The scaling limits of dynamical and near-critical planar percolation

Gábor Pete
http://www.math.bme.hu/∼gabor

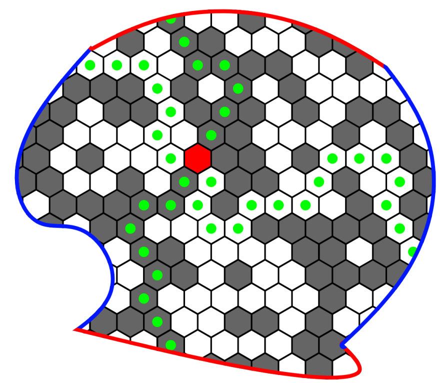
Joint work with Christophe Garban (ENS Lyon)
and Oded Schramm (Microsoft Research, †2008)

Pivotal, cluster and interface measures for critical planar percolation,
[arXiv:1008.1378 math.PR];
The scaling limit of the Minimal Spanning Tree — a preliminary report,
[arXiv:0909.3138 math.PR];
and two more papers in preparation.

The critical window of percolation

Standard coupling: to each site $x \in \Delta_\eta$, assign $V(x)$ i.i.d. Unif$[0,1]$, and let x be **open at level** p if $V(x) \leqslant p$.

In $\mathcal{Q} \cap \Delta_\eta$, when raising p from p_c, when does it become well-connected?



A site is pivotal in ω if flipping it changes the existence of a left-right crossing. Equivalent to having alternating 4 arms. For nice quads, there are not many pivotals close to $\partial\mathcal{Q}$, hence

$$\mathbf{E}|\mathrm{Piv}_\eta| \asymp \eta^{-2}\,\alpha_4(\eta,1) = \eta^{-3/4+o(1)} \text{ on } \Delta_\eta.$$

If $p - p_c \gg \eta^{3/4+o(1)}$, we have opened many critical pivotals, hence already supercritical. But maybe new pivotals appeared on the way, hence the change was actually faster?

Stability by Kesten (1987): multi-arm probabilities stay comparable inside this regime, hence this is the critical window. And $\theta(p_c + \epsilon) = \epsilon^{5/36+o(1)}$.

Kesten's proof of stability and $\beta = \frac{\xi_1}{2-\xi_4} = 5/36$

For $p > 1/2$, let $L_\delta(p) := \min\{n : \mathbf{P}_p\big[\,\mathsf{LR}(n)\,\big] > 1-\delta\}$, correlation length.

For small enough δ, there is a dense infinite cluster above this scale $L(p)$. In particular, $\mathbf{P}_p\big[\,0 \leftrightarrow L(p)\,\big] \asymp \mathbf{P}_p\big[\,0 \leftrightarrow \infty\,\big]$.

Russo's inequality: $\frac{d}{dp}\mathbf{P}_p\big[\,\mathcal{A}\,\big] = \sum_{x\in\mathcal{Q}} \mathbf{P}_p\big[\,x \text{ is pivotal for } \mathcal{A}\,\big]$.

$\frac{d}{dp}\mathbf{P}_p\big[\,\mathsf{LR}(n)\,\big] \asymp n^2\,\mathbf{P}_p\big[\,\mathcal{A}_4(n)\,\big]$ and $\left|\frac{d}{dp}\mathbf{P}_p\big[\,\mathcal{A}_4(n)\,\big]\right| \leqslant O(1)\,n^2\,\mathbf{P}_p\big[\,\mathcal{A}_4(n)\,\big]^2$, thus $\left|\frac{d}{dp}\log\mathbf{P}_p\big[\,\mathcal{A}_4(n)\,\big]\right| \leqslant O(1)\frac{d}{dp}\mathbf{P}_p\big[\,\mathsf{LR}(n)\,\big]$.

