Stochastic Differential Equations

Problem set No 6 — April 24, 2012

▷ **Exercise 1.** Recall that we defined $A : \mathscr{C}(A) \longrightarrow \mathbf{B}$ to be **dissipative**, or $-A$ to be **accretive** if $\mathscr{C}(A)$ is dense in $\mathbf{B}$ and $\forall \varphi \in \mathscr{C}(A)$ exists a normalized tangent functional $\ell \in \mathbf{B}^*$ such that $\ell(-A\varphi) \geq 0$. We showed in class that this implies that

$$\|(\lambda - A)\varphi\| \geq \lambda \|\varphi\| \qquad \text{for all } \lambda > 0\,, \tag{1}$$

and, if A is the infinitesimal generator of a contraction semigroup, then also the other way around.

(a) Show that (1) implies that A is closable in $\mathbf{B}$.

(b) Consider $Af = \frac{1}{2} f''$ on $\tilde{\mathscr{C}} = C_\infty[0,\infty) \cap C^2_\infty[0,\infty)$, a wannabe generator of reflecting BM on $[0,\infty)$. Show that A does not satisfy (1) on $\tilde{\mathscr{C}}$ and is not closable in $\mathbf{B} = C_\infty[0,\infty)$. On the other hand, it does satisfy (1) on $\mathscr{C} = C_\infty[0,\infty) \cap C^2_\infty[0,\infty) \cap \{f'(0) = 0\}$.

▷ **Exercise 2.** Young's inequality for convolutions says that if $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, then $\|f * g\|_r \leq \|f\|_p \, \|g\|_q$. Using this, show that $e^{\Delta t}$ is a continuous L^p-contractive semigroup.

▷ **Exercise 3** (Bonus)**.** For Δ on $\mathbf{B} = C_\infty(\mathbb{R})$, we have seen that $\mathscr{D}(\Delta) = C_\infty(\mathbb{R}) \cap C^2_\infty(\mathbb{R})$, i.e., vanishing value and vanishing 2nd derivative at infinity. We have also seen that on $\mathbb{R}^d$, $d \geq 2$, the Schwarz space $\mathscr{S}(\mathbb{R}^d)$ is a good core $\mathscr{C}(\Delta)$: the operator $-\Delta$ is accretive there, hence closable, and $\overline{\mathrm{Ran}(I - \Delta)} = C_\infty(\mathbb{R}^d)$, and thus Δ is indeed an infinitesimal generator, as we already knew. But what is $\mathscr{D}(\Delta)$, i.e., what domain do we get when we close the operator from $\mathscr{S}(\Delta)$? It contains $C_\infty(\mathbb{R}) \cap C^2_\infty(\mathbb{R})$, but isn't it larger?

▷ **Exercise 4.**

(a) Let ψ be a bounded continuous function on $\mathbb{R}^n$, and $\alpha > 0$. Find a bounded solution u of the equation

$$(\alpha - \frac{1}{2}\Delta)u = \psi \qquad \text{on } \mathbb{R}^n\,.$$

Prove that the solution is unique.

(b) Let B_t be n-dimensional Brownian motion ($n \geq 1$) and let F be a Borel set in $\mathbb{R}^n$. Prove that the expected total length of times t that B_t stays in F is zero if and only if the Lebesgue measure of F is zero. (Hint: Consider the resolvent R_α for $\alpha > 0$ and then let $\alpha \to 0$.)

▷ **Exercise 5.** In connection with the deduction of the Black-Scholes formula for the price of an option, the following partial differential equation appears for $u = u(t, x)$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\rho\, u + \alpha\, x\, \frac{\partial u}{\partial x} + \frac{1}{2}\, \beta^2\, x^2\, \frac{\partial^2 u}{\partial x^2}\,; \qquad t > 0,\ x \in \mathbb{R} \\ u(0, x) &= (x - K)^+\,; \qquad x \in \mathbb{R}\,, \end{aligned}$$

where $\rho > 0$, α, β, $K > 0$ are constants. Use the Feynman-Kac formula to prove that the solution u of this equation is given by

$$u(t, x) = \frac{e^{-\rho t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} \Big(x \,\exp\big\{(\alpha - \beta^2/2)t + \beta y\big\} - K\Big)^+ \exp\left(-\frac{y^2}{2t}\right) dy\,; \qquad t > 0\,.$$

▷ **Exercise 6.**

(a) I have seen the following result called Feynman-Kac formula, e.g., by L. Craig Evans: the unique solution for the PDE

$$\begin{aligned} -\frac{1}{2}\Delta u + c\, u &= f \qquad \text{in } U \subset \mathbb{R}^n \text{ bounded} \\ u &= 0 \qquad \text{on } \partial U\,, \end{aligned}$$

where c, f are smooth functions, $c \geq 0$, is given by

$$u(x) = \mathbf{E}_x \left[\int_0^\tau f(X_t)\, e^{-\int_0^t c(X_s)\, ds} dt \right] \qquad \text{for } x \in U\,,$$

where τ is the first hitting time of ∂U. Prove this formula.

(b) I have found the following exercise in the same source, but I haven't managed to do it in the way suggested by Craig Evans. Let f be a positive smooth function on $\mathbb{R}^n$. Use the above Feynman-Kac formula to prove that

$$M(t) := f(B_t) \exp\Big\{ -\frac{1}{2} \int_0^t \Delta f(B_s)\, ds \Big\}$$

is a martingale.