Stochastic Differential Equations

Problem set No 4 — March 20, 2012

▷ **Exercise 1.** Check that the following processes solve the corresponding SDE's, where B_t is 1-dimensional Brownian motion:

(a) $X_t = e^{B_t}$ solves $dX_t = \frac{1}{2}\,X_t\,dt + X_t\,dB_t$.

(b) $X_t = \frac{B_t}{1+t}$, with $B_0 = 0$, solves

$$dX_t = \frac{-X_t}{1+t}\,dt + \frac{1}{1+t}\,dB_t\,, \qquad X_0 = 0\,.$$

(c) $X_t = \sin(B_t)$, with $B_0 = a \in (-\frac{\pi}{2}, \frac{\pi}{2})$, solves

$$dX_t = -\frac{1}{2}\,X_t\,dt + \sqrt{1 - X_t^2}\,dB_t\,, \qquad t < \inf\Big\{s > 0 : B_s \notin \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]\Big\}\,.$$

(d) $\big(X_1(t),\, X_2(t)\big) = \big(\cosh(B_t),\, \sinh(B_t)\big)$ solves

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} dB_t\,.$$

▷ **Exercise 2.** Solve the following two-dimensional SDE for $X_t = (U_t, V_t)$, driven by a one-dimensional Brownian motion B_t:

$$\begin{aligned} dU_t &= -\frac{1}{2}\,U_t\,dt - V_t\,dB_t \\ dV_t &= -\frac{1}{2}\,V_t\,dt + U_t\,dB_t\,, \end{aligned}$$

or in vector notation,

$$dX_t = -\frac{1}{2}\,X_t\,dt + K\,X_t\,dB_t\,, \qquad \text{where } K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,,$$

and observe that it is Brownian motion on a circle in $\mathbb{R}^2$. (Hint: observe the similarity of the equation with the one for geometric Brownian motion, hence try $Z_t := U_t + iV_t$.)

▷ **Exercise 3.** The **mean-reverting Ornstein-Uhlenbeck process** is the solution of the SDE

$$dX_t = (\mu - X_t)\, dt + \sigma\, dB_t\,,$$

with $\mu, \sigma \in \mathbb{R}$ constants and B_t 1-dim BM. (We saw this in the special case of $\mu = 0$.)

(a) Solve the equation.

(b) Find $\mathbf{E}[X_t]$ and $\mathrm{Var}[X_t]$.

(c) Let $\{X_i\}_{i\geq 0}$ be SRW on the hypercube $\{0,1\}^n$, and let $|X_i|$ be the number of 1's among the coordinates. What does

$$\frac{|X_{\lfloor nt \rfloor}| - n/2}{\sqrt{n}}\,, \qquad t \geq 0\,,$$

have to do with the Ornstein-Uhlenbeck process?

▷ **Exercise 4.** Solve the following SDE's, where B_t is 1-dimensional Brownian motion:

(a) $dX_t = -X_t\, dt + e^{-t}\, dB_t$.

(b) $dX_t = r\, dt + \alpha\, X_t\, dB_t$, with $r, \alpha \in \mathbb{R}$ constants. (Hint: multiply by $\exp\big(-\alpha B_t + \frac{\alpha^2}{2} t\big)$.)

(c) With $X(t) = (X_1(t), X_2(t))$, and a two-dimensional Brownian motion $B(t) = (B_1(t), B_2(t))$,

$$\begin{aligned} dX_1(t) &= X_2(t)\, dt + \alpha\, dB_1(t) \\ dX_2(t) &= -X_1(t)\, dt + \beta\, dB_2(t)\,, \end{aligned}$$

or in vector notation,

$$dX(t) = JX(t)\, dt + A\, dB(t)\,, \qquad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

(Same hint again: multiply by e^{-Jt}. Don't leave the answer in matrix notation, but write out the coordinates using simple 1-dimensional Itô-integrals.)

▷ **Exercise 5.** Recall that any continuous Gaussian process X_t is determined by its means $\mathbf{E}X_t$ pairwise covariances $\mathsf{cov}(s,t) := \mathbf{E}[\,X_s X_t\,] - \mathbf{E}X_s\,\mathbf{E}X_t$. For $a, b \in \mathbb{R}$, the **one-dimensional Brownian bridge** from a to b is such a process for $t \in [0,1]$, with $\mathbf{E}X_t = a(1-t) + bt$ and $\mathsf{cov}(s,t) = s \wedge t - st$. Prove that the law of this process is also given by any of the following definitions:

(a) $X_t := a(1-t) + bt + B_t - tB_1$ for $t \in [0,1]$, with BM started at $B_0 = 0$.

(b) $X_t := a(1-t) + bt + (1-t)B_{t/(1-t)}$. Note that it requires a tiny argument that this definition makes sense at $t = 1$ and gives what we want.

(c) $X_t := a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}\,dB_s$. Note again that $t = 1$ requires care. (Hint for that: use Doob's martingale inequality to bound the probability that $\sup\big\{(1-t)\int_0^t \frac{1}{1-s}\,dB_s : t \in [1-2^{-n}, 1-2^{-n-1})\big\} > \epsilon$.)

(d) Part (c) is in fact the strong solution of the SDE

$$dX_t = \frac{b - X_t}{1-t}\,dt + dB_t\,, \qquad t \in [0,1)\,, \quad X_0 = a\,.$$

▷ **Exercise 6** (Bonus on Tanaka)**.** Recall that **Tanaka's SDE** $dX_t = \mathsf{sign}(X_t)\,dB_t$ has a weak solution but no strong solutions: X_t is a Brownian motion which cannot be measurable w.r.t. $\sigma\{B_s : 0 \le s \le t\}$. In the proof, we used two ingredients: **Tanaka's formula**

$$|B_t| - |B_0| = \int_0^t \mathsf{sign}(B_s)\,dB_s + L_0(t)\,,$$

and that the integral term on the right hand side, denoted by Y_t from now on, is a Brownian motion.

(a) Prove that Y_t is indeed a standard BM. (Hint: use the definition of Itô integrals and the fact that the zero set of BM is closed with zero Lebesgue measure.)

(b) Using part (a), show **Lévy's theorem** relating local time at zero and the maximum process $M_t := \sup\{B_s : 0 \le s \le t\}$ to each other:

$$\big(|B_t|,\, L_0(t)\big)_{t\ge 0} \stackrel{d}{=} \big(M_t - B_t,\, M_t\big)_{t\ge 0}\,.$$

(c) Show the following **discrete Tanaka formula** for SRW $S_n := \sum_{j=1}^n X_j$ on $\mathbb{Z}$:

$$|S_n| - |S_0| = \sum_{j=0}^{n-1} \mathsf{sign}(S_j)\,(S_{j+1} - S_j) + L_0(n)\,,$$

where $L_0(n) := |\{0 \le j \le n-1 : S_j = 0\}|$.