Near-critical scaling limits in 2d statistical physics

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(based on joint works with Oded Schramm
and Hugo Duminil-Copin)

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Plan

Part I. Near-critical geometry

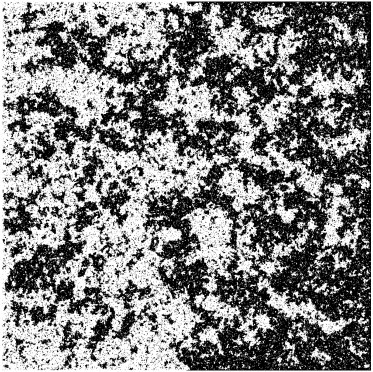
- Near-critical scaling limits
- Our main statement on the near-critical limit of **planar percolation**
- Idea of proof

Part II. Applications

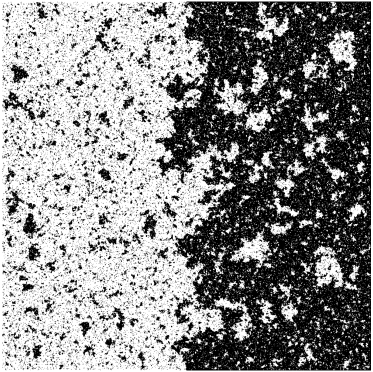
- **Minimal spanning tree** in the plane
- Near-critical geometry in **FK percolation** (with H. Duminil-Copin)
- **Dynamics** at the critical point

Near-critical geometry in general

Ising model near its critical point

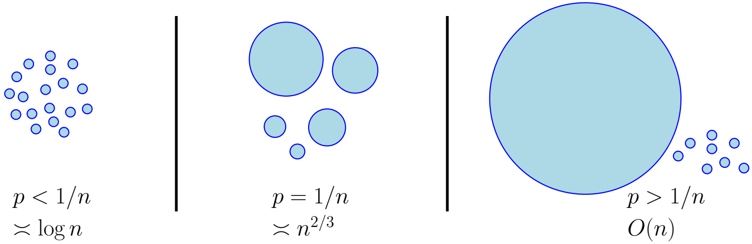


$T = T_c$



$T = T_c - \delta T$

Erdős-Rényi random graphs $G(n, p)$



"Everything happens" in the near-critical window

$$p_{\lambda,n} = \frac{1}{n} + \lambda \frac{1}{n^{4/3}}$$

Alon and Spencer (2002): " With $\lambda = -10^6$, say we have feudalism. Many components (castles) are each vying to be the largest. As λ increases . . . and by $\lambda = 10^6$ it is very likely that a giant component, the Roman Empire, has emerged. "

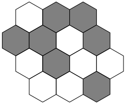
Theorem (Addario-Berry, Broutin, Goldschmidt, 2012)

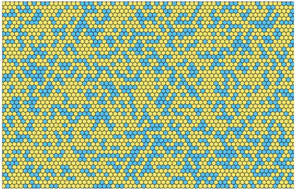
$$\left(G(n, p_{\lambda,n}), \frac{1}{n^{1/3}} d_{graph}\right) \xrightarrow{law} G_\infty(\lambda)$$

under (a slight generalization of) the Gromov-Hausdorff topology.

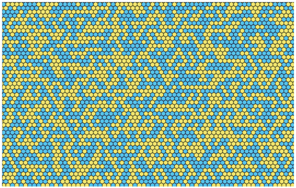
Near-critical percolation in the plane

Site percolation on the triangular lattice $\mathbb{T}$:





"feudalism"
$p < 1/2$



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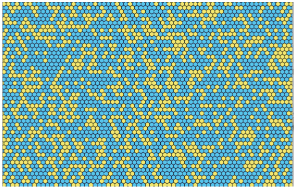


"Roman empire"
$p > 1/2$

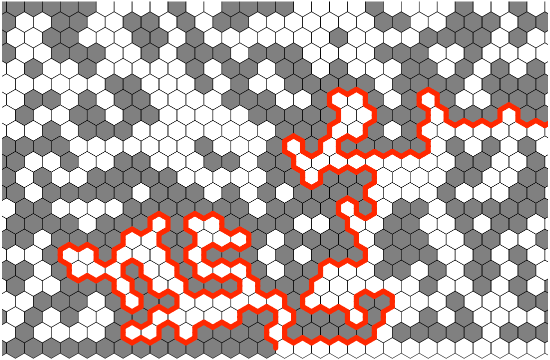
Renormalise the lattice as follows: $\eta\mathbb{T}$

$\eta \to 0$??

Scaling limit of percolation

Theorem (Smirnov, 2001)

Critical site percolation on $\eta\mathbb{T}$ is asymptotically (as $\eta \searrow 0$) conformally invariant.

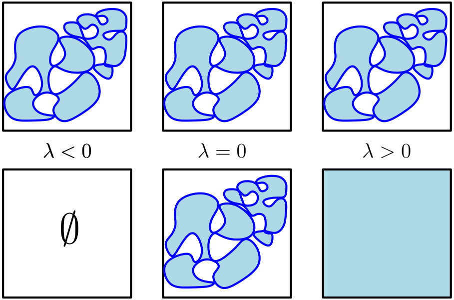


Convergence to SLE_6 (Schramm-Loewner-Evolution with $\kappa = 6$)

looking for the right ZOOMING

We shall now zoom around p_c as follows:

$$\boxed{p = p_c + \lambda\, r(\eta)}$$



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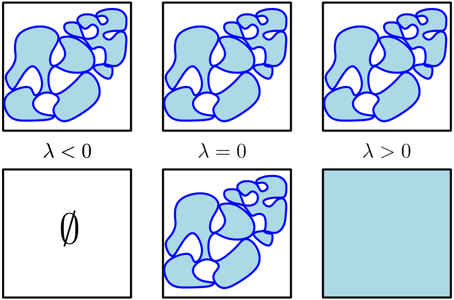


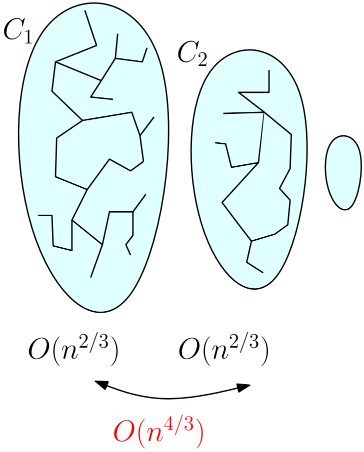
Theorem (Kesten, 1987)

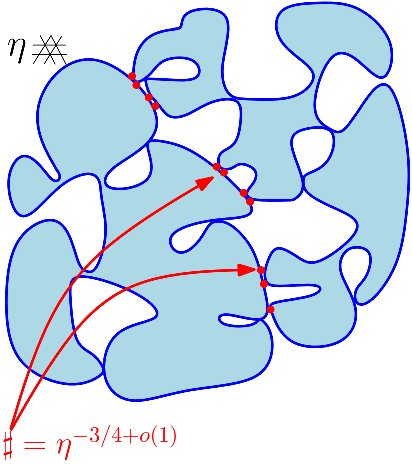
The right zooming factor is given by

$$\begin{aligned} r(\eta) &:= \eta^2 \alpha_4(\eta, 1)^{-1} \\ &= \eta^{3/4+o(1)} \end{aligned}$$

Heuristics behind these scalings

$p = 1/n + \lambda n^{-4/3}$ versus $p_c + \lambda \eta^{3/4+o(1)}$





Scaling limit ?

