compact set $C\subset M$ such that $M\setminus C = E$ and one can find a diffeomorphism $\phi: S_g\times(0,\infty)\to E$ where S_g is a compact, oriented surface of genus g. An initial value data set (M,g,k) is called *asymptotically flat along* E if M has an end $E\cong S^2\times(0,\infty)$ and one can find a coordinate system along $S^2\times(0,\infty)$ such that the following asymptotical fall-off conditions hold for the complete metric g and the field k (r parameterizes $\mathbb{R}$):

$$\begin{aligned} &\phi^*(g|_E)_{ij} = \delta_{ij} + O(1/r), \quad && \phi^*(k|_E)_{ij} = O(1/r^2), \\ &\partial_l\phi^*(g|_E)_{ij} = O(1/r^2), && \partial_l\phi^*(k|_E)_{ij} = O(1/r^3), \\ &\partial_l\partial_k\phi^*(g|_E)_{ij} = O(1/r^3). \end{aligned} \tag{2}$$

Notice that the definition of a manifold with an end does not exclude the possibility that M still has a boundary. However geodesic completeness of g requires that this extra boundary must be empty, in other words M is diffeomorphic to the punctured manifold $\widetilde{M}\setminus\{*\}$ where $\widetilde{M}$ is a connected, compact, oriented three-manifold without boundary. Moreover geodesic completeness implies the Heine–Borel property, namely talking about "compact sets" is the same as talking about "bounded and closed sets" in M (with respect to the metric).

Finally, the *support* of a tensor field $T\in C^\infty(T^{(m,n)}W)$ is the closed set

$$\mathrm{supp}\, T := \overline{\{x\in W \mid T(x)\neq 0\}}.$$

After this background material, we can present our theorem.

3 Rigidity for non-vacuum initial data

Now we prove a theorem which states that on open Riemannian manifolds with everywhere non-positive scalar curvature the extrinsic curvature field of a non-vacuum initial data cannot be compactly supported i.e. it has a "tail" at infinity although this tail may have sufficiently fast fall-off to make such an initial data still asymptotically flat. The proof of this theorem is elementary but analytic in its nature as consequence of using arbitrary matter fields. It is based on the following idea.

By using the initial data set (M, g, k) and the assumption that supp$k$ is compact, we construct another "universal" initial data set $(M, g, \varphi g)$ where $\varphi : M \to \mathbb{R}$ is a compactly supported at least once continuously differentiable (or C^1-) function on M (with a little more effort this function could be smoothened but we do not need this). However this leads us to a contradiction if the scalar curvature of g is non-positive everywhere. In other words, we deform the original initial data set into a standard one whose properties are easier to understand.

Theorem (Rigidity for non-vacuum initial data). *Let (M, g) be a connected, oriented, complete Riemannian three-manifold with an end $S^2 \times (0, \infty) \cong E \subset M$. Suppose the scalar curvature s_g of g is non-positive everywhere and there is a non-vacuum initial data set (M, g, k) on it which is asymptotically flat along the end E. Then* supp k *is non-compact.*

Proof. Since the scalar curvature is non-positive, the set supp s_g consists of the closure of those points where s_g is negative. Then the first and third (in)equalities of (1) show that supp$\,s_g \subseteq$ supp$\,k$ therefore if the scalar curvature is negative everywhere the statement is trivially true, consequently we may assume that supp $s_g \subset M$. In the same fashion, since (M, g, k) is a non-vacuum data set, there is a point $x_0 \in M$ such that $\rho(x_0) \neq 0$. This yields supp $\rho \neq \emptyset$. Being the scalar curvature non-positive, via the first and third (in)equalities of (1) again we have $k(x_0) \neq 0$ i.e., supp $\rho \subseteq$ supp k. Therefore if the energy density is supported everywhere the theorem is again trivially valid consequently we may assume supp $\rho \subset M$. Consider a subset $C \subset M$ such that supp $s_g \subset C$ and supp $\rho \subset C$ and suppose the decomposition $M = C \cup E$ is valid where E denotes the end of M. Consequently by the structure of M we may assume that C is compact. This shows that there is a constant

$$0 < a := \sup_{x \in C} \left(-|k(x)|_g^2 + \mathrm{tr}^2 k(x) \right) < \infty.$$

