Geometric construction of new Yang–Mills instantons over Taub-NUT space

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Abstract

In this paper we exhibit a one-parameter family of new Taub-NUT instantons parameterized by a half-line. The endpoint of the half-line will be the reducible Yang–Mills instanton corresponding to the Eguchi–Hanson–Gibbons L^2 harmonic 2-form, while at an inner point we recover the Pope–Yuille instanton constructed as a projection of the Levi–Civitá connection onto the positive $\mathfrak{su}(2)^+ \subset \mathfrak{so}(4)$ subalgebra. Our method imitates the Jackiw–Nohl–Rebbi construction originally designed for flat $\mathbb{R}^4$. That is we find a one-parameter family of harmonic functions on the Taub-NUT space with a point singularity, rescale the metric and project the obtained Levi–Civitá connection onto the other negative $\mathfrak{su}(2)^- \subset \mathfrak{so}(4)$ part. Our solutions will possess the full $U(2)$ symmetry, and thus provide more solutions to the recently proposed $U(2)$ symmetric ansatz of Kim and Yoon.

1 Introduction

Motivated by Sen's S-duality conjecture [27] from 1994, recently there have been some interest in understanding the L^2-cohomology of certain hyper-Kähler moduli spaces of magnetic monopoles. Probably the strongest evidence in Sen's paper for his conjecture was an explicit construction of an L^2 harmonic 2-form on the universal double cover of the Atiyah–Hitchin manifold. Sen's conjecture also predicted an L^2 harmonic 2-form on the Euclidean Taub-NUT space (in what follows we call this space systematically the Taub-NUT space). This was found later by Gibbons in 1996 [14]. He constructed it as the exterior derivative of the dual of the Killing field of a canonical $U(1)$-action.

In [10] we imitated Gibbons' construction and found one self-dual (and one anti-self-dual) L^2 harmonic 2-form on the Euclidean Schwarzschild manifold. We then went on and interpreted such a 2-form with integer cohomology class, as the curvature of a $U(1)$ Maxwell connection. (We note that in this Maxwell form our solution was written down by Pope in [24] while the above Gibbons' 2-form was discovered by Eguchi and Hanson [9], both in 1978.) This way we found reducible $SU(2)$ Yang–Mills connections on the Euclidean Schwarzschild manifold, and showed that they agree with $SU(2)$ Yang–Mills connections found by Charap and Duff in 1978 [6] and were supposed to be irreducible solutions.

In this paper we can start by interpreting the above Eguchi–Hanson–Gibbons L^2 harmonic 2-form as a reducible $SU(2)$ Yang–Mills instanton on the Taub-NUT space. We will show how to deform the one with unit energy to get a one-dimensional family of $SU(2)$ Yang-Mills instantons on Taub-NUT space.

There are only very few $SU(2)$ Yang–Mills instantons on Taub-NUT space found in the literature compared to the case of flat $\mathbb{R}^4$ (see [18] and [1]) and in general to ALE spaces (see [21] and also [4]). There is a complete ADHM-Nahm data for the $U(1)$-invariant self-dual configuration on Taub-NUT space described in [7]. However apart from the above reducible ones apparently only the Pope–Yuille instanton [25] is known explicitly, which is of unit energy. In fact Pope and Yuille constructed it as the projection of the Levi–Civitá connection onto the positive $\mathfrak{su}(2)^+ \subset \mathfrak{so}(4)$ part (see next section for details).

The Pope-Yuille solution was reinvented by Kim and Yoon recently [19]. They wrote down the full $U(2)$ symmetric ansatz, and came up with two coupled differential equations. They were able to find a solution of unit energy to this system, which turns out to be identical with the above mentioned Pope–Yuille solution.

In the present paper we will use the method of Jackiw, Nohl and Rebbi [18] to construct a one-dimensional family of $SU(2)$ Yang–Mills instantons of unit energy on Taub-NUT space. Namely we find all $U(2)$-invariant harmonic functions on Taub-NUT space. They will be of the form

$$f(r) = 1 + \frac{\lambda}{r-m}$$

for positive λ's and will have a one-point singularity at the NUT. We produce our one-parameter (i.e. $\lambda \in (0,\infty]$) family of instantons (see (13) for an explicit form) via rescaling the metric by f and then projecting the obtained Levi–Civitá connection onto the negative $\mathfrak{su}(2)^- \subset \mathfrak{so}(4)$ subalgebra, and finally removing the singularity at the NUT by applying Uhlenbeck's theorem [28].

When $\lambda = 2m$ we recover the Pope–Yuille solution. When $\lambda = \infty$, that is, after rescaling the metric with the harmonic function $1/(r-m)$, we will find a reducible connection, namely the Eguchi–Hanson–Gibbons L^2 harmonic form of unit energy. Thus we can interpret this result as a new intrinsically geometric construction of this harmonic 2-form. Also as a check we show that the explicitly calculated $SU(2)$-instantons are satisfying the Kim–Yoon ansatz [19].