Definition

Define $\omega_\eta^{\text{nc}}(\lambda)$ to be the percolation configuration on $\eta\mathbb{T}$ of parameter

$$\boxed{p = p_c + \lambda\, r(\eta)}$$

For all $\eta > 0$, we define a monotone càdlàg process

$$\lambda \in \mathbb{R} \mapsto \omega_\eta^{\text{nc}}(\lambda) \in \{0,1\}^{\eta\mathbb{T}}$$

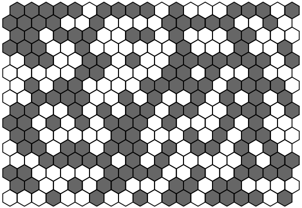
Question

Does the process $\lambda \in \mathbb{R} \mapsto \omega_\eta^{\text{nc}}(\lambda)$ *converge (in law) as* $\eta \searrow 0$ *to a limiting process*

$$\lambda \mapsto \omega_\infty^{\text{nc}}(\lambda) \ \ ?$$

- For which **topology** ?? Find an appropriate Polish space (E, d) whose points $\omega \in E$ are naturally identified to **percolation configurations**.

The first natural idea which comes to mind



This configuration on $\eta\mathbb{T}$ may be coded by the distribution

$$X_\eta := \eta \sum_{x \in \eta\mathbb{T}} \sigma_x \, \delta_x$$

$\{X_\eta\}_\eta$ is **tight** in $\mathcal{H}^{-1-\varepsilon}$ and converge to the **Gaussian white noise** on $\mathbb{R}^2$.

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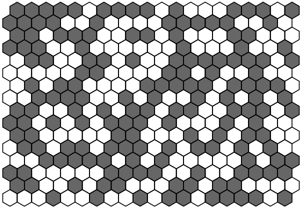

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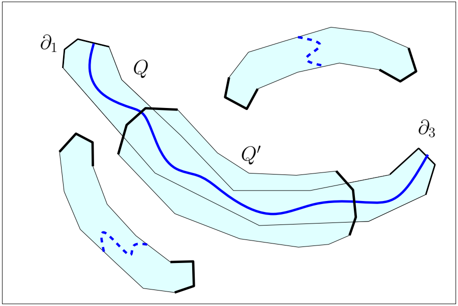
Theorem (Benjamini, Kalai, Schramm, 1999)

This setup is ***NOT*** *appropriate to handle percolation: natural observables for percolation are highly discontinuous under the topology induced by* $\|\cdot\|_{\mathcal{H}^{-1-\varepsilon}}$ *and in fact are not even measurable in the limit.*

Some other historical approaches

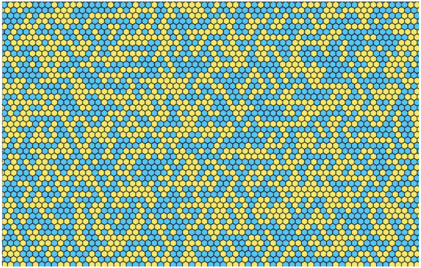
1. Aizenman 1998 and Aizenman, Burchard 1999.
2. Camia, Newman 2006.
3. The **Schramm-Smirnov space** $\mathscr{H}$, 2011



- $\mathscr{H} \subset \{0,1\}^{\mathcal{Q}}$
- $\mathscr{H}$ can be endowed with a natural topology $\mathcal{T}$ ($\approx$ Fell's topology) for which, $(\mathscr{H}, \mathcal{T})$ is compact, Hausdorff and metrizable

The "critical slice" $\omega_\infty \sim \mathbb{P}_\infty$

View $\omega_\eta \sim \mathbb{P}_\eta$ as a random point in the compact space $(\mathscr{H}, d_{\mathscr{H}})$



Theorem (Smirnov 2001, CN 2006, GPS 2013)

As $\eta \searrow 0$, $\omega_\eta \sim \mathbb{P}_\eta$ converges in law in $(\mathscr{H}, d_{\mathscr{H}})$ to a **continuum percolation**

$$\omega_\infty \sim \mathbb{P}_\infty$$

$\Rightarrow$ this handles the case $\lambda = 0$

Main results

Theorem (Garban, Pete, Schramm 2013)

Fix $\lambda \in \mathbb{R}$.

$$\omega_\eta^{\mathrm{nc}}(\lambda) \xrightarrow{(d)} \omega_\infty^{\mathrm{nc}}(\lambda)$$

The convergence in law holds in the space $(\mathscr{H}, d_{\mathscr{H}})$.

Theorem (Garban, Pete, Schramm 2013)

The càdlàg process $\lambda \mapsto \omega_\eta^{\mathrm{nc}}(\lambda)$ converges in law to $\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\lambda)$ for the ***Skorohod topology*** *on $\mathscr{H}$.*
The limit is a Non-Feller Markov process and is conformally covariant.