Let $p_0 > 1/2$, $n = L(p_0)$. Integrate from $1/2$ to p_0. Here $\mathbf{P}_p\big[\,\mathsf{LR}(L(p_0))\,\big]$ is almost a constant. Hence $\mathbf{P}_{1/2}\big[\,\mathcal{A}_4(L(p_0))\,\big] \asymp \mathbf{P}_{p_0}\big[\,\mathcal{A}_4(L(p_0))\,\big]$.

From this we also get
$$\frac{d}{dp}\mathbf{P}_p\big[\,\mathsf{LR}(L(p_0))\,\big] \asymp L(p_0)^2\,\mathbf{P}_p\big[\,\mathcal{A}_4(L(p_0))\,\big] \asymp L(p_0)^2\,\mathbf{P}_{1/2}\big[\,\mathcal{A}_4(L(p_0))\,\big],$$
and integrating from $1/2$ to p_0 now gives
$$1 \asymp (p_0 - 1/2)\,L(p_0)^2\,\mathbf{P}_{1/2}\big[\,\mathcal{A}_4(L(p_0))\,\big].$$

From the 4-arm exponent $5/4$, get $L(p) = (p - 1/2)^{-4/3+o(1)}$.

As above, also get $\mathbf{P}_{1/2}\big[\,0 \leftrightarrow L(p)\,\big] \asymp \mathbf{P}_p\big[\,0 \leftrightarrow L(p)\,\big]$, hence
$$\begin{aligned}\mathbf{P}_p\big[\,0 \leftrightarrow \infty\,\big] &\asymp \mathbf{P}_p\big[\,0 \leftrightarrow L(p)\,\big] \asymp \mathbf{P}_{1/2}\big[\,0 \leftrightarrow L(p)\,\big]\\ &\asymp \left((p-1/2)^{-4/3+o(1)}\right)^{-5/48+o(1)} = (p-1/2)^{5/36+o(1)}.\end{aligned}$$

Later, more precise finite-size scaling results by Borgs-Chayes-Kesten-Spencer (2001). The system looks critical below the scale $L(p)$; e.g., the sized of largest clusters are not concentrated.

Taking the scaling limit

So, take $p = 1/2 + \lambda\, r(\eta)$, with $r(\eta) := \eta^2 \alpha_4(\eta, 1)^{-1} \overset{\triangle}{=} \eta^{3/4+o(1)}$ and $\lambda \in (-\infty, \infty)$. The standard coupling in this range is the near-critical ensemble. Might hope to get interesting scaling limit as $\eta \to 0$.

Nolin-Werner (2008): Subsequential limits of the near-critical interface exist, and are singular w.r.t. the critical interface SLE_6.

What about similar limit in dynamical percolation? As we saw, if each clock has a rate $r(\eta)$ (as opposed to RW $\to$ BM, need to slow down time!), then the expected number of pivotal switches for $\mathsf{LR}_{\mathcal{Q},\eta}$ in unit time is $\Theta_{\mathcal{Q}}(1)$. So, again hope for nice scaling limit. Moreover, [GPS'08]:

$$\mathbf{E}\big[\, \mathsf{LR}_{\mathcal{Q},\eta}(\omega_0)\, \mathsf{LR}_{\mathcal{Q},\eta}(\omega_{tr(\eta)})\, \big] - \mathbf{E}[\, f_{\mathcal{Q},\eta}\,]^2 \asymp_{\mathcal{Q}} t^{-2/3} \quad \text{as } t \to \infty.$$

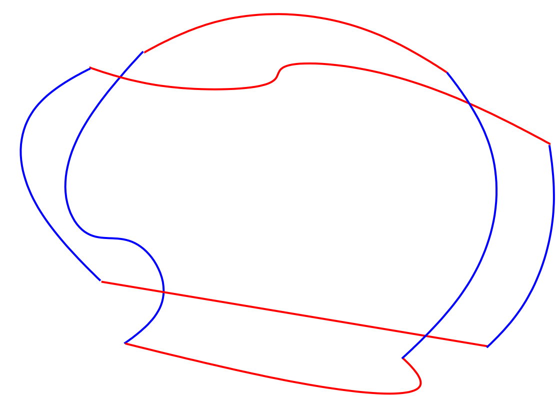
Relation between NCE and DP: whenever a clock rings, **open** it. At time t, each site is open with probability $\sim 1/2 + t\, r(\eta)$. May also take $t < 0$.