Finally we note that the case of the multi-centered metrics of Gibbons–Hawking (cf. [15]) will be treated elsewhere [11].

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2 The Atiyah–Hitchin–Singer theorem

First we recall the general theory from [5]. Let (M,g) be a four-dimensional Riemannian spin-manifold. Remember that via $\mathrm{Spin}(4)\cong SU(2)\times SU(2)$ we have a Lie algebra isomorphism $\mathfrak{so}(4)\cong\mathfrak{su}(2)^+\oplus\mathfrak{su}(2)^-$. Consider the Levi–Civitá connection which is locally represented by an $\mathfrak{so}(4)$-valued 1-form ω on TM. Because M is spin and four-dimensional, we can consistently lift this connection to the spin connection, locally given by ω_S, on the spin bundle SM (which is a complex bundle of rank four) and can project it to the $\mathfrak{su}(2)^\pm$ components. The projected connections $A^\pm$ live on the chiral spinor bundles $S^\pm M$ where the decomposition $SM=S^+M\oplus S^-M$ corresponds to the above splitting of $\mathrm{Spin}(4)$. One can raise the question what are the conditions on the metric g for either A^+ or A^- to be self-dual (seeking for antiself-dual solutions is only a matter of reversing the orientation of M).

Consider the curvature 2-form $R\in C^\infty(\Lambda^2M\otimes\mathfrak{so}(4))$ of the metric. There is a standard linear isomorphism $\mathfrak{so}(4)\cong\Lambda^2\mathbb{R}^4$ given by $A\mapsto\alpha$ with $xAy=\alpha(x,y)$ for all $x,y\in\mathbb{R}^4$. Therefore we may regard R as a *2-form-valued 2-form* in $C^\infty(\Lambda^2M\otimes\Lambda^2M)$ i.e. for vector fields X,Y over M we have $R(X,Y)\in C^\infty(\Lambda^2M)$. Since the space of four dimensional curvature tensors, acted on by $SO(4)$, is 20 dimensional, one gets a 20 dimensional reducible representation of $SO(4)$ (and of $\mathrm{Spin}(4)$, being M spin). The decomposition into irreducible components is (see [3], pp. 45-52 or [5], pp. 344-348)

$$R=\frac{1}{12}\begin{pmatrix}s&0\\0&s\end{pmatrix}+\begin{pmatrix}0&B\\B^T&0\end{pmatrix}+\begin{pmatrix}W^+&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&W^-\end{pmatrix},\tag{1}$$

where s is the scalar curvature, B is the traceless Ricci tensor, $W^\pm$ are the Weyl tensors. The splitting of the Weyl tensor is a special four-dimensional phenomenon and is related with the above splitting of the Lie algebra $\mathfrak{so}(4)$. There are two Hodge operations which can operate on R. One (denoted by $*$) acts on the *2-form part of* R while the other one (denoted by $\star$) acts on the *values* of R (which are also 2-forms). In a local coordinate system, these actions are given by

$$(\star R)_{ijkl}=\frac{1}{2}\sqrt{\det g}\,\varepsilon_{ijmn}R^{mn}{}_{kl},$$

$$(*R)_{ijkl}=\frac{1}{2}\sqrt{\det g}\,R_{ij}{}^{mn}\varepsilon_{mnkl}.$$

It is not difficult to see that the projections $p^\pm:\mathfrak{so}(4)\to\mathfrak{su}(2)^\pm$ are given by $R\mapsto F^\pm:=\frac{1}{2}(1\pm\star R)$, and $F^\pm$ are self-dual with respect to g if and only if $*(1\pm\star R)=(1\pm\star R)$. Using the previous representation for the decomposition of R suppose $\star$ acts on the left while $*$ on the right, both of them via

$$\begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix}.$$

In this case the previous self-duality condition looks like ($\overline{W}^{\pm} := W^{\pm} + \frac{1}{12}s$)

$$\begin{pmatrix} \overline{W}^+ \pm \overline{W}^+ & -(B \pm B) \\ B^T \mp B^T & -\left(\overline{W}^- \mp \overline{W}^-\right) \end{pmatrix} = \begin{pmatrix} \overline{W}^+ \pm \overline{W}^+ & B \pm B \\ B^T \mp B^T & \overline{W}^- \mp \overline{W}^- \end{pmatrix}.$$

From here we can immediately conclude that F^+ is self-dual if and only if $B = 0$ i.e. g is *Einstein* while F^- is self-dual if and only if $\overline{W}^- = 0$ i.e. g is *half-conformally flat (i.e. self-dual) with vanishing scalar curvature.* Hence we have proved [2]:

Theorem 2.1 (Atiyah–Hitchin–Singer) *Let (M, g) be a four-dimensional Riemannian spin manifold. Then*
(i) *F^+ is the curvature of an self-dual $SU(2)$-connection on S^+M if and only if g is Einstein, or*
(ii) *F^- is the curvature of a self-dual $SU(2)$-connection on S^-M if and only if g is half conformally flat (i.e. self-dual) with vanishing scalar curvature.* $\diamond$

Remember that both the (anti)self-duality equations

$$*F = \pm F$$

and the action

$$\|F\|^2_{L^2(M)} = \frac{1}{8\pi^2}\int\limits_M |F|^2_g = -\frac{1}{8\pi^2}\int\limits_M \mathrm{tr}(F \wedge *F)$$

are conformally invariant in four dimensions; consequently if we can rescale g with a suitable positive function f producing a metric $\tilde{g}$ which satisfies one of the properties of the previous theorem then we can construct instantons over the original manifold (M, g). This idea was used by Jackiw, Nohl and Rebbi to construct instantons over the flat $\mathbb{R}^4$ [18].

3 Projecting onto the positive side

First consider the case of F^+, i.e. part (i) of the above theorem. Let (M, g) be a Riemannian manifold of dimension $n > 2$. Remember that $\psi : M \to M$ is a *conformal isometry* of (M, g) if there is a function $f : M \to \mathbb{R}$ such that $\psi^* g = f^2 g$. Notice that being ψ a diffeomorphism, f cannot be zero anywhere i.e. we may assume that it is positive, $f > 0$. Ordinary isometries are the special cases with $f = 1$. The vector field X on M, induced by the conformal isometry, is called a *conformal Killing field.* It satisfies the *conformal Killing equation* ([30], pp. 443-444)

$$L_X g - \frac{2\mathrm{div}(X)}{n} g = 0$$

where L is the Lie derivative while div is the divergence of a vector field. If $\xi = \langle X, \cdot\rangle$ denotes the dual 1-form to X with respect to the metric, then consider the following *conformal Killing data*:

$$(\xi, \;\; \mathrm{d}\xi, \;\; \mathrm{div}(X), \;\; \mathrm{d}\mathrm{div}(X)). \tag{2}$$

These satisfy the following equations (see [12] or [13]):

$$\begin{aligned}
&\nabla\xi = (1/2)\mathrm{d}\xi + (1/n)\mathrm{div}(X)g,\\
&\nabla(\mathrm{d}\xi) = (1/n)(g\otimes \mathrm{ddiv}(X) - \mathrm{ddiv}(X)\otimes g) + 2R(\,\cdot\,,\,\cdot\,,\,\cdot\,,\xi),\\
&\nabla(\mathrm{div}(X)) = \mathrm{ddiv}(X),\\
&\nabla(\mathrm{ddiv}(X)) = -(n/2)\nabla_X P - \mathrm{div}(X)P - (n/2)\mathrm{tr}(P\otimes \mathrm{d}\xi + \mathrm{d}\xi\otimes P).
\end{aligned} \tag{3}$$

Here R is understood as the $(3,1)$-curvature tensor while

$$P = r - \frac{s}{(n-1)(n-2)}g$$

with r being the Ricci-tensor. If γ is a smooth curve in M then fixing conformal Killing data in a point $p = \gamma(t)$ we can integrate (3) to get all the values of X along γ. Actually if X is a conformal Killing field then by fixing the above data in one point $p \in M$ we can determine the values of X over the *whole* M, provided that M is connected. Consequently, if these data vanish in one point, then X vanishes over all the M.

Furthermore a Riemannian manifold (M,g) is called *irreducible* if the holonomy group, induced by the metric, acts irreducibly on each tangent space of M.

Now we can state:

Proposition 3.1 *Let (M,g) be a connected, irreducible, Ricci-flat Riemannian manifold of dimension $n>2$. Then $(M,\tilde g)$ with $\tilde g = \varphi^{-2}g$ is Einstein if and only if φ is a non-zero constant function on M.*

Remark. Notice that the above proposition is not true for reducible manifolds: the already mentioned Jackiw–Nohl–Rebbi construction [18] provides us with non-trivial Einstein metrics, conformally equivalent to the flat $\mathbb{R}^4$.