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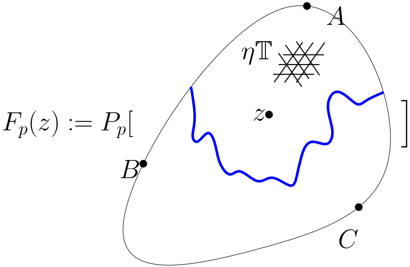
Theorem (Nolin, Werner 2007)

Fix $\lambda \neq 0$. All the subsequential scaling limits of $\omega_{\eta_k}^{\mathrm{nc}}(\lambda) \overset{(d)}{\to} \tilde{\omega}_\infty(\lambda)$ are such that their **interfaces** *are* **singular** *w.r.t the SLE_6 curves !*

Two possible approaches

Smirnov's approach to handle the critical case ($\lambda = 0$):



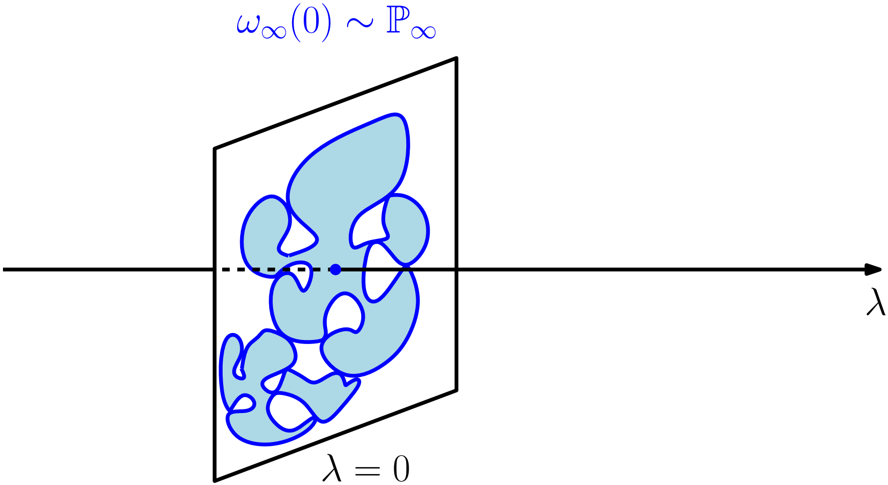
1. This suggests the following approach to handle the case $\lambda \neq 0$: for all $p \neq p_c(\mathbb{T}) = 1/2$, find a **massive harmonic observable** F_p:

$$\boxed{\Delta F_p(z) \approx m(p) F_p(z)}$$

The "mass" $m(p)$ should then scale as $|p - p_c|^{8/3}$.

2. A "perturbative" approach.

Naïve Strategy to build $\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\lambda)$



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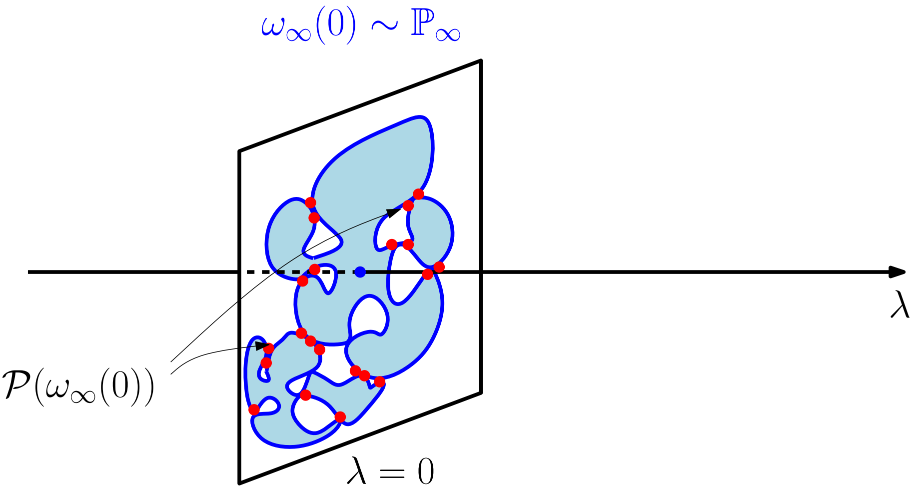
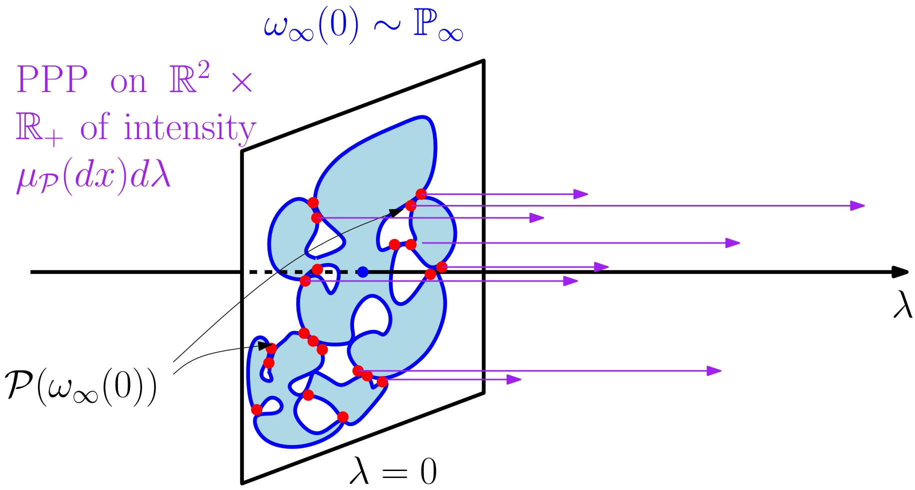


Naïve Strategy to build $\lambda \mapsto \omega_\infty^{\text{nc}}(\lambda)$

PPP on $\mathbb{R}^2 \times \mathbb{R}_+$ of intensity $\mu_{\mathcal{P}}(dx)d\lambda$



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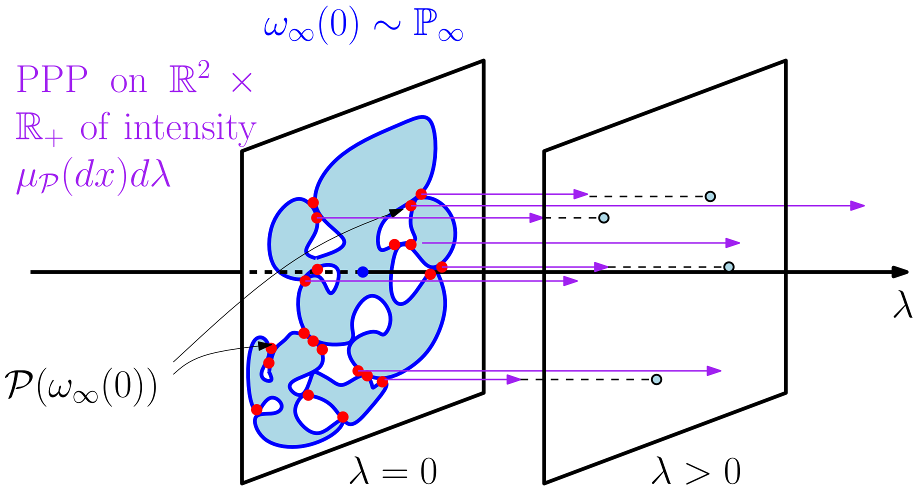

