Stochastic Differential Equations

Problem set No 5 — April 3, 2012

▷ **Exercise 1.** Find the generator of the following Itô diffusions:

(a) $dX_t = r\,dt + \alpha\,X_t\,dB_t$.

(b) $dY_t = \begin{pmatrix} dt \\ dX_t \end{pmatrix}$, where $dX_t = \gamma\,X_t\,dt + \sigma\,dB_t$ is an Ornstein-Uhlenbeck process.

(c) $\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^{X_1} \end{pmatrix} dB_t$.

(d) $\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & X_1 \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}$.

▷ **Exercise 2.** Find an Itô diffusion (i.e., write down the SDE for it) whose generator is the following:

(a) $Af(x) = f'(x) + f''(x)$, $f \in C_0^2(\mathbb{R})$.

(b) $Af(t,x) = \frac{\partial f}{\partial t} + cx\frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2}$, $f \in C_0^2(\mathbb{R}^2)$.

▷ **Exercise 3.**

(a) Let B_t be 1-dimensional BM, with $B_0 = x > 0$, and let $\tau := \inf\{t : B_t = 0\}$. Using Dynkin's formula, show that $\tau < \infty$ a.s., but $\mathbf{E}\tau = \infty$.

(b) Let X_t be a geometric BM, i.e.,

$$dX_t = r\,X_t\,dt + \alpha\,X_t\,dB_t\,, \qquad X_0 = x > 0\,.$$

Find the generator A of X_t and compute $Af(x)$ when $f(x) = x^\gamma$, $x > 0$, γ constant.

(c) Recall that if $r < \alpha^2/2$, then $X_t \to 0$ a.s., as $t \to \infty$. For $R > x$, using Dynkin's formula with $f(x) = x^{\gamma_1}$, $\gamma_1 = 1 - 2r/\alpha^2$, prove that

$$\mathbf{P}_x[\,X_t \text{ ever hits } R\,] = \left(\frac{x}{R}\right)^{\gamma_1}\,.$$

▷ **Exercise 4.**

(a) Find the generator of the d-dimensional Bessel process $dX_t = \frac{d-1}{2X_t}\,dt + dB_t$ on $\mathbb{R}$.

(b) Using Dynkin's formula, show that the Bessel(d) process is transient iff $d > 2$.

(c) (**Bonus**) Show that the Bessel(d) process is recurrent for $0 < d < 2$ in the sense that all points (including zero) are visited infinitely often. (Clear for strictly positive points and $d > 2 \geq 1$, when the drift is non-negative.) For $d \leq 0$, the point 0 is absorbing.

▷ **Exercise 5.** Let $C(\mathbb{N}) := \big\{ f : \mathbb{N} \longrightarrow \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0 \big\}$ with norm $\|f\| := \sup_{x\in\mathbb{N}} f(x)$. Give examples of continuous time Markov chains P_t on $\mathbb{N}$ with the following properties:

(a) it is not a contraction semigroup from $C(\mathbb{N})$ into $C(\mathbb{N})$.

(b) it is a contraction semigroup on $C(\mathbb{N})$, whose generator is defined on the entire $C(\mathbb{N})$ but is not bounded there.

(c) (**Bonus**) If P_t is a contraction semigroup on $C(\mathbb{N})$, is it necessarily strongly continuous? What about $C(\mathbb{N}^*)$, where $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$? (This allows the Markov chain to make infinitely many jumps in finite time and then stay at ∞ forever.) And what if P_t is a contraction semigroup on $C(\mathbb{Q})$?

▷ **Exercise 6.** Show that the solution $u(t,x)$ of the initial value problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2} + \alpha\, x\, \frac{\partial u}{\partial x}\,, \qquad t > 0,\ x \in \mathbb{R}\,,$$

$$u(0,x) = f(x)\,, \qquad (f \in C_0^2(\mathbb{R}) \text{ given})$$

can be expressed as follows:

$$\begin{aligned} u(t,x) &= \mathbf{E}\Big[\, f\Big(x\, \exp\big\{\beta\, B_t + (\alpha - \beta^2/2)\, t\big\}\Big)\Big] \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f\Big(x\, \exp\big\{\beta\, y + (\alpha - \beta^2/2)\, t\big\}\Big) \exp\left(-\frac{y^2}{2t}\right)\, dy\,, \qquad t > 0\,. \end{aligned}$$