What kind of limit? One interface is not enough for a Markovian DPSL.

What kind of limit?

A good definition: a configuration in the scaling limit is the collection of all pw-smooth quads that are crossed.

A much better one by Schramm, with Smirnov:



The set of crossed quads is closed and hereditary.

The collection $\mathcal{S}$ of all closed hereditary sets of quads is a compact Hausdorff space in an appropriate topology. “Dedekind cuts” in a poset.

For each mesh η, percolation is a probability measure on $\mathcal{S}$. Take convergence in law (weak convergence).

Other definitions: all open paths Aizenman (1995); all interface loops Camia-Newman (2006); exploration trees Sheffield (2009).

Uniqueness first proved for and by Camia-Newman. Uniqueness in quad-crossing topology follows.

The results

Theorem (GPS 2010-12). On Δ_η, with rate $r(\eta)$ clocks,
∗ ∃ DPSL
∗ ∃ NCESL
∗ both are Markov
∗ both are conformally covariant: if the domain is changed by $\phi(z)$, then time is scaled locally by $|\phi'(z)|^{3/4}$
∗ DPSL is ergodic (by GPS 2008)

In either case, the process is a random map $\gamma_\eta : \mathbb{R} \mapsto \mathcal{S}$. Not continuous. For the scaling limit, we take Skorohod topology of càdlàg functions.

DPSL question was asked by Schramm, ICM lecture (2006).
Results were conjectured by Camia-Newman-Fontes (2006).
NCESL results refine Bo-Ch-Ke-Sp (2001) and Nolin-Werner (2008).

Near-critical interface (the "massive SLE_6") should have a driving process involving a self-interacting drift term: $dW_t = \sqrt{6}\, dB_t + c\,\lambda\,|d\gamma_t|^{3/4}\, dt^{1/2}$. But is this useful? Near-critical Cardy?

The first main ingredient

Pivotal switches govern the dynamics. If we know the number of pivotal sites for each quad at any given moment, then know the rates at which pivotal switches occur. However, no pivotal sites in scaling limit any more!

Quantity of microscopic pivotals can be seen from macroscopic information:

Theorem 1 (Measurability). For any pw-smooth quad $\mathcal{Q}$, let $\mu_\eta^{\mathcal{Q}}$ be the number of $\mathcal{Q}$-pivotal sites normalized by $\eta^{-2}\alpha_4(\eta,1)$. Then there is a limit of the joint law $(\mu_\eta^{\mathcal{Q}},\omega_\eta) \to (\mu^{\mathcal{Q}},\omega)$, where $\mu^{\mathcal{Q}}$ is a function of ω.
Similar statement for μ_η^ρ, the normalized number of ρ-important sites.

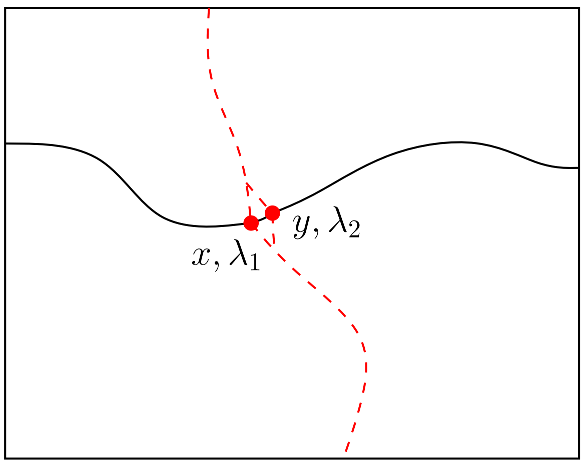
A similar proof almost gives natural time-parametrizations for SLE_6 and $SLE_{8/3}$: questions studied for general κ by Lawler, Sheffield, Alberts, Zhou.

