Standard monomials and Gröbner bases of vanishing ideals of finite sets

Bálint Felszeghy

Abstract

Let $\mathbb{F}$ be a field, $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over $\mathbb{F}$ and $\prec$ be a complete order on the monomials of $\mathbb{F}[\mathbf{x}]$, such that it is compatible with multiplication of monomials and the minimal element is the monomial 1. The *leading monomial* $\mathrm{lm}(f)$ of a nonzero polynomial $f \in \mathbb{F}[\mathbf{x}]$ is the largest monomial in f with respect to $\prec$. The *Gröbner basis* of an ideal $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ is a finite set $G \subseteq I$, such that for all $f \in I$ there is a $g \in G$ with the property that $\mathrm{lm}(g)$ divides $\mathrm{lm}(f)$. The *standard monomials* of I are those monomials of $\mathbb{F}[\mathbf{x}]$ which are not leading monomials of any $f \in I$. Let V be a finite subset $V \subseteq \mathbb{F}^n$. The polynomials that vanish at all the points of V form an ideal $I(V)$ of $\mathbb{F}[\mathbf{x}]$. We will simply refer to the standard monomials and Gröbner basis of $I(V)$ as the standard monomials and Gröbner basis of V respectively.

The concept of Gröbner basis was introduced by Bruno Buchberger 40 years ago, who put down the foundations of the theory. His theory found applications in various fields of mathematics for example in statistics, differential equations and numerical methods.

In my thesis I deal with an application different from the above, I use Gröbner bases of finite sets of points as a combinatorial tool. The main example is the following. Let $V = V_{\mathcal{F}}$ be the finite set of points consisting of the characteristic vectors of a family of sets $\mathcal{F}$ of $\{1, 2, \ldots, n\}$. Lots of combinatorial properties can be formulated in terms of polynomial functions. The main objective of my thesis is to introduce Gröbner basis as a combinatorial tool and to give computational methods for determining Gröbner basis of finite sets of points of combinatorial relevance.

A possible choice for the order $\prec$ is the lexicographic order of the exponent vector of the monomials, the *lex* order for short. I introduce a game played by two players, which has a finite set of points V and an exponent vector $\mathbf{w}$ as parameters in its rules. I show that it can be decided if the monomial $x_1^{w_1} \ldots x_n^{w_n}$ is a lex standard mononomial of V or not by determining the player who has winning strategy in the lex game. This gives a purely combinatorial description of the lex standard monomials which together with its immediate corollary about the *recursive structure of the lex*

standard monomials is the central theorem of the thesis. The other new results are mainly consequences of this description. I endeavored to give new and simpler proofs for known results as well with the help of the lex game.

I introduce algorithms in connection with Gröbner bases: I show the original algorithm of Buchberger which works for general ideals, and two other methods for vanishing ideals of finite sets of points. I suggest an algorithm for computing the standard monomials of a finite set V which is faster then the known methods. I provide detailed analysis of the computational costs of the algorithms.

As a consequence of the description of the lex standard monomials, I compute standard monomials of some finite sets which in certain cases allows us to get the reduced Gröbner basis as well. Some of these were already known and others are new results. The choice of these finite sets are always motivated by combinatorial importance. In the last section I exhibit beautiful combinatorial applications of standard monomials and Gröbner basis picking from articles of Gábor Hegedűs and Lajos Rónyai.