Proof. If η is a 1-form on (M,g) with dual vector $Y = \langle\eta,\,\cdot\,\rangle$, then the $(0,2)$-tensor $\nabla\eta$ can be decomposed into antisymmetric, trace and traceless symmetric parts respectively as follows (e.g. [23], pp. 200, Ex. 5.):

$$\nabla\eta = \frac{1}{2}\mathrm{d}\eta - \frac{\delta\eta}{n}g + \frac{1}{2}\left(L_Y g + \frac{2\delta\eta}{n}g\right) \tag{4}$$

where δ is the exterior codifferentiation on (M,g) satisfying $\delta\eta = -\mathrm{div}(Y)$. Being g an Einstein metric, it has identically zero traceless Ricci tensor i.e. $B=0$ from the decomposition (1). We rescale g with the function $\varphi : M\to\mathbb{R}^+$ as $\tilde g := \varphi^{-2}g$. Then the traceless Ricci part of the new curvature is (see [3], p. 59)

$$\widetilde{B} = \frac{n-2}{\varphi}\left(\nabla^2\varphi + \frac{\triangle\varphi}{n}g\right).$$

Here $\triangle$ denotes the Laplacian with respect to g. From here we can see that if $n>2$, the condition for $\tilde g$ to be again Einstein is

$$\nabla^2\varphi + \frac{\triangle\varphi}{n}g = 0.$$

However, if $X := \langle {\rm d}\varphi,\ \cdot\ \rangle$ is the dual vector field then we can write by (4) that

$$\nabla^2\varphi=\frac{1}{2}{\rm d}^2\varphi-\frac{\delta({\rm d}\varphi)}{n}g+\frac{1}{2}\left(L_Xg+\frac{2\delta({\rm d}\varphi)}{n}g\right)=-\frac{\triangle\varphi}{n}g+\frac{1}{2}\left(L_Xg+\frac{2\triangle\varphi}{n}g\right).$$

We have used ${\rm d}^2=0$ and $\delta{\rm d}=\triangle$ for functions. Therefore we can conclude that $\varphi^{-2}g$ is Einstein if and only if

$$L_Xg+\frac{2\triangle\varphi}{n}g=L_Xg-\frac{2{\rm div}(X)}{n}g=0$$

i.e. X is a conformal Killing field on (M,g) obeying $X=\langle{\rm d}\varphi,\ \cdot\ \rangle$. The conformal Killing data (2) for this X are the following:

$$({\rm d}\varphi,\quad {\rm d}^2\varphi=0,\quad -\triangle\varphi,\quad -{\rm d}(\triangle\varphi)). \tag{5}$$

Now we may argue as follows: the last equation of (3) implies that

$$\nabla({\rm d}(\triangle\varphi))=0$$

over the Ricci-flat (M,g). By virtue of the irreducibility of (M,g) this means that actually ${\rm d}(\triangle\varphi)=0$ (cf. e.g. [3], p. 282, Th. 10.19) and hence $\triangle\varphi$ =const. over the whole (M,g). Consequently, the second equation of (3) shows that for all Y,Z,V we have

$$R(Y,Z,V,{\rm d}\varphi)=0.$$

Taking into account again that (M,g) is irreducible, there is a point where R_p is non-zero. Assume that the previous equality holds for all Y_p,Z_p,V_p but ${\rm d}\varphi_p\neq0$. This is possible only if a subspace, spanned by ${\rm d}\varphi_p$ in T^*_pM, is invariant under the action of the holonomy group. But this contradicts the irreducibility assumption. Consequently ${\rm d}\varphi_p=0$. Finally, the first equation of (3) yields that $\triangle\varphi(p)=0$ i.e. $\triangle\varphi=0$. Therefore we can conclude that in that point all the conformal data (5) vanish implying $X=0$. In other words φ is a non-zero constant. $\diamond$

In light of this proposition, general Ricci-flat manifolds cannot be rescaled into Einstein manifolds in a non-trivial way. Notice that the Taub-NUT space (see below) is an irreducible Ricci-flat manifold. If this was not the case, then, taking into account its simply connectedness and geodesic completeness, it would split into a Riemannian product $(M_1\times M_2, g_1\times g_2)$ by virtue of the de Rham theorem [26]. But it is easily checked that this is not the case. We just remark that the same is true for the Euclidean Schwarzschild manifold.

Consequently constructing instantons in this way is not very productive.