The constraint equations (1) can be rewritten as follows by using the decomposition of k into trace- and tracefree parts $k = \varphi g + k_0$ and noticing that $< \varphi g, k_0 >= 0$:

$$\begin{cases} s_g + 6\varphi^2 = 16\pi\rho + |k_0|_g^2, \\ \\ \mathrm{div}(-2\varphi g) = 8\pi J - \mathrm{div} k_0, \\ \\ \rho \geq |J|_g \geq 0. \end{cases}$$

Consider a triple $(M, g, \varphi g)$ where $\varphi : M \to \mathbb{R}$ is a C^1-function. Hence this triple is a weak initial data set if it obeys the constraint equations with "energy density" $\rho + (1/16\pi)|k_0|_g^2$ and "current" $J - (1/8\pi)\mathrm{div}k_0$ (here by "weak" we mean that the initial data set in question is not smooth, only C^k for some $k \in \mathbb{N}$). However these modified matter has to satisfy the dominant energy condition

$$\rho + \frac{1}{16\pi}|k_0|_g^2 \geq \left| J - \frac{1}{8\pi}\mathrm{div}k_0 \right|_g$$

yielding a first order partial differential inequality for the unknown function φ:

$$\frac{1}{4}(s_g + 6\varphi^2) \geq |\mathrm{d}\varphi|_g \tag{3}$$

Assume $\emptyset \neq$ supp $k \subset M$ is compact i.e., the theorem is not true. In this case we construct a compactly supported function φ, out of the original data (M, g, k) such that $(M, g, \varphi g)$ is a weak initial data set. We achieve this in three steps.

(i) *Construction of φ in the compact interior of M.* Let us identify the end $E \subset M$ with $S^2 \times (\mathbb{R}^+ \setminus \{0\})$. By assumption $\operatorname{supp} k$ is compact in M consequently there is an $R_1 \in \mathbb{R}^+$ satisfying $S^2 \times (R_1, \infty) \not\subset \operatorname{supp} k$. Note that this is possible only if $s_g|_{S^2\times(R_1,\infty)} = 0$. We can take the choice $C := M \setminus (S^2 \times (R_1, \infty))$ for the compact set used in the definition of the constant a. We construct the function φ in C as follows:

$$\varphi(x) := -\sqrt{a}, \quad x \in C.$$

In other words φ is a constant negative function on M except the infinite tube $S^2 \times (R_1, \infty)$. Note that with this function (3) is trivially satisfied in C because (M, g, k) is an initial data set on C.

(ii) *Construction of φ along an annulus in E.* Consider an inner point $x_0 \in C \subset M$ where $\rho(x_0) > 0$ and $k(x_0) \neq 0$. There is an open (geodesic) ball $B_\varepsilon(x_0) \subset M$ of radius $\varepsilon > 0$ such that $\rho|_{B_\varepsilon(x_0)} > 0$ and $k|_{B_\varepsilon(x_0)} \neq 0$. Consider the annulus $U_\varepsilon := \overline{B_\varepsilon(x_0) \setminus B_{\frac{\varepsilon}{2}}(x_0)} \cong S^2 \times [\frac{\varepsilon}{2}, \varepsilon]$. Take another constant $R_1 < R_2 < \infty$ and the diffeomorphism

$$\beta : U_\varepsilon \longrightarrow S^2 \times [R_1, R_2], \quad x_t = (p, t) \longmapsto \left(p\,,\ R_1 + \frac{2t - \varepsilon}{\varepsilon}(R_2 - R_1)\right) = (p, r)$$

where $p \in S^2$ and the point $x_t \in U_\varepsilon$ is identified with $(p, t) \in S^2 \times \left[\frac{\varepsilon}{2}, \varepsilon\right]$. Here $S^2 \times [R_1, R_2]$ is also an annulus in the tube E. By assumption g is asymptotically flat i.e., the function $\sqrt{g^{11}} \geq 0$ is bounded consequently there is a constant