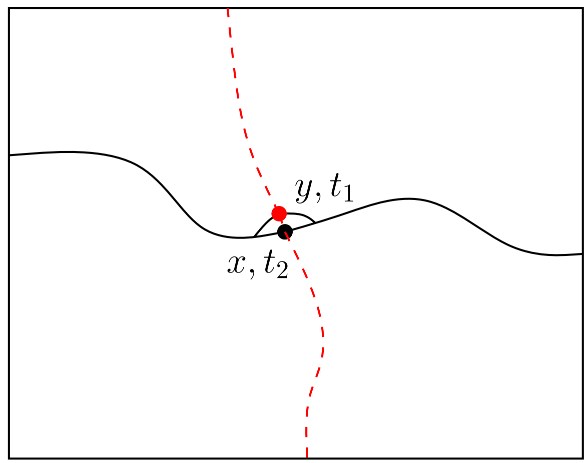
So, can hope that scaling limit of dynamics is given by $\omega_{t=0}$ plus a "filtered" Poisson point process $(\mathscr{P}^\rho)_{\rho>0}$ of flips from μ^ρ(domain) $\times$ Lebesgue(time). This was suggested by Camia-Fontes-Newman.

The second main ingredient

But if we follow all these changes, will we know later what quads are crossed? During the dynamics, no new macroscopic information appears:

Theorem 2 (Stability). Quad $\mathcal{Q}$. Set of sites switched in $[0, t]$ is X_t. The probability that a configuration ω can be changed on X_t into ω', ω'' such that they agree on any site that is at least ϵ-important in ω, but $\mathcal{Q}$ is crossed by ω' while not crossed by ω'', is small if ϵ is small.





Such scenarios of "cascade of importance" do not happen.

Strengthening and simplifying Kesten (1987), saying that in the near-critical window the 4-arm probabilities remain comparable.

Measure on pivotals is measurable in scaling limit

$X = X_\eta^\rho$ is the number of ρ-important sites in Ω, with mesh η.

Intermediate scale: $Y = Y_\eta^{\rho,\epsilon}$ is number of ρ-important ϵ-boxes in a lattice.

$\beta = \beta_\eta^{\rho,\epsilon} := \mathbf{E}\big[\,\rho\text{-important sites in } \epsilon\text{-box } B \;\big|\; B \text{ is } \rho\text{-important}\,\big]$.

Hence $\mathbf{E}[\,X\,] \sim \beta\,\mathbf{E}[\,Y\,]$.

Want that $\lim_{\eta\to 0} \frac{X_\eta^\rho}{\eta^{-2}\alpha_4(\eta,1)}$ exists, and the limit can be read off from macroscopic information (measurable w.r.t. the percolation scaling limit).

This will follow from $\mathbf{E}\big[\,(X - \beta\,Y)^2\,\big] = o(1)\,\mathbf{E}\big[\,X^2\,\big]$ as ϵ and $\eta/\epsilon \to 0$.

Second moment control

X_i := #ρ-important sites in B_i , $\qquad$ $Y_i := 1_{\{B_i \text{ is } \rho\text{-important}\}}$.

$$\mathbf{E}\Big[\big(X - \beta Y\big)^2\Big] = \textstyle\sum_{i,j} \mathbf{E}\big[\,(X_i - \beta\, Y_i)(X_j - \beta\, Y_j)\,\big]$$

Near-diagonal terms are insignificant.