4 Constructing new Taub-NUT instantons

Therefore we turn our attention to the condition on the F^- part of the metric curvature in the special case of the Taub-NUT space. Consider the Taub-NUT metric

$${\rm d}s^2=\frac{r+m}{r-m}{\rm d}r^2+4m^2\frac{r-m}{r+m}({\rm d}\tau+\cos\Theta{\rm d}\phi)^2+(r^2-m^2)({\rm d}\Theta^2+\sin^2\Theta{\rm d}\phi^2), \tag{6}$$

$$m\leq r<\infty,\quad 0\leq\tau<4\pi,\quad 0\leq\phi<2\pi,\quad 0\leq\Theta<\pi,$$

where m is a positive real number. If one introduces the left-invariant 1-forms σ_x, σ_y, σ_z on S^3 given by

$$\sigma_x = -\sin\tau \mathrm{d}\Theta + \cos\tau\sin\Theta \mathrm{d}\phi, \quad \sigma_y = \cos\tau \mathrm{d}\Theta + \sin\tau\sin\Theta\mathrm{d}\phi, \quad \sigma_z = \mathrm{d}\tau + \cos\Theta\mathrm{d}\phi,$$

then the metric can be re-written in another useful form as

$$\mathrm{d}s^2 = \frac{r+m}{r-m}\mathrm{d}r^2 + (r^2-m^2)\left(\sigma_x^2+\sigma_y^2+\left(\frac{2m}{r+m}\right)^2\sigma_z^2\right).$$

As it is well known (see e.g. [24]) this metric, despite its apparent singularity in the origin, extends analytically to a metric on $\mathbb{R}^4$. Moreover it is Ricci-flat i.e. $B=0$ and $s=0$ as well. Also notice that the Taub-NUT space is (anti)self-dual ($W^+=0$ or $W^-=0$, depending on the orientation).

Our aim is to find metrics $\tilde{g}$ (as much as possible), conformally equivalent to the original Taub-NUT metric g, such that $\tilde{g}$'s are self-dual and have vanishing scalar curvature: in this case the metric instantons in $\tilde{g}$ provide (anti)self-dual connections in the Taub-NUT case, as we have seen. Taking into account that the $(3,1)$-Weyl tensor W is invariant under conformal rescalings i.e. $\widetilde{W}=W$, the condition $\widetilde{W}^- + \tilde{s}/12 = 0$ for the $\tilde{g}$'s settles down for having vanishing scalar curvature $\tilde{s}=0$. Consider the rescaling

$$g \longmapsto \tilde{g} := f^2 g$$

where $f:\mathbb{R}^4\to\mathbb{R}^+$ is a positive function. In this case the scalar curvature transforms as $s\mapsto\tilde{s}$ where $\tilde{s}$ satisfies (see [3], pp. 58-59):

$$f^3\tilde{s} = 6\triangle f + fs.$$

Taking into account that the Taub-NUT space is Ricci-flat, i.e. $s=0$, our condition for the scaling function amounts to the simple condition (cf. pp. 366-367 of [5])

$$\triangle f = 0 \tag{7}$$

i.e. it must be a harmonic function (with respect to the Taub-NUT geometry). From (6) one easily calculates $\det g = 4m^2(r^2-m^2)^2\sin^2\Theta$ and by using the local expression

$$\triangle = \sum_{i,j}\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij}\frac{\partial}{\partial x^j}\right),$$

where g^{ij} are the components of the inverse matrix, the Taub-NUT Laplacian looks like

$$\triangle = \left(\frac{r+m}{4m^2(r-m)}+\frac{\cot^2\Theta}{r^2-m^2}\right)\frac{\partial^2}{\partial\tau^2} + \frac{r-m}{r+m}\frac{\partial^2}{\partial r^2} + \frac{2}{r+m}\frac{\partial}{\partial r} +$$
$$+\frac{1}{r^2-m^2}\left(\frac{\partial^2}{\partial\Theta^2}+\cot\Theta\frac{\partial}{\partial\Theta}\right) + \frac{1}{(r^2-m^2)\sin^2\Theta}\left(\frac{\partial^2}{\partial\phi^2} - 2\cos\Theta\frac{\partial^2}{\partial\tau\partial\phi}\right).$$

The easiest thing is to seek for solutions depending only on the radial coordinate r, in other words functions invariant under the full $U(2)$ symmetry of the Taub-NUT space. In this case the Laplace equation (7) cuts down to the ordinary differential equation

$$(r-m)f'' + 2f' = 0$$

where prime denotes differentiation with respect to r. The general solution to this equation is $f(r) = c_1 + c_2(r-m)^{-1}$. Hence if we rescale the original metric via

$$f(r) = 1 + \frac{\lambda}{r-m} \tag{8}$$

with $\lambda \geq 0$ then the metric (anti)instantons in $\tilde{g} = f^2 g$ will be (anti)instantons in the original space. Although the metric $\tilde{g}$ is singular in the origin, the resulting instantons will automatically appear in a gauge in which they are smooth everywhere.