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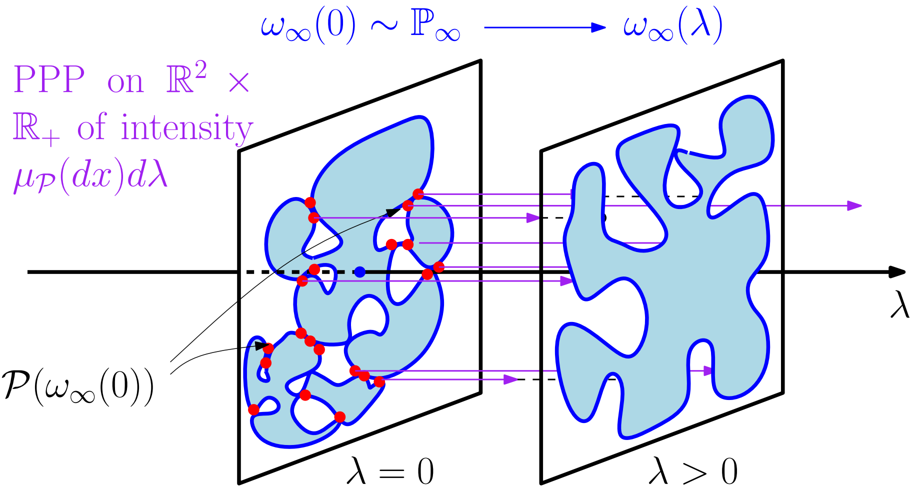
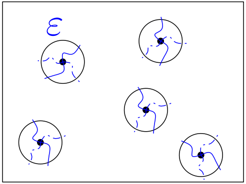


Some difficulties along the way

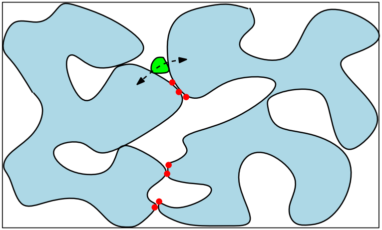
1. Too many pivotals!

   The mass measure μ is degenerate (∞)

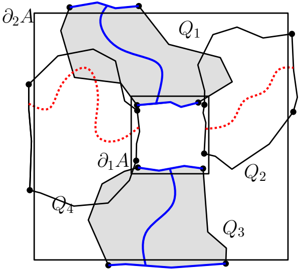
   $\Rightarrow$ introduce a cut-off $\varepsilon > 0$



2. Stability question as $\varepsilon \to 0$



3. Measurability issues on the **Schramm-Smirnov space** $(\mathscr{H}, d_{\mathscr{H}})$

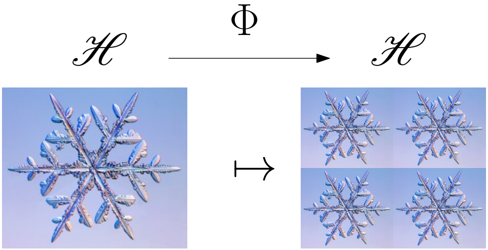


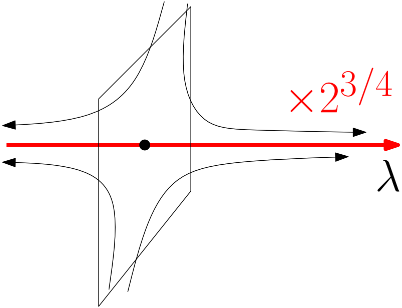
Scaling covariance of our limiting object

Theorem

Near-critical percolation behaves as follows under the scaling $z \mapsto \alpha \cdot z$:

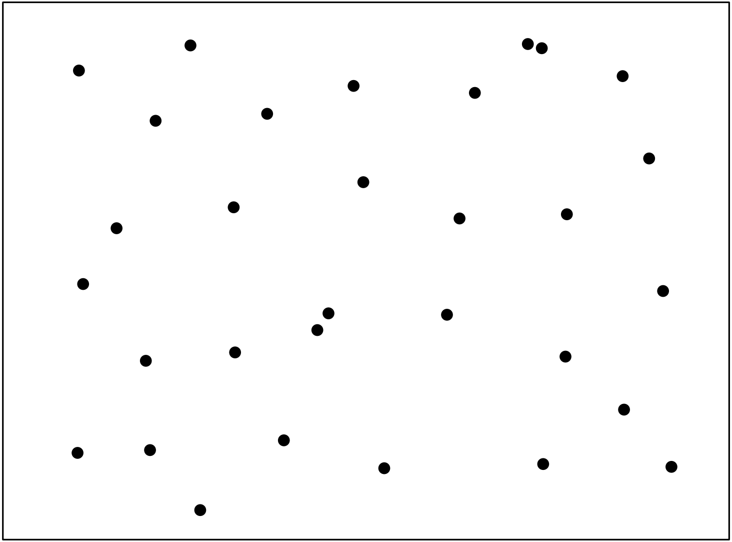
$$\Big(\lambda \mapsto \alpha \cdot \omega_\infty^{\mathrm{nc}}(\lambda)\Big) \overset{(d)}{=} \Big(\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\alpha^{-3/4}\lambda)\Big)$$