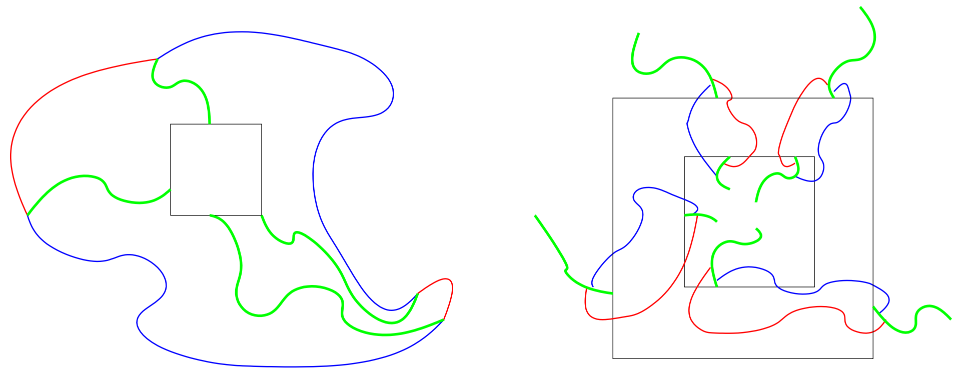
For i, j corresponding to boxes that at distance at least $\sqrt{\epsilon}$ apart:

$\mathcal{F}_i$: σ-field generated by exterior of the $\sqrt{\epsilon}$-box C_i around B_i.

$$\begin{aligned}
&\mathbf{E}\big[\,(X_i - \beta\, Y_i)(X_j - \beta\, Y_j)\;\big|\;\mathcal{F}_i, Y_i = Y_j = 1, \text{connection in } C_i\,\big] \\
&\qquad = (X_j - \beta)\mathbf{E}\big[\,X_i - \beta\;\big|\;\mathcal{F}_i, Y_i = Y_j = 1,\ \text{connection in } C_i\,\big] \\
&\qquad\qquad \leqslant (X_j + \beta)\Big|\mathbf{E}\big[\,X_i - \beta\;\big|\;\mathcal{F}_i, Y_i = Y_j = 1,\ \text{connection in } C_i\,\big]\Big|.
\end{aligned}$$

Now: Loss of information when zooming in to smaller scales, proved by a coupling argument.

Strong Separation Lemma and the coupling argument



Strong Separation Lemma. For $\mathrm{d}(B, \partial\mathcal{Q}) > \mathrm{diam}(B)$, conditioned on the 4 interfaces to reach ∂B, with *arbitrary starting points*, with a uniformly positive conditional probability the interfaces are well-separated around ∂B.

[Simple proof by Damron-Sapozhnikov (2009), following Kesten (1987). See also GPS Pivotal measure (2010) Appendix.]

And given two well-separated 4-tuples of interfaces, using RSW, there is a uniformly positive probability that they couple.

So, going down from the $\sqrt{\epsilon}$-box C_i to the ϵ-box B_i, on each scale we have a uniformly positive probability that the coupling has happened.

Where is the $+o(1)$ from the covariance exponent?

Ratio limit result: $\lim_{\eta\to 0}\frac{\alpha_4^\eta(\eta,r)}{\alpha_4^\eta(\eta,1)} = \lim_{\epsilon\to 0}\frac{\alpha_4(\epsilon,r)}{\alpha_4(\epsilon,1)} = r^{-5/4}$.

The limits ℓ^η and ℓ^ϵ exist by coupling interfaces started from different positions.

Then, given $\lim_{n\to\infty}\frac{\log\alpha_4(r^n,1)}{n} = \log(r^{5/4})$, let us write $\alpha_4(r^n,1)$ as:

$$\alpha_4(r^n,1) = \frac{\alpha_4(r^n,1)}{\alpha_4(r^n,r)}\frac{\alpha_4(r^{n-1},1)}{\alpha_4(r^{n-1},r)}\cdots\frac{\alpha_4(r,1)}{1}\,.$$

$$\frac{\log\alpha_4(r^n,1)}{n} = \frac{1}{n}\sum_{j=1}^{n}\log\frac{\alpha_4(r^j,1)}{\alpha_4(r^j,r)}\,.$$

By the convergence of the Cesàro mean, the right hand side converges to $\log\frac{1}{\ell^\epsilon}$, and we are done.