Do not hesitate, let us construct these instantons explicitly in order to determine their energy! Choose a global trivialization of the spinor bundle SM induced by the obvious orthonormal tetrad

$$\xi^0 = \sqrt{\frac{r+m}{r-m}}\,\mathrm{d}r, \quad \xi^1 = \sqrt{r^2-m^2}\,\sigma_x, \quad \xi^2 = \sqrt{r^2-m^2}\,\sigma_y, \quad \xi^3 = 2m\sqrt{\frac{r-m}{r+m}}\,\sigma_z. \tag{9}$$

(Note that in four dimensions we have $SM \cong TM \otimes \mathbb{C}$). In this gauge the Levi–Civitá connection of the re-scaled space is globally represented by an $\mathfrak{so}(4)$-valued 1-form $\tilde{\omega}$ whose components obey the Cartan equation

$$\mathrm{d}\tilde{\xi}^i = -\tilde{\omega}^i_j \wedge \tilde{\xi}^j$$

where $\tilde{\xi}^i = f\xi^i$. Consequently we can write

$$\mathrm{d}\xi^i + \mathrm{d}(\log f) \wedge \xi^i = -\tilde{\omega}^i_j \wedge \xi^j. \tag{10}$$

The basis (9) shows that

$$\mathrm{d}(\log f) = \sqrt{\frac{r-m}{r+m}}(\log f)'\xi^0.$$

Putting this and $\mathrm{d}\xi^i = -\omega^i_j \wedge \xi^j$ into (10) we get

$$\omega^i_j \wedge \xi^j + \sqrt{\frac{r-m}{r+m}}(\log f)'\xi^i \wedge \xi^0 = \tilde{\omega}^i_j \wedge \xi^j.$$

Therefore, the components of the new connection form are given by

$$\tilde{\omega}^i_0 = \omega^i_0 + \sqrt{\frac{r-m}{r+m}}(\log f)'\xi^i, \quad \tilde{\omega}^k_l = \omega^k_l \tag{11}$$

(here and only here i, k, l run over 1,2,3). Now by using the original connection (see [8], p. 351) and substituting (8) into (11) we have the explicit form

$$\tilde{\omega}^1_0 = \left(\frac{r}{r+m} - \frac{\lambda}{r-m+\lambda}\right)\sigma_x, \quad \tilde{\omega}^3_2 = -\frac{m}{r+m}\sigma_x,$$

$$\tilde{\omega}^2_0 = \left(\frac{r}{r+m} - \frac{\lambda}{r-m+\lambda}\right)\sigma_y, \quad \tilde{\omega}^1_3 = -\frac{m}{r+m}\sigma_y,$$

$$\tilde{\omega}^3_0 = \left(\frac{2m^2}{(r+m)^2} - \frac{2m\lambda}{(r-m+\lambda)(r+m)}\right)\sigma_z, \quad \tilde{\omega}^2_1 = \left(\frac{2m^2}{(r+m)^2} - 1\right)\sigma_z.$$

If we wish to project this connection to the self-dual part A^-_λ we have to evoke the 't Hooft matrices which span the $\mathfrak{su}(2)^- \subset \mathfrak{so}(4)$ subalgebra:

$$\overline{\eta}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \overline{\eta}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \overline{\eta}_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The projected connection A^-_λ in the gauge (9) is given by

$$A^-_\lambda = \frac{1}{4}\sum_{a=1}^{3}\sum_{i,j=0}^{3}\left(\overline{\eta}^i_{a,j}\tilde{\omega}^i_j\right)\,\overline{\eta}_a.$$

The normalization factor 1/4 comes from $\overline{\eta}^{ij}_a\overline{\eta}_{a,ij} = 4$ (summation is over i, j while a is fixed). This eventually yields (by using the identification $\mathfrak{su}(2)^- \cong \mathrm{Im}\mathbb{H}$ via $(\overline{\eta}_1, \overline{\eta}_2, \overline{\eta}_3) \mapsto (\mathbf{i}, \mathbf{j}, \mathbf{k})$ and the straightforward gauge (9))

$$A^-_\lambda = -\frac{\mathbf{i}}{2}\Psi\sigma_x + \frac{\mathbf{j}}{2}\Psi\sigma_y + \frac{\mathbf{k}}{2}\Phi\sigma_z \tag{12}$$

where we have introduced the notations

$$\Phi(r) := 1 - \frac{2m\lambda}{(r-m+\lambda)(r+m)}, \qquad \Psi(r) := 1 - \frac{\lambda}{r-m+\lambda}. \tag{13}$$

These connections are self-dual with respect to the orientation $\varepsilon_{\tau r\Theta\phi} = 1$ and are non gauge-equivalent, as we will see in a moment. We also find that for $\lambda = 0$ this ansatz describes a singular flat connection. But this isolated singularity can be removed by a gauge transformation, as guaranteed by the Theorem of Removable Singularities by Uhlenbeck [28]. That is, A^-_0 is gauge-equivalent to the trivial connection, since $\mathbb{R}^4$ is simply connected. Otherwise A^-_λ is regular in the origin since $\Phi(m) = 0$ and $\Psi(m) = 0$ if $\lambda > 0$. In particular, for $\lambda = 2m$ it reduces to the Pope–Yuille instanton [25]. Last but not least, if $\lambda = \infty$, that is when the rescaling function is $1/(r-m)$, the solution (12) takes the form

$$A^-_\infty = \frac{\mathbf{k}}{2}\frac{r-m}{r+m}\sigma_z,$$

which coincides with the Abelian instanton found by Eguchi–Hanson [9] and reinvented in mathematical terms by Gibbons [14] (see also [24]).