$$0 < b := \sup_{x\in M} \sqrt{g^{11}(x)} < \infty$$

(here $x^1 = r$). Choose a smooth function $\psi : [R_1, R_2] \to \mathbb{R}^-$. Viewing it as a function on $S^2 \times [R_1, R_2]$ (i.e. a function depending only on r), one obtains the estimate

$$b|\psi'| \geq |\sqrt{g^{11}}\psi'| = |\mathrm{d}\psi|_g \tag{4}$$

where prime denotes differentiation with respect to r. Now we define ψ as follows:

$$\psi(\beta(x_t)) := \begin{cases} -\sqrt{a} & \text{if } t = \frac{\varepsilon}{2}, \\ \text{arbitrary but the derivative of } \psi \text{ is small} & \text{if } t \in (\frac{\varepsilon}{2}, \varepsilon), \\ 0 & \text{if } t = \varepsilon. \end{cases}$$

In this definition the smallness of ψ' means the following. Consider a differentiable curve $\gamma : [\frac{\varepsilon}{2}, \varepsilon] \to U_\varepsilon$ given by

$$t \longmapsto x_t := \left(\Theta_{\frac{\varepsilon}{2}} + A\sin(R_2 - R_1)t\,,\ \phi_{\frac{\varepsilon}{2}} + A\sin(R_2 - R_1)t\,,\ t\right).$$

This is a high-speed curve because it oscillates rapidly inside $B_\varepsilon(x_0)$. More precisely, for its speed $|\dot\gamma(t)|_g \sim R_2 - R_1$ is valid (dot denotes differentiation with respect to t). We can take a choice for the point x_0 and the amplitude A, the initial phases $\Theta_{\frac{\varepsilon}{2}}$ and $\phi_{\frac{\varepsilon}{2}}$ of the curve γ such that

$$\mathrm{d}\left(\sqrt{-|k(x_t)|^2_g + \mathrm{tr}^2 k(x_t)}\right)(\dot\gamma(t)) \sim -(R_2 - R_1) < 0$$

holds for each $t \in \left[\frac{\varepsilon}{2}, \varepsilon\right]$. Then we suppose

$$0 \leq \psi'(\beta(x_t)) \leq \min\left(-\frac{\varepsilon}{4(R_2-R_1)}\mathrm{d}\left(\sqrt{-|k(x_t)|^2_g + \mathrm{tr}^2 k(x_t)}\right)(\dot\gamma(t))\,,\ \frac{16\pi}{b}\rho(x_t)\right). \qquad (5)$$

We emphasize that the right hand side of (5) is independent of the quantity $R_2 - R_1$ by construction of the curve γ.

It is also clear that such a function exists if R_2 is suitable large: let ψ be an arbitrary smooth, negative-valued function $\psi : S^2 \times [R_1, R_2] \to \mathbb{R}^-$ with initial value $\psi(p, R_1) = \psi(\beta(x_{\frac{\varepsilon}{2}})) = -\sqrt{a}$. Suppose there is an interval $[R, R+T] \subset [R_1, R_2]$ such that ψ' obeys (5) but there is a constant $c > 0$, independent of $R_2 - R_1$, with $\psi'(p, r) \geq c$ if $r \in [R, R+T]$. In this case we can estimate for large R_1 and R_2 as follows:

$$\psi(p, R_2) \geq -\sqrt{a} + \int\limits_R^{R+T} \mathrm{d}\psi(p,r)(\beta'(x_t))\mathrm{d}r =$$

$$-\sqrt{a} + \int\limits_R^{R+T} \psi'(p,r)\left(g^{11}(p,r) + A\frac{\varepsilon}{2}(g^{12}(p,r) + g^{13}(p,r))\cos\frac{\varepsilon}{2}(r + R_2)\right)\mathrm{d}r \geq$$

$$-\sqrt{a} + \frac{1}{2}\int\limits_R^{R+T} \psi'(p,r)\mathrm{d}r \geq -\sqrt{a} + \frac{cT}{2}.$$

In other words if T that is, $R_2 - R_1$ is sufficiently large we can achieve that $\psi(p, R_2) = 0$. We choose φ on $S^2 \times [R_1, R_2]$ to be the ψ just constructed.