Stability: important points suffice

Fix $0 < \eta < 1$. Static configuration ω.

$X = X_t$: i.i.d. set of bits each chosen with probability $t\, r(\eta)$.

$\Omega(X, \omega)$: set of ω' that are equal to ω off of X.

$\mathcal{W}_z(r, r')$: the event $\mathcal{A}_4(z, r, r')$ holds for some $\omega' \in \Omega(X, \omega)$.

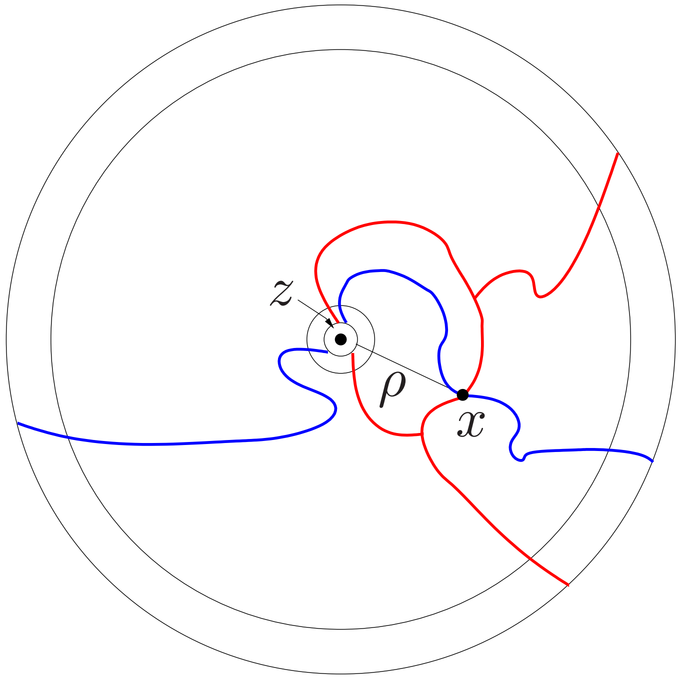
Key Lemma: For $0 < i < j$ with $r_j := 2^j\, \eta < 1$,

$$\mathbf{P}\big[\, \mathcal{W}_z(r_i, r_j)\, \big] \leqslant C_t\, \alpha_4(r_i, r_j).$$

Proof

Assume
$\{\omega \notin \mathcal{A}_4(z, r_{i+1}, r_{j-1})\} \cap \mathcal{W}_z(r_i, r_j)$.

Then $\exists x \in X \cap A(z, r_{i+1}, r_{j-1})$
such that $\mathcal{W}_x(\eta, \rho_x)$ holds.



Hence recursion for $b_i^j := \mathbf{P}\big[\, \mathcal{W}_z(r_i, r_j)\,\big]$:

$$b_i^j \leqslant \alpha_4(r_{i+1}, r_{j-1}) + \sum_{n=i+1}^{j-2} \sum_{x \in A(z, r_n, r_{n+1})} t\, r(\eta)\, b_i^{n-1}\, b_0^{n-1}\, b_{n+2}^j$$

Completed with double induction. ∎

Note that we could have switched the sites in the random set X in any way, say, always to open, hence it's a strengthening of Kesten's stability.

