The near-critical planar Ising Random Cluster model

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and Christophe Garban (ENS Lyon, CNRS)

arXiv:1111.0144

Rough outline

The Fortuin-Kasteleyn random cluster model FK(p, q) on a finite or infinite graph is, in some sense, a joint generalization of three well-known models:

$q = 0$, $p = 0$: Uniform Spanning Tree, UST

$q = 1$: Bernoulli(p) bond percolation

$q = 2$: The Ising model of magnetization

The most interesting is always the critical behavior: p at and around $p_c(q)$.

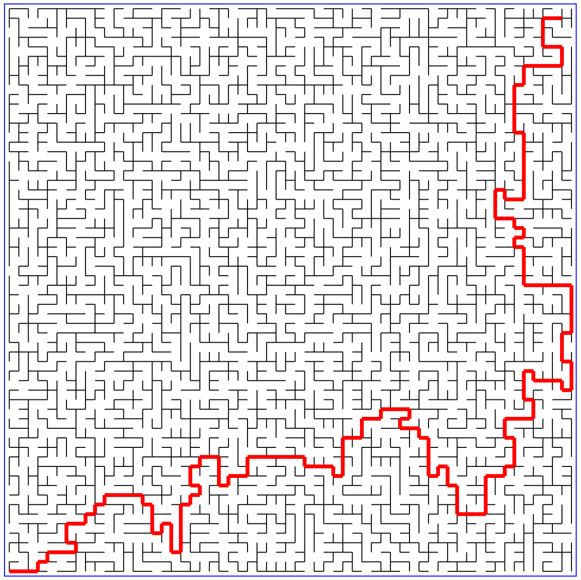
Critical models on planar lattices are getting well-understood, following Oded Schramm '00 and Stas Smirnov '01, '07, and others.

For $q = 1$, building on the critical behavior, the near-critical regime can also be understood (Kesten '87, Garban-P.-Schramm '10-12). Similarly for all q? We have found some unexpected things for $q = 2$.

The Uniform Spanning Tree

On a finite graph, take one uniformly from all spanning trees.

Paths inside are loop-erased random walk paths (David Wilson's algorithm '96).



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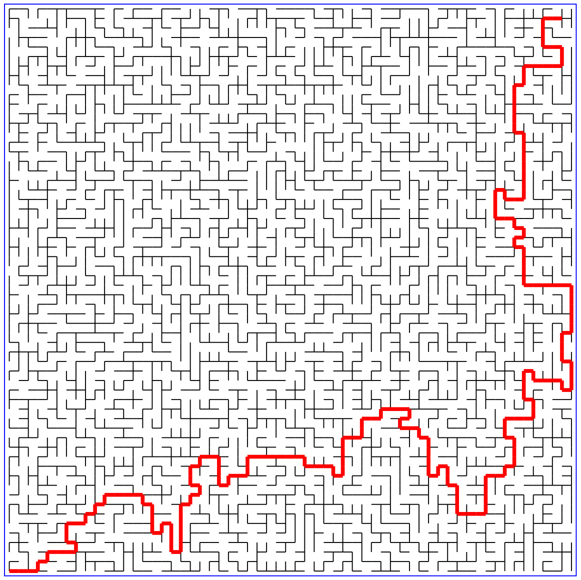

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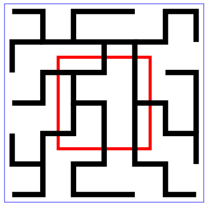

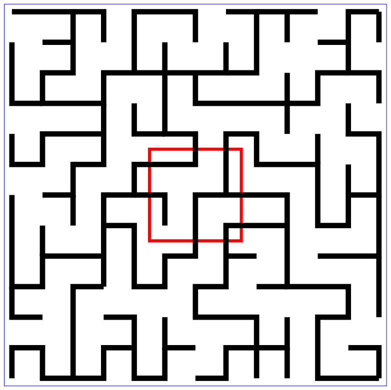

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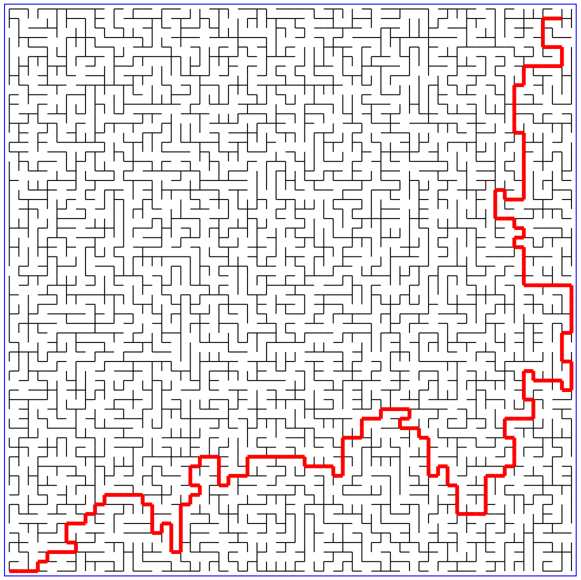

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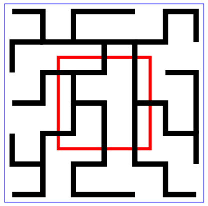
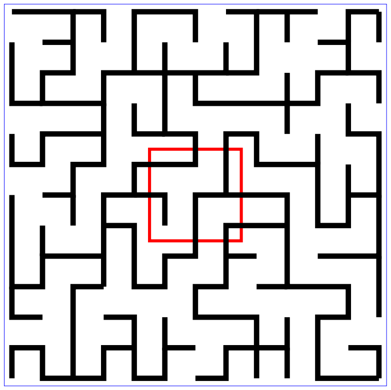