It is not difficult to check that φ obeys (3) in $S^2 \times [R_1, R_2]$. Indeed, by the definition of the constant a we have

$$\varphi(p, R_1) = \varphi(\beta(x_{\frac{\varepsilon}{2}})) = -\sqrt{a} \leq -\sqrt{-\left|k\left(x_{\frac{\varepsilon}{2}}\right)\right|^2_g + \mathrm{tr}^2 k\left(x_{\frac{\varepsilon}{2}}\right)}\,.$$

Taking suitable large R_1 and R_2, exploiting the decay of the metric g and using (5) this implies that for each $t \in \left[\frac{\varepsilon}{2}, \varepsilon\right]$ we have

$$\varphi(\beta(x_t)) = \varphi(p, r) = -\sqrt{a} + \int\limits_{R_1}^{r} \mathrm{d}\varphi(p,\varrho)(\beta'(x_\tau))\mathrm{d}\varrho =$$

$$-\sqrt{a} + \int\limits_{R_1}^{r} \varphi'(p,\varrho)\left(g^{11}(p,\varrho) + A\frac{\varepsilon}{2}(g^{12}(p,\varrho) + g^{13}(p,\varrho))\cos\frac{\varepsilon}{2}(\varrho + R_2)\right)\mathrm{d}\varrho \leq$$

$$-\sqrt{a}+2\int\limits_{R_1}^{r}\varphi'(p,\varrho)\mathrm{d}\varrho=-\sqrt{a}+\frac{4(R_2-R_1)}{\varepsilon}\int\limits_{\frac{\varepsilon}{2}}^{t}\varphi'(\beta(x_\tau))\mathrm{d}\tau\leq$$

$$-\sqrt{-\left|k\left(x_{\frac{\varepsilon}{2}}\right)\right|_g^2+\mathrm{tr}^2k\left(x_{\frac{\varepsilon}{2}}\right)}-\int\limits_{\frac{\varepsilon}{2}}^{t}\mathrm{d}\left(\sqrt{-|k(x_\tau)|_g^2+\mathrm{tr}^2k(x_\tau)}\right)(\dot\gamma(\tau))\mathrm{d}\tau=-\sqrt{-|k(x_t)|_g^2+\mathrm{tr}^2k(x_t)}.$$

Consequently

$$\varphi^2(\beta(x_t))\geq-\left|k(x_t)\right|_g^2+\mathrm{tr}^2k(x_t).$$

Therefore, since $s_g(\beta(x_t))=0$ and $0\geq s_g(x_t)$, we can write

$$\frac{1}{4}\left(s_g(\beta(x_t))+6\varphi^2(\beta(x_t))\right)=\frac{3}{2}\varphi^2(\beta(x_t))\geq s_g(x_t)-|k(x_t)|_g^2+\mathrm{tr}^2k(x_t)=16\pi\rho(x_t).$$

Moreover, also by (5), we have for the same $x_t\in U_\varepsilon$ that $16\pi\rho(x_t)\geq b\varphi'(\beta(x_t))$. This gives rise to our key inequality

$$\frac{3}{2}\varphi^2(\beta(x_t))\geq b\varphi'(\beta(x_t))\tag{6}$$

showing via (4) that (3) is again satisfied in the annulus $S^2\times[R_1,R_2]$.

(iii) *Construction of φ along the remaining part of the infinitely long tube in M.* Finally, define

$$\varphi(x):=0\quad\text{if } x\in S^2\times[R_2,\infty).$$

Again, (3) is trivially valid.

Consider the function $\varphi: M\to\mathbb{R}^-$ defined through (i)-(iii). This is a continuous negative function on M and is compactly supported: it is equal to zero for all $r\geq R_2$ and equal to the constant $-\sqrt{a}$ if $r\leq R_1$. Its derivative is also compactly supported in $S^2\times[R_1,R_2]$ and is positive. Moreover φ can be adjusted to be C^1 on M (note that φ is smooth except the junction points): it is clearly C^1 at $r=R_2$ by (6). However, by exploiting the freedom in the construction of φ in the inner points of the annulus, we can deform it to be C^1 at $r=R_1$ as well (i.e., we may assume that $\varphi'(p,r)\to0$ as $r\to R_1$). In this way we have constructed a weak C^1 initial data set $(M,g,\varphi g)$ (with a little effort we could smooth this data but we do not need this).