Gradient percolation

Traveling Salesman Problem



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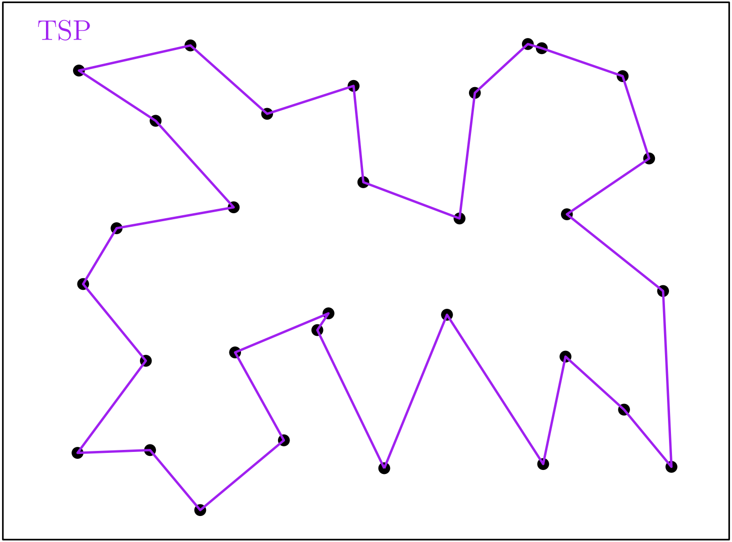
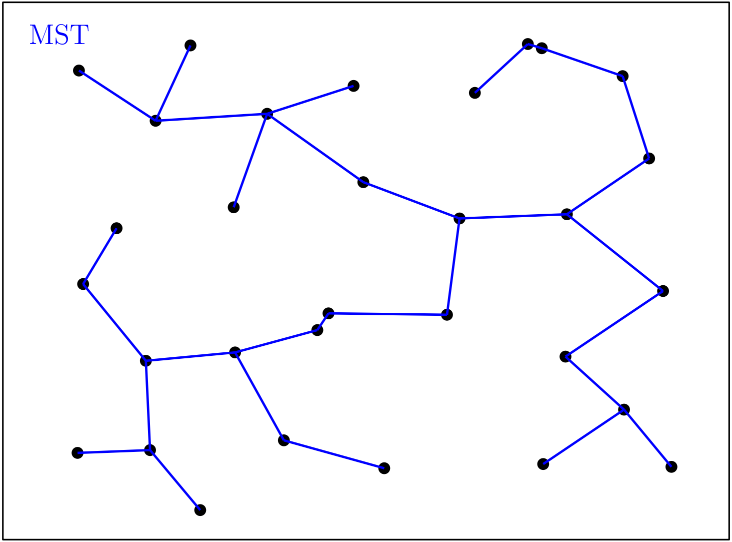
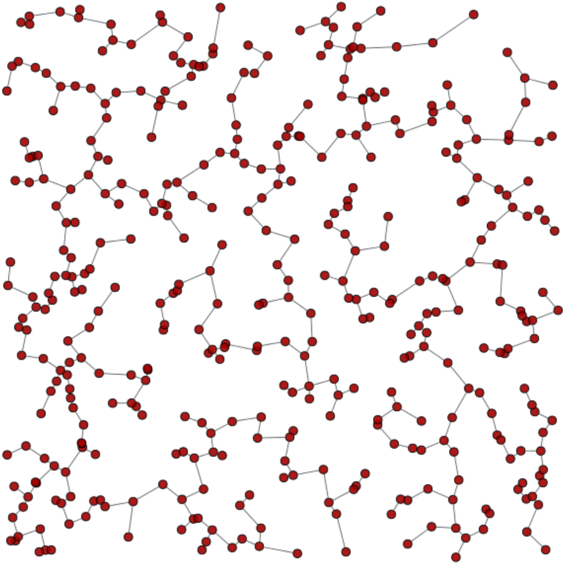


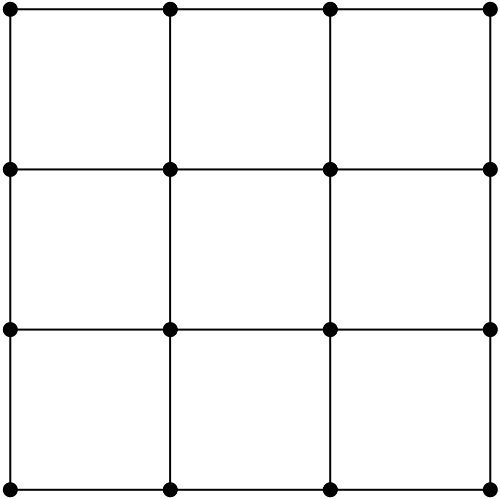
Minimal Spanning Tree (MST)



MAIN QUESTION: scaling limit of the planar MST ?



Minimal Spanning Tree on $\mathbb{Z}^2$



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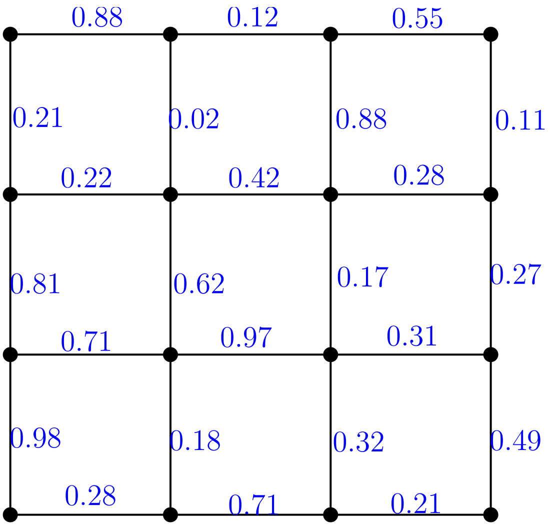
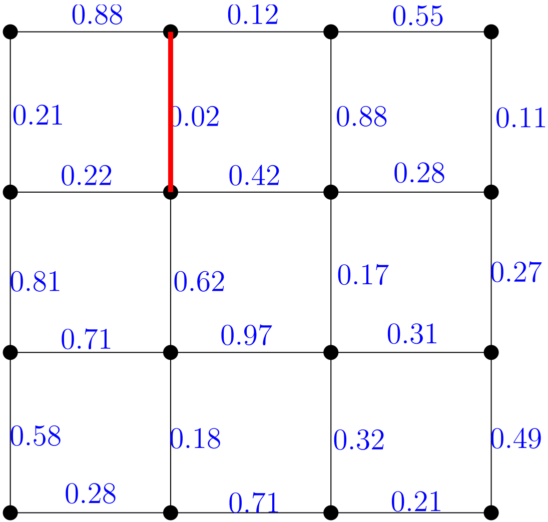


Kruskal's algorithm on $\mathbb{Z}^2$



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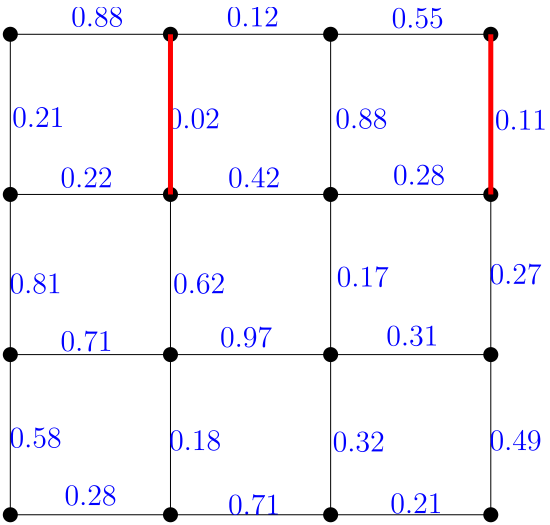