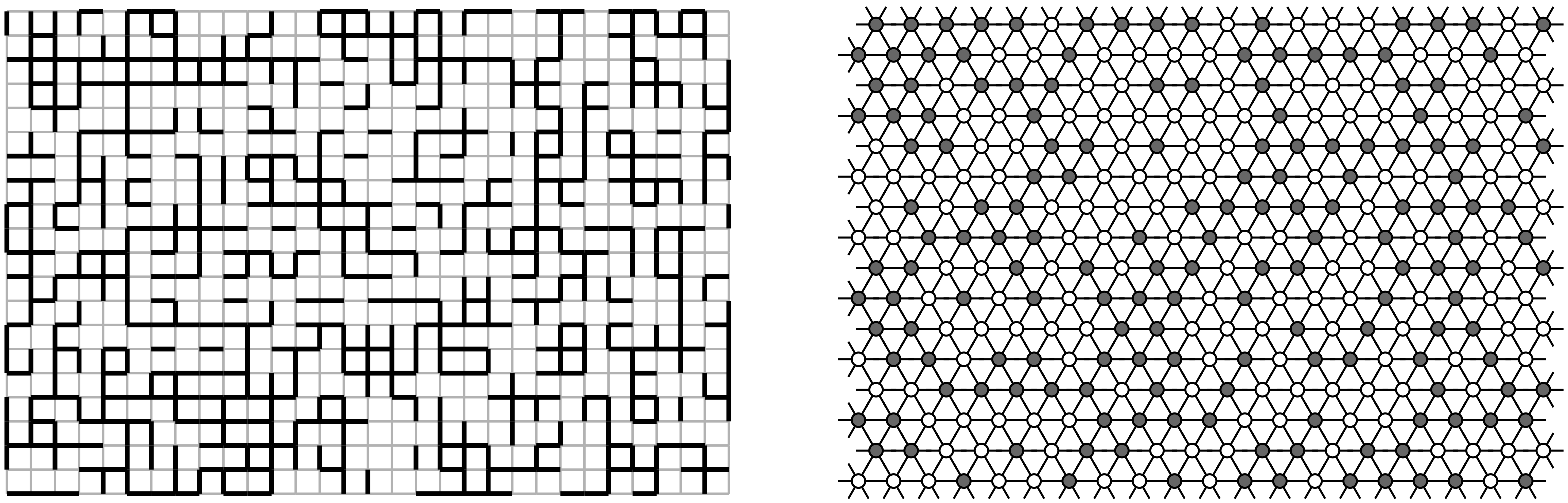
On an infinite G, take exhaustion $G_n \nearrow G$ by finite subgraphs, and hope that distribution converges in all finite windows. Using electric networks, there is some monotonicity, hence indeed there is a limit distribution, independent of the exhaustion. Might be a forest!

On planar Euclidean lattices, the limit is a single tree (Pemantle '91).

Bernoulli(p) bond and site percolation

Graph $G(V,E)$ and $p \in [0,1]$. Each site (or bond) is open with probability p, closed with $1-p$, independently. Consider open connected clusters.

$$p_c(G) := \inf\left\{p : \mathbf{P}_p[0 \longleftrightarrow \infty] > 0\right\} = \inf\left\{p : \mathbf{P}_p[\exists\ \infty \text{ cluster}] = 1\right\}$$



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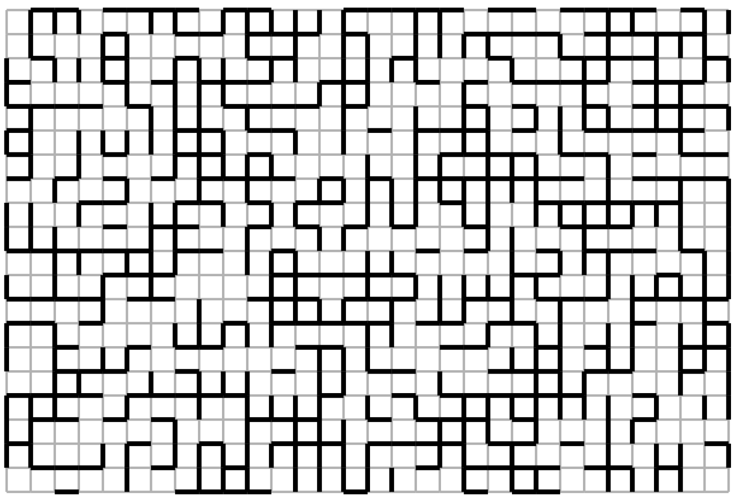
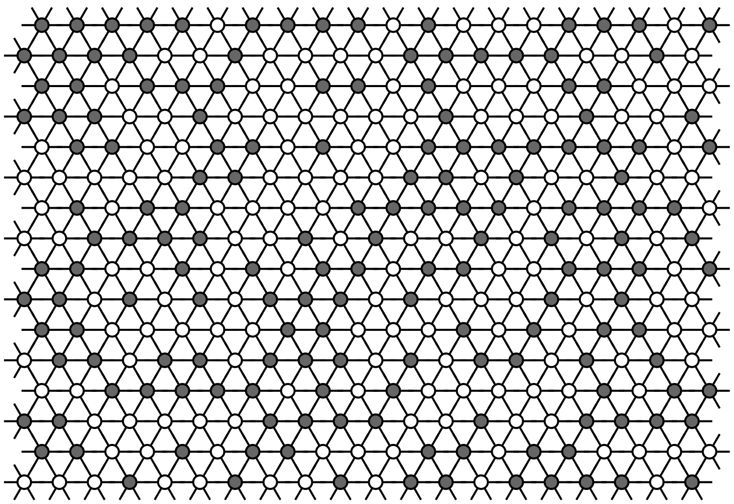