Now we calculate the curvature $F^-_\lambda = \mathrm{d}A^-_\lambda + \frac{1}{2}[A^-_\lambda, A^-_\lambda] = \mathrm{d}A^-_\lambda + A^-_\lambda \wedge A^-_\lambda$. Taking into account the identities $\mathrm{d}\sigma_x = \sigma_y \wedge \sigma_z$, etc. (the indices x, y, z are permuted cyclically), a straightforward calculation yields the shape

$$\begin{aligned} F^-_\lambda = &\frac{\mathbf{i}}{2}\left(-\Psi'\mathrm{d}r\wedge\sigma_x - (\Psi - \Psi\Phi)\sigma_y\wedge\sigma_z\right) + \\ &+\frac{\mathbf{j}}{2}\left(\Psi'\mathrm{d}r\wedge\sigma_y - (\Psi - \Psi\Phi)\sigma_x\wedge\sigma_z\right) + \\ &+\frac{\mathbf{k}}{2}\left(\Phi'\mathrm{d}r\wedge\sigma_z + (\Phi - \Psi^2)\sigma_x\wedge\sigma_y\right). \end{aligned} \tag{14}$$

If $A, B \in \mathfrak{su}(2)$ whose images are $x, y \in \mathrm{Im}\mathbb{H}$ under the above identification, then, as easily checked, the Killing form $-\operatorname{tr}(AB)$ is given by $2\mathrm{Re}(x\overline{y})$. Bearing this in mind, the energy-density 4-form is

$$|F^-_\lambda|^2_g = -\operatorname{tr}(F^-_\lambda \wedge F^-_\lambda) = \left(\Phi'\Phi + 2\Psi'\Psi - 2\Psi'\Psi\Phi - \Phi'\Psi^2\right) \mathrm{d}r \wedge \sigma_x \wedge \sigma_y \wedge \sigma_z =$$
$$= \left(\Phi'\Phi + 2\Psi'\Psi - 2\Psi'\Psi\Phi - \Phi'\Psi^2\right) \sin\Theta \mathrm{d}\tau \wedge \mathrm{d}r \wedge \mathrm{d}\Theta \wedge \mathrm{d}\phi. \tag{15}$$

Using the parameterization (6), the energy is the integral

$$\|F^-_\lambda\|^2_{L^2(\mathbb{R}^4)} =$$
$$= \frac{1}{8\pi^2} \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_m^{\infty}\int\limits_0^{4\pi} \left(\Phi'(r)\Phi(r) + 2\Psi'(r)\Psi(r) - 2\Psi'(r)\Psi(r)\Phi(r) - \Phi'(r)\Psi^2(r)\right) \sin\Theta \mathrm{d}\tau \mathrm{d}r \mathrm{d}\Theta \mathrm{d}\phi.$$

By performing the only non-trivial integral

$$\int\limits_m^{\infty} \left(\Phi'(r)\Phi(r) + 2\Psi'(r)\Psi(r) - 2\Psi'(r)\Psi(r)\Phi(r) - \Phi'(r)\Psi^2(r)\right) \mathrm{d}r = \begin{cases} 0 & \text{if } \lambda = 0 \\ 1/2 & \text{if } \lambda > 0 \end{cases}$$

we eventually find the energy to be

$$\|F^-_\lambda\|^2_{L^2(\mathbb{R}^4)} = \begin{cases} 0 & \text{if } \lambda = 0 \\ 1 & \text{if } \lambda > 0. \end{cases}$$

To conclude this section, we show that the connections A^-_λ just constructed do indeed satisfy the self-duality equations and are not gauge equivalent.