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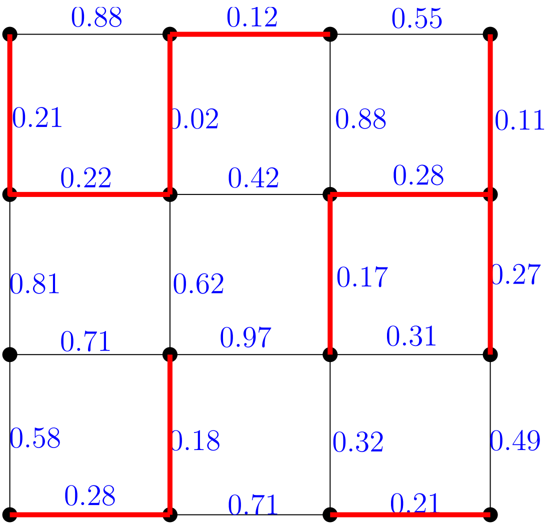

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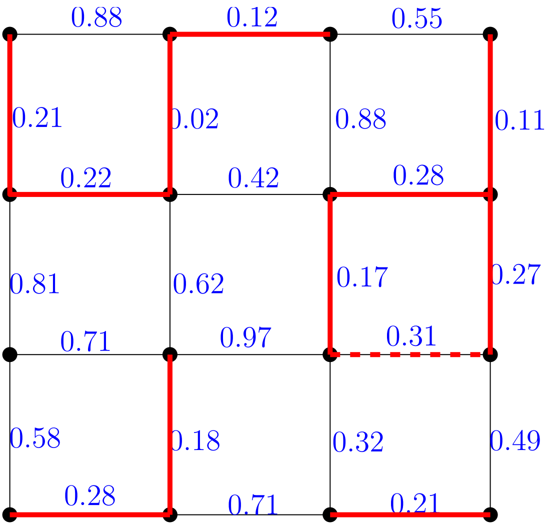

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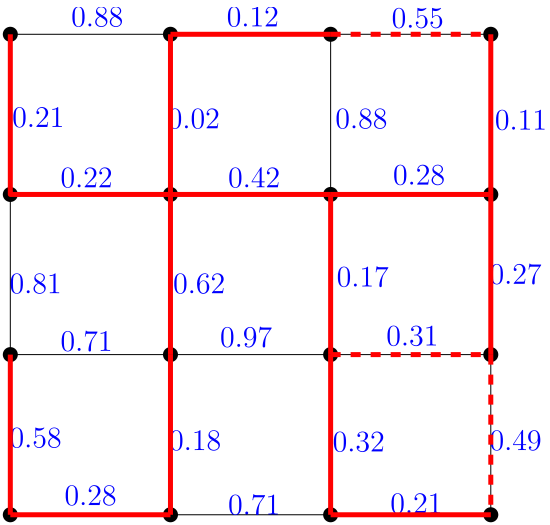

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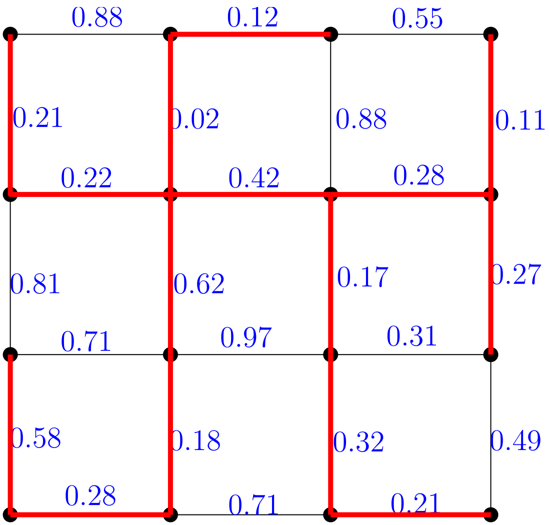
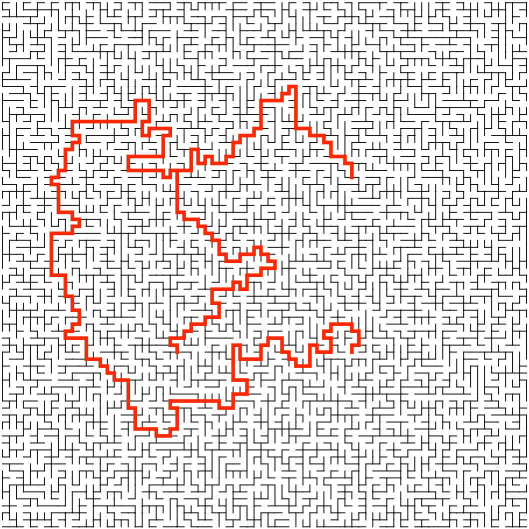
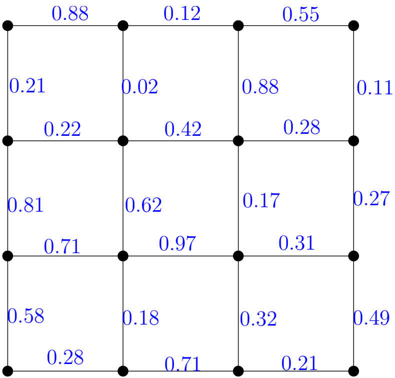


The MST on $\mathbb{Z}^2$ seen from further away ...





Definition (Standard coupling)

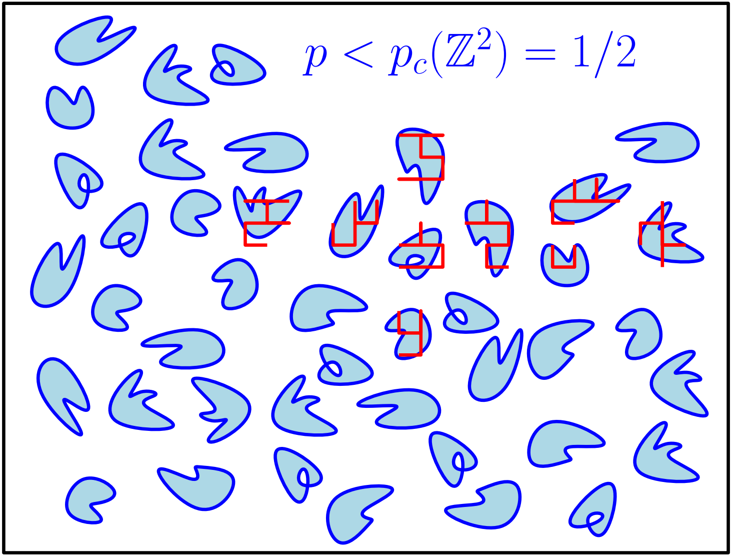
For all $e \in \mathbb{Z}^2$, sample $u_e \sim \mathrm{Unif}[0,1]$.
For any fixed $p \in [0,1]$, let

$$\omega_p(e) := 1_{u_e \le p}$$

Then $\omega_p \sim \mathbb{P}_p$ for all p, and

$$\omega_p \le \omega_{p'} \quad \text{if } p \le p'.$$

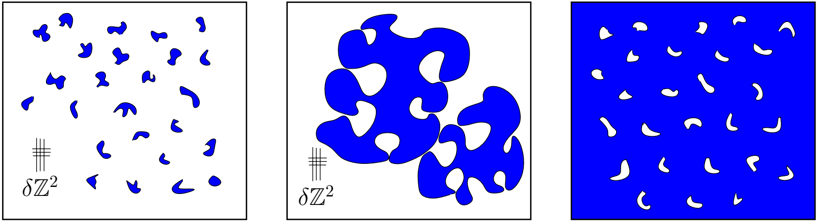
Kruskal's algorithm on $\mathbb{Z}^2$



Minimal Spanning Tree in the plane

Raising p from 0 to 1, the edges where the percolation p-clusters coalesce are exactly the MST.