Second lemma. Let $Z_\omega(z)$:= importance of z in ω and $Z_\omega^X(z) := \max\{Z_{\omega'}(z) : \omega' \in \Omega(\omega, X)\}$. For $\eta < \epsilon < 4\,\epsilon < r < 1$:

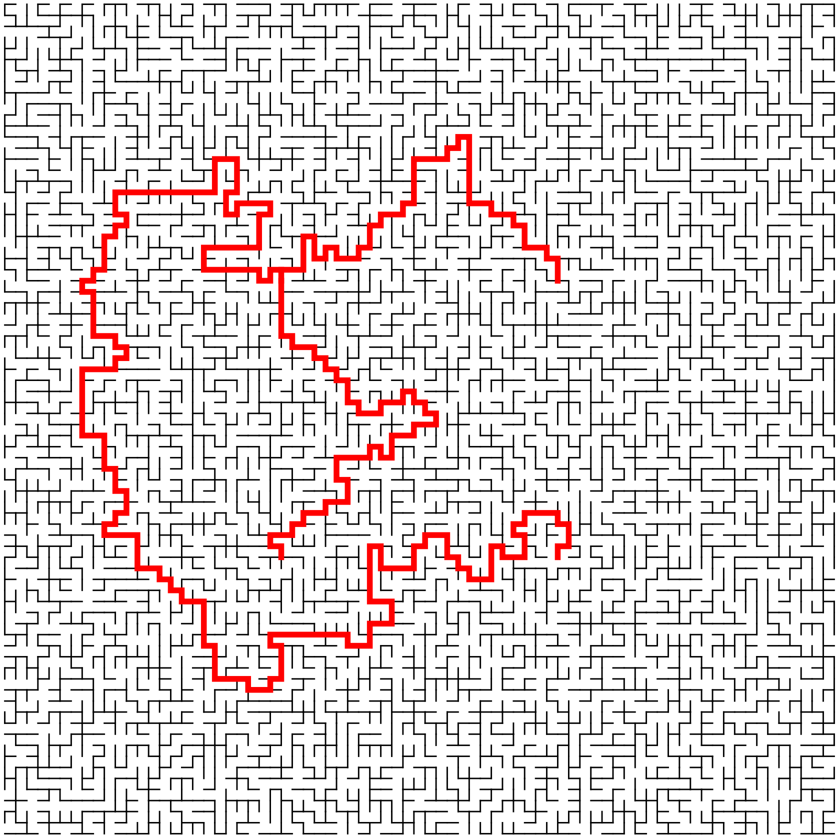
$$\mathbf{P}\Big[\, Z_\omega(z) < \epsilon < r < Z_\omega^X(z) \,\Big] \leqslant C_t \, \frac{\epsilon^2 \, \alpha_4(\eta, \epsilon)}{\alpha_4(r, 1)} \, .$$

Therefore,

$$\mathbf{P}\Big[\, \exists_{z \in [0,1]^2 \cap X} \;\; Z_\omega(z) < \epsilon < r < Z_\omega^X(z) \,\Big] \leqslant C_t \, \alpha_4(r, 1)^{-1} \, \epsilon^2 \, \alpha_4(\epsilon, 1)^{-1} \, .$$

This goes to 0 as $\epsilon \longrightarrow 0$, uniformly in η.