Using the orthonormal tetrad (9), it follows that the Hodge-operator on 2-forms is given as follows:

$$*(\mathrm{d}r \wedge \sigma_x) = *\left(\frac{1}{r+m}\xi^0 \wedge \xi^1\right) = \frac{1}{r+m}\xi^2 \wedge \xi^3 = \frac{2m(r-m)}{r+m}\sigma_y \wedge \sigma_z,$$
$$*(\mathrm{d}r \wedge \sigma_y) = *\left(\frac{1}{r+m}\xi^0 \wedge \xi^2\right) = -\frac{1}{r+m}\xi^1 \wedge \xi^3 = -\frac{2m(r-m)}{r+m}\sigma_x \wedge \sigma_z,$$
$$*(\mathrm{d}r \wedge \sigma_z) = *\left(\frac{1}{2m}\xi^0 \wedge \xi^3\right) = \frac{1}{2m}\xi^1 \wedge \xi^2 = \frac{r^2-m^2}{2m}\sigma_x \wedge \sigma_y.$$

Applying this to the original curvature expression (14) we get

$$*F^-_\lambda = \frac{\mathbf{i}}{2}\left(-\frac{2m(r-m)}{r+m}\Psi'\sigma_y \wedge \sigma_z - \frac{r+m}{2m(r-m)}(\Psi - \Psi\Phi)\mathrm{d}r \wedge \sigma_x\right) +$$
$$+\frac{\mathbf{j}}{2}\left(-\frac{2m(r-m)}{r+m}\Psi'\sigma_x \wedge \sigma_z + \frac{r+m}{2m(r-m)}(\Psi - \Psi\Phi)\mathrm{d}r \wedge \sigma_y\right) +$$
$$+\frac{\mathbf{k}}{2}\left(\frac{r^2-m^2}{2m}\Phi'\sigma_x \wedge \sigma_y + \frac{2m}{r^2-m^2}(\Phi - \Psi^2)\mathrm{d}r \wedge \sigma_z\right).$$

Therefore comparing the original curvature (14) with the above expression term by term, the self-duality equations reduce to the two equations

$$\Psi' = \frac{r+m}{2m(r-m)}(\Psi - \Psi\Phi),$$

$$\Phi' = \frac{2m}{r^2 - m^2}(\Phi - \Psi^2).$$

One checks by hand that the previously constructed functions obey these equations. We note that the above equations were previously obtained in [19].

Also notice that our solutions are non-gauge equivalent. The simplest way to see this is by looking at the energy-density, which is a gauge independent 4-form on the space. Putting the functions (13) into the energy-density 4-form (15) then for small λ we get

$$|F^-_\lambda|^2_g \sim \frac{4\lambda^2 m(3r^3 - r^2m - rm^2 - m^3)}{(r - m + \lambda)^4(r+m)^3} \mathrm{d}r \wedge \sigma_x \wedge \sigma_y \wedge \sigma_z.$$

Consequently the energy-density approaches a Dirac-delta concentrated at the origin $r = m$ of $\mathbb{R}^4$ as $\lambda \to 0$. This shows that the connections with different λ's are non-gauge equivalent because their curvatures are different. This observation also provides a natural interpretation of the parameter λ, namely it measures the "concentration" of the instanton around the origin of $\mathbb{R}^4$. The limit connection as $\lambda \to 0$ is an "ideal", Dirac-delta-type instanton, concentrated at the NUT. This ideal connection compactifies our half line of solutions into a closed segment.

5 Concluding remarks

In this paper we constructed a one-parameter family of new Taub-NUT instantons of unit energy depending on a parameter $\lambda \in (0, \infty]$. When $\lambda = 2m$ we recovered the previously known Pope–Yuille instanton and when $\lambda = \infty$ we obtained a reducible Maxwell instanton corresponding to the Eguchi–Hanson–Gibbons L^2 harmonic 2-form.

Our hope is that there exist nice moduli spaces of Yang–Mills instantons on Taub-NUT space like for the flat $\mathbb{R}^4$ (cf. [2]). Unfortunately at present even the Atiyah–Singer index theorem is not fully understood on manifolds with ends like Taub-NUT (however cf. [24], [22] and [29]). So the infinitesimal description of the moduli space lacks some fundamental analysis. Nevertheless we hope that the half-line of solutions we found will actually turn out to be the full moduli space of unit charge $SU(2)$ Yang–Mills instantons on Taub-NUT space. To go further one might conjecture that the moduli space of *framed* $SU(2)$ instantons of unit charge on Taub-NUT space will be the Taub-NUT space itself, and the half-line we found will arise as Taub-NUT space modulo the isometric action of $SU(2) \subset U(2)$ (moving the frame). In this description the NUT would correspond to the reducible instanton, the end-point of the half-line.

Another intriguing possibility is to analyse what happens if we consider our $U(1)$-invariant Yang–Mills solutions as solutions of the Bogomolny equations on $\mathbb{R}^3 = \mathbb{R}^4/U(1)$ (c.f. [20]). We hope to return to this problem in the near future.

Finally we mention that the analogous constructions on the Gibbons–Hawking multi-centered spaces will appear elsewhere [11]. There we will see a clearer understanding of the role of harmonic functions and their singularities in the problem. Also we will find there nice geometrical constructions of the L^2-harmonic forms on these multi-centered spaces, whose existence is proven by different means in [16].

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