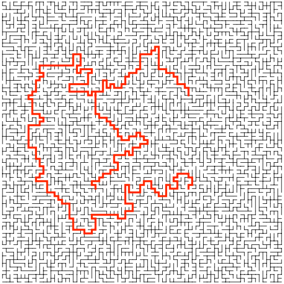
Thus, the macroscopic structure of MST might be understood from near-critical percolation: at $p = p_c + \lambda r(\eta)$, how clusters coalesce as λ increases from $-\infty$ to ∞.



Minimal Spanning Tree in the plane

Theorem (Aizenman, Burchard, Newman, Wilson, 1999)

*The Minimal Spanning Tree on $\eta\mathbb{Z}^2$ is **tight** as $\eta \to 0$.*
*Results on the **geometry** of any limit; e.g., degrees are a.s. less than some k_0.*



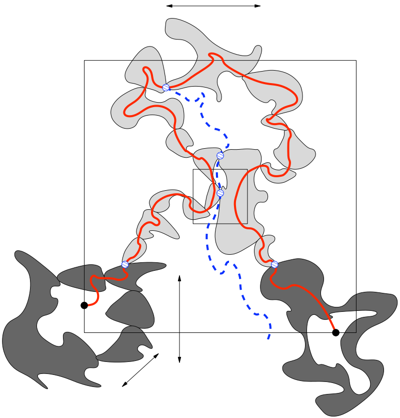
Main results on the scaling limit of MST

Theorem (GPS 2013)

On the rescaled triangular lattice $\eta\mathbb{T}$, MST_η converges in law to MST_∞, in the topology of [ABNW 1999]

Theorem (GPS 2013)

1. ***Invariant under rotations, scalings, translations***
2. *The Hausdorff dimension of the branches lies in* $(1+\varepsilon, 7/4-\varepsilon)$
3. *All points have* ***degree*** ≤ 4
4. *There are no pinching points*

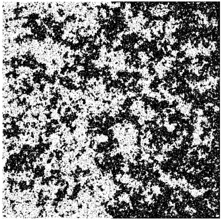


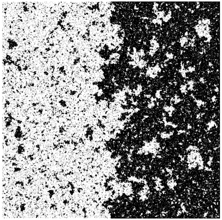
Near-critical FK-Ising

Ising model

For $\sigma \in \{-1,1\}^V$,
$\mathbb{P}_T[\sigma] \propto \exp(\frac{1}{T} \sum_{i\sim j} \sigma_i \sigma_j)$

$T > T_c$: correlations decay quickly.
T_c: they decay slowly.
$T < T_c$: they do not decay.





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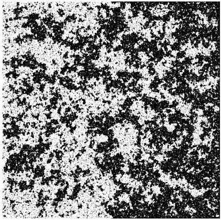
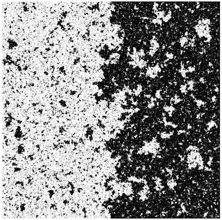
For $\sigma \in \{-1,1\}^V$,
$\mathbb{P}_T[\sigma] \propto \exp(\frac{1}{T}\sum_{i\sim j}\sigma_i\sigma_j)$

$T > T_c$: correlations decay quickly.
T_c: they decay slowly.
$T < T_c$: they do not decay.

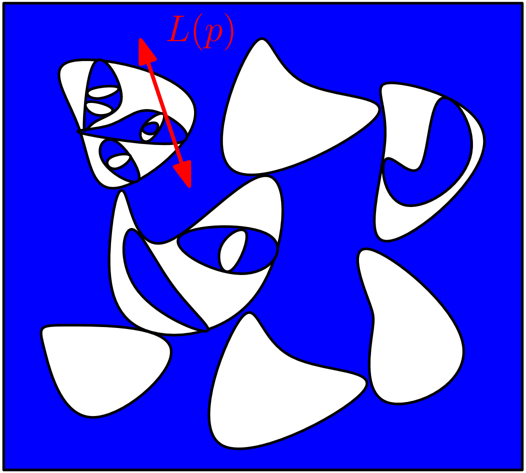
Fortuin-Kasteleyn's FK(p, q) random cluster model (1972)

$$\omega \in \{0,1\}^E \qquad \mathbb{P}_{p,q}[\omega] \propto p^{\sharp\text{open}(\omega)}(1-p)^{\sharp\text{closed}(\omega)}\, q^{\sharp\text{clusters}(\omega)}$$

Edwards-Sokal coupling for $q=2$: toss a fair coin for each ω-cluster. Get Ising with $\frac{2}{T} = -\ln(1-p)$, thus $\text{Correl}_T[\sigma(x),\sigma(y)] = \mathbb{P}_{\text{FK}(p,2)}[x \longleftrightarrow y]$.

- Study **FK percolation phase transition** at $p_c(\mathbb{Z}^2, q=2) = \frac{\sqrt{2}}{1+\sqrt{2}}$.