The compactly supported φ depends nontrivially only on r with $(p,r)=\beta(x_t)\in S^2\times[R_1,R_2]$ and satisfies the ordinary differential inequality (6). Now we demonstrate that it is impossible. Dividing by φ'^2 and taking reciprocies in (6) we get

$$\left(\frac{\varphi'}{\varphi}\right)^2\leq\frac{3\varphi'}{2b},$$

which is nothing but

$$-\sqrt{\frac{3\varphi'}{2b}}\leq\frac{\varphi'}{\varphi}\leq\sqrt{\frac{3\varphi'}{2b}}.$$

By integrating the left inequality from R_1 to $r<R_2$ we arrive at the following estimate:

$$\log\sqrt{a}-\sqrt{\frac{3}{2b}}\int\limits_{R_1}^{R_2}\sqrt{\varphi'(p,\varrho)}\,\mathrm{d}\varrho\leq\log(-\varphi(p,r)).$$

At this point we have used the inequality

$$0<\int\limits_{R_1}^{r}\sqrt{\varphi'(p,\varrho)}\,\mathrm{d}\varrho\leq\int\limits_{R_1}^{R_2}\sqrt{\varphi'(p,\varrho)}\,\mathrm{d}\varrho<\infty$$

for the non-negaitve function φ'. This shows that the logarithm of φ is bounded from below. However being φ compactly supported, $\log(-\varphi(p,r))$ is unbounded, as r approaches R_2. Consequently the last but one inequality shows a contradiction yielding our original assumption, that $\mathrm{supp}\, k$ is compact, was wrong. We finished the proof. $\diamond$

Remarks. 1.We would like to summarize here how the original initial data (M, g, k) was used in the construction because apparently its behaviour has been taken into account only in a particular small ball $B_\varepsilon(x_0)$. But in fact the construction is sensitive for the global characteristics of the original initial data. In step (i) we considered (M, g, k) in the whole interior C by exploiting the existence of the constant a which is in some sense the maximum of k in the whole compact C. This enabled us to "pump up" the original initial data in C into a standard one which corresponds to the extremal point(s) of the original extrinsic curvature in some sense. Concerning part (iii), we have seen in the beginning of the proof that the only interesting possibility for our would-be initial data with compactly supported extrinsic curvature was the case where both the scalar curvature and energy-density were compactly supported. Consequently all fields in the initial data vanish along the tube for very large r yielding the hypothetical initial data did not carry "information" along an infinitely long part of the end E. This is in accordance with the fact that our adjusted universal initial data $(M, g, \varphi g)$ was also trivial on this portion. Finally, part (ii) which is the descending regime, is nothing but a magnification of the behaviour of (M, g, k) in a small ball where matter is present via the diffeomorphism β. Indeed this small ball is responsible for the details of the fall-off of φ (we could have used equally well any other ball) however the fact that this function can vanish within a finite distance, is again guaranteed by the global properties of the original would-be initial data set: namely the only interesting case was when all fields were compactly supported.

2. Note that even if $\mathrm{supp}\, k$ is non-compact the non-vacuum data (M, g, k) may be asymptotically flat, as it is shown by Witt [20] who constructs non-vacuum, asymptotically flat initial data for every three-manifold with an end. But the above theorem is sharp in the following sense. If we allow for a Riemannian manifold (M, g) to have positive scalar curvature in a suitable region in M, it is possible to construct non-vacuum asymptotically flat initial data with compactly supported second fundamental form. An example is the Tolman–Bondi solution. This is because in this case the key inequality (6) can be written in the form

$$\frac{1}{4}(s_g+6\varphi^2)\geq b\varphi'$$

with $s_g>0$ in the positive scalar curvature regime and it may have compactly supported solutions. But if s_g is still negative somewhere, then k is non-zero in that point, consequently the initial surface is not a maximal slice in this case.

3. Notice that the above considerations do not remain valid for *vacuum initial data.* For example, the Schwarzschild space-time has initial data with non-positive scalar curvature (namely it is identically zero) but the extrinsic curvature of the initial surface is compactly supported (namely identically zero i.e., the initial surface is a maximal slice).

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