Minimal spanning tree

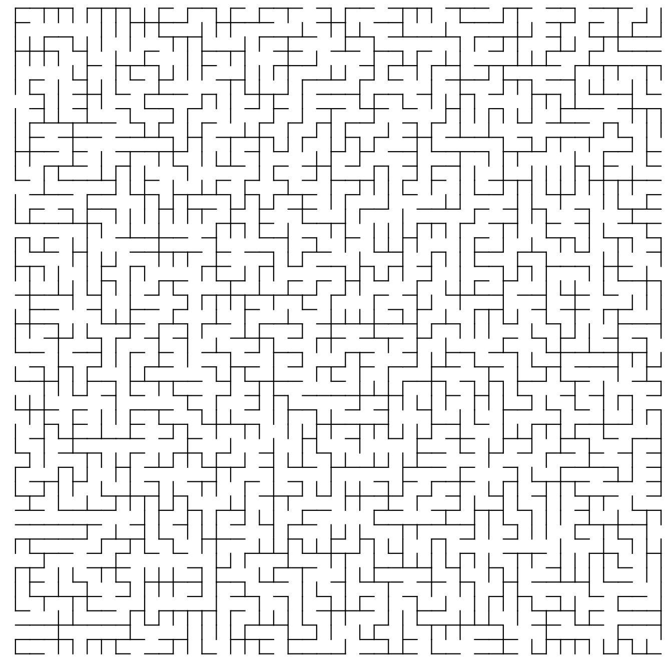


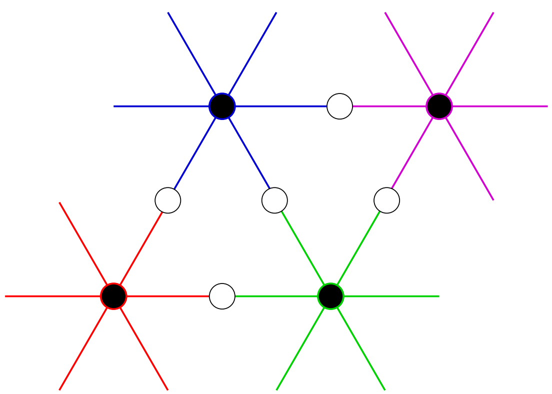
See 6 page ICMP lecture at [arXiv:0909.3138 math.PR].

Minimal spanning tree

For each edge of a finite graph, say $e \in E(\mathbb{Z}_n^2)$, let $U(e)$ be i.i.d. Unif$[0, 1]$. The Minimal Spanning Tree is the tree T for which $\sum_{e \in T} U(e)$ is minimal.

Same as deleting from each cycle the edge with highest U. Or the collection of lowest level paths between all pairs of vertices.



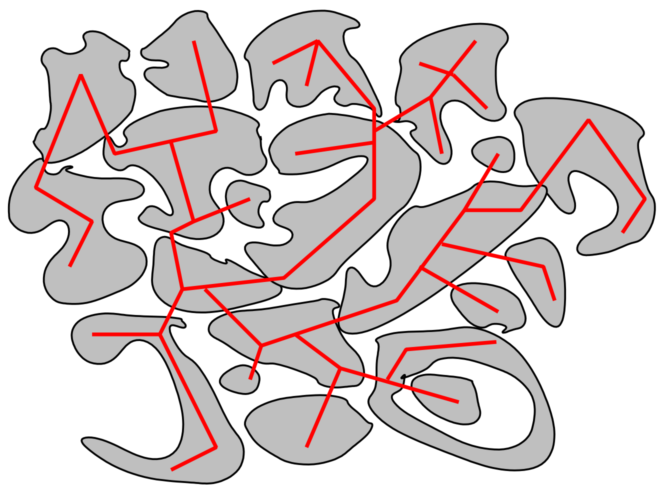


Version adapted to site percolation on Δ: replace each edge by two in series, and for each such edge e, let $U(e) := V(e^*)$, the old vertex endpoint.

MST coupled with NCE

Connection to NCE: macroscopic structure is determined by the cluster tree $T_{\geqslant\lambda}$ between the level λ clusters, $p = 1/2 + \lambda\, r(\eta)$, as $\lambda \to -\infty$.

And the collection of cluster trees $T_{\geqslant\lambda}$ is determined by the collection of λ-clusters over all $\lambda \in (-\infty, \infty)$.



Also, T is the union of the invasion trees of Invasion Percolation. Alexander 1995, Aizenman-Burchard-Newman-Wilson 1999, Häggström-Peres-Schonmann 1999, Lyons-Peres-Schramm 2006.

Theorem GPS. Scaling limit of MST exists, and is rotationally and scaling invariant.

We do **not** expect conformal invariance, because the conformal *co*variance of the NCESL suggests that MST will feel that $|\phi(z)|$ is changing. For example, simulations by D. Wilson (2002).