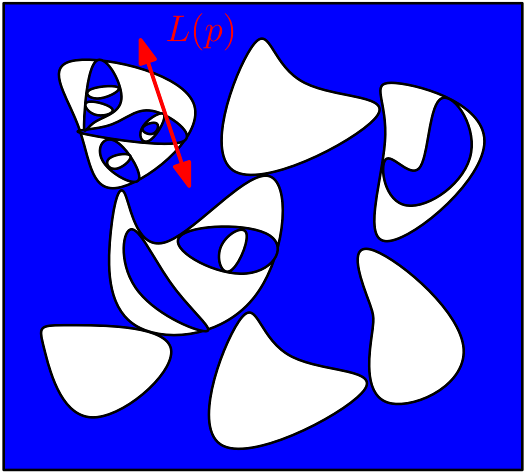
Notion of correlation length $L(p)$



$$p = p_c + \delta p$$

$$L(p) = \left|\frac{1}{p - p_c}\right|^{\nu + o(1)}$$

Notion of correlation length $L(p)$



$$p = p_c + \delta p$$

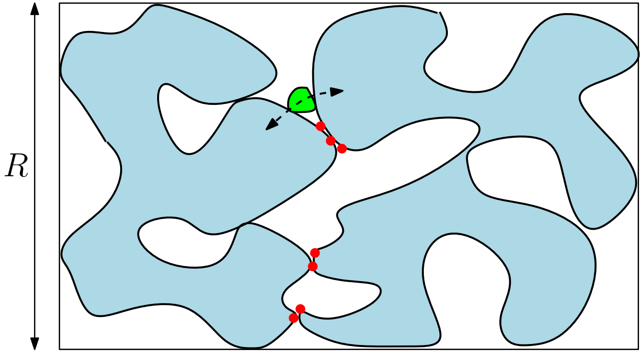
$$L(p) = |\frac{1}{p-p_c}|^{\nu+o(1)}$$

Example (critical percolation):

Theorem (Smirnov-Werner 2001):

$$L(p) = |\frac{1}{p-p_c}|^{4/3+o(1)}$$

Recipe to guess the correlation length

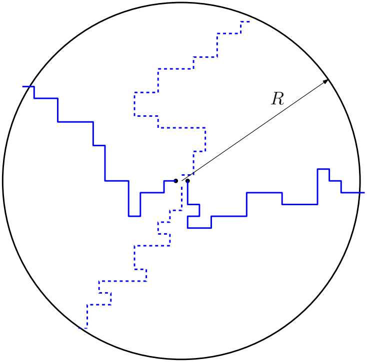


Take $p = p_c + \delta p$. Find the scale $R = L(p)$ for which

$$|\delta p|\, R^2 \alpha_4(R) \asymp 1\,.$$

This works for percolation: Kesten's near-critical scaling relation (1987).

Correlation length of FK-Ising



Using conformal invariance and SLE (Smirnov, Chelkak, et al):

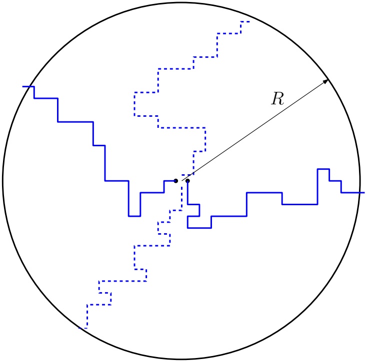
Theorem (Hao Wu, 2015)

$$\alpha_4^{\mathrm{FK}(q=2)}(R) = R^{-\frac{35}{24}+o(1)}$$

The above recipe would then give

$$L(p) = \left|\frac{1}{p - p_c(2)}\right|^{24/13+o(1)} .$$

Correlation length of FK-Ising



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The above recipe would then give

$$L(p) = \left| \frac{1}{p - p_c(2)} \right|^{24/13+o(1)}.$$

But this contradicts closely related results of Onsager (1944), suggesting

$$L(p) \approx \left| \frac{1}{p - p_c} \right| \ll \left| \frac{1}{p - p_c} \right|^{24/13} !!$$

What is wrong with the recipe?

- To make sense of the recipe, need a monotone coupling as p varies, i.e., random $Z \in [0,1]^{E(G)}$ labeling such that $Z_{\leq p} \sim \mathrm{FK}(p,q)$.

  Shown to exist by Grimmett (1995), but not very explicit.
- The density of edges in $Z_{\leq p_c+\delta p} \setminus Z_{\leq p_c}$ is not $\asymp \delta p$, but $\delta p \log \frac{1}{\delta p}$ for $q = 2$, and polynomial blowup for $q > 2$.
- If they were arriving in a Poissonian way, our stability proof would work, and get the same exponent! But changes are much faster!

  There are clouds of open bonds appearing together, with some clever self-organization, to create long connections.

What we can prove

Theorem (Duminil-Copin, G., P., 2014)

For $q = 2$, there are constants $c, C > 0$ s.t.

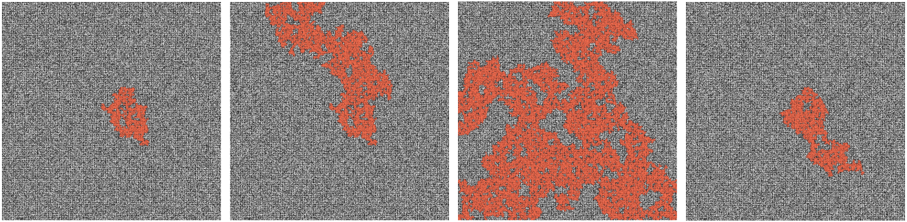
$$c \, \frac{1}{|p - p_c|} \leq L(p) \leq C \frac{1}{|p - p_c|} \sqrt{\log \frac{1}{|p - p_c|}}$$

for all $p \neq p_c$.

Proof technique: Do not understand self-organization enough. Instead, massive version of Smirnov's fermionic observable, building on work of Beffara & Duminil-Copin (2012).

Stationary dynamics

- Each edge of $\mathbb{Z}^2$ has an independent Poisson clock, resampling its state according to $\mathrm{FK}(p_c, q)$, given all the other edges.
- For $1 \leq q \leq 4$, at any given time t, the system **almost surely** has no infinite cluster. (Duminil-Copin, Sidoravicius, Tassion 2015).
- But there could exist exceptional random times with an infinite cluster!



Exceptional times in dynamical FK

Theorem (GPS 2010)

For $q = 1$, they exist, and their Hausdorff dimension is 31/36.

Upper bound is easy, using comparison with near-critical dynamics. Lower bound needs strong noise sensitivity, proved via discrete Fourier analysis.

Theorem (GP, in preparation)

For $q = 2$, using fake Poissonian near-critical dynamics, H-dim $< \frac{10}{13}$.
For $q > 4$, using discontinuity of phase transition, no exceptional times.

Conjecture (GP)

Upper bound is correct for all $q \geq 1$.
There are no exceptional times iff $q > q^* := 4\cos^2(\frac{\pi}{4}\sqrt{14}) \approx 3.83$.

Some open questions

- Show that MST_∞ is not conformally invariant!
- Or at least that $\mathsf{MST}_\infty \neq \mathsf{UST}_\infty$. The latter is given by SLE_8.
- Find the Hausdorff dimension of branches
- Describe the massive SLE_6 we obtained
- Prove noise sensitivity of dynamical FK-Ising.
- Prove the conjectures on exceptional times.