Exotica or the failure of the strong cosmic censorship in four dimensions

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The meaning and physical formulation of the strong cosmic censor conjecture (SCCC)

We have the strong conviction that in the *classical* physical world at least, every physical event (possibly except the Big Bang) has a cause which is another and preceding physical event. R. Penrose’ original aim in the 1960-1970’s with formulating the *strong cosmic censor conjecture* (or hypothesis) was to protect this concept of causality in general gravitational situations. Mathematically speaking space-times having this property are *globally hyperbolic* therefore

**SCCC.** *A physically relevant, generic space-time is globally hyperbolic.*

**Remark**

By a *space-time* we will always mean an *inextendible*, oriented and time-oriented Lorentzian 4-manifold \((M, g)\).
(i) Without the “genericity” assumption the SCCC is trivially not true, almost all basic solutions (AdS, Taub–NUT, Gödel, Kerr–Newman, etc.) are counterexamples. But problem: what is “genericity”?

(ii) Conventional approach: using the initial value formulation “genericity” can be defined rigorously. Within this framework during the past five decades several partial results appeared and gave a support for the validity of the SCCC in certain classes of space-times (Gowdy space-times, space-times with compact Cauchy horizon, scalar field space-times, etc.).
A conceptional problem with the initial value approach to the SCCC

(i) In this approach “genericity” of three dimensional initial data are defined rigorously and considered. However, when thinking about the SCCC, certainly one should be able to talk about the “genericity” of the four dimensional space-time itself.

(ii) In light of discoveries in low dimensional differential topology during the 1980’s typical smooth 4-manifolds have unexpected properties (called exotica) not detectable from a 3-manifold perspective. Therefore one has to be cautious when applies the genuinely 3 dimensional initial value paradigm, investigated during the 1950-60’s, to genuinely 4 dimensional space-time problems.
The idea of a 4 dimensional counterexample

Definition
Let \((M, g)\) be a maximal extension of the maximal Cauchy development \((D(S), g|_{D(S)})\) of an initial data set \((S, h, k)\). The (continuous) Lorentzian manifold \((M', g')\) is a perturbation of \((M, g)\) relative to \((S, h, k)\) if

(i) \(M'\) has the structure

\[
M' := \text{the connected component of } M \setminus \mathcal{H} \text{ containing } S
\]

where, for a connected open subset \(S \subset U \subseteq M\) containing the initial surface, \(\emptyset \subseteq \mathcal{H} \subseteq \partial U\) is a closed subset in the boundary of \(U\);

(ii) \((M', g')\) does not admit further extensions and \((S, h') \subset (M', g')\) with \(h' := g'|_S\) is a spacelike complete sub-3-manifold.
Definition
Let \((M, g)\) be a maximal extension of the maximal Cauchy development \((D(S), g|_{D(S)})\) of an initial data set \((S, h, k)\). Then \((M, g)\) is a robust counterexample to the \textbf{SCCC} if it is very stably non-globally hyperbolic i.e., all of its perturbations \((M', g')\) relative to \((S, h, k)\) are not globally hyperbolic.

Remark

(i) Perturbations of the whole 4 dimensional space-time are considered;

(ii) The concept of a robust counterexample is logically stronger than the “generic” one.
The Gompf–Taubes family of large exotic $\mathbb{R}^4$’s

Strongly motivated by the Bernal–Sánchez smooth splitting theorem (2003) and the Chernov–Nemirovski smooth censorship theorem (2013) we consider a family of fake $\mathbb{R}^4$’s:

Theorem (Gompf 1993, Taubes 1987)

There exists a pair $(\mathbb{R}^4, K)$ consisting of a differentiable 4-manifold $\mathbb{R}^4$ homeomorphic but not diffeomorphic to the standard $\mathbb{R}^4$ and a compact oriented smooth 4-manifold $K \subset \mathbb{R}^4$ such that

1. $\mathbb{R}^4$ cannot be smoothly embedded into the standard $\mathbb{R}^4$ i.e., $\mathbb{R}^4 \not\subseteq \mathbb{R}^4$ but it can be smoothly embedded as a proper open subset into the complex projective plane i.e., $\mathbb{R}^4 \subsetneq \mathbb{C}P^2$;

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(ii) Take a homeomorphism $f : \mathbb{R}^4 \to \mathbb{R}^4$, let $0 \in B_t^4 \subset \mathbb{R}^4$ be the standard open 4-ball of radius $t \in \mathbb{R}^+$ centered at the origin and put $R_t^4 := f(B_t^4)$ and $R_{+\infty}^4 := \mathbb{R}^4$. Then

$$\left\{ R_t^4 \mid r \leq t \leq +\infty \text{ such that } 0 < r < +\infty \text{ satisfies } K \subset R_r^4 \right\}$$

is an uncountable family of nondiffeomorphic exotic $\mathbb{R}^4$'s none of them admitting a smooth embedding into $\mathbb{R}^4$ i.e., $R_t^4 \not\subseteq \mathbb{R}^4$ for all $r \leq t \leq +\infty$. ◇
Theorem (Etesi 2015)

The space $\mathbb{R}^4$ carries a smooth Lorentzian Ricci-flat metric $g$. Moreover there exists an open (i.e., non-compact without boundary) contractible spacelike and complete sub-3-manifold $(S, h) \subset (\mathbb{R}^4, g)$ in it such that $h = g|_S$.

The Ricci-flat Lorentzian manifold $(\mathbb{R}^4, g)$ might be timelike and (or) null geodesically incomplete.

Idea of the proof.

(i) Apply Penrose’ non-linear graviton construction (i.e., twistor theory) for the embedding $\mathbb{R}^4 \subset \mathbb{C}P^2$ to obtain a hyper-Kähler metric $g_1$ on $\mathbb{R}^4$;

(ii) Use Wick rotation (in a precise way) to get a Ricci-flat Lorentzian metric $g$ on $\mathbb{R}^4$. ◇
Lemma (Etesi 2015)

Consider the pair \((R^4, K)\) and the Ricci-flat Lorentzian manifold \((R^4, g)\) with its open contractible spacelike and complete sub-3-manifold \((S, h) \subset (R^4, g)\). Let \((S, h, k)\) be the initial data set inside \((R^4, g)\) induced by \((S, h)\) and let \((M', g')\) be a perturbation of \((R^4, g)\) relative to \((S, h, k)\).

Assume that \(K \subset M'\) holds. Then \((M', g')\) is not globally hyperbolic.

Idea of the proof. Assume \((M', g')\) was globally hyperbolic. Then \(M' \cong R^4\). Moreover there exists \(0 < r \leq t_0 \leq +\infty\) such that

\[
K \subsetneq R^4_{t_0} \subseteq M' \subseteq R^4_{+\infty} = R^4
\]

but we know \(R^4_{t_0} \not\subseteq R^4\) and this is a contradiction. ♦
Although \((R^4, g)\) is not a robust counterexample as we defined, it is still very reasonable to consider the Ricci-flat Lorentzian space \((R^4, g)\) as a generic counterexample to the SCCC.

**Remark**
The resolution of the apparent conflict between the initial value approach (providing affirmative answers for the SCCC) and this more global one (providing results against the SCCC) is that the initial value approach probes only the tubular neighbourhood of 3 dimensional spacelike submanifolds in the ambient 4 dimensional space-time hence cannot detect exotica at all!
For further details please check:


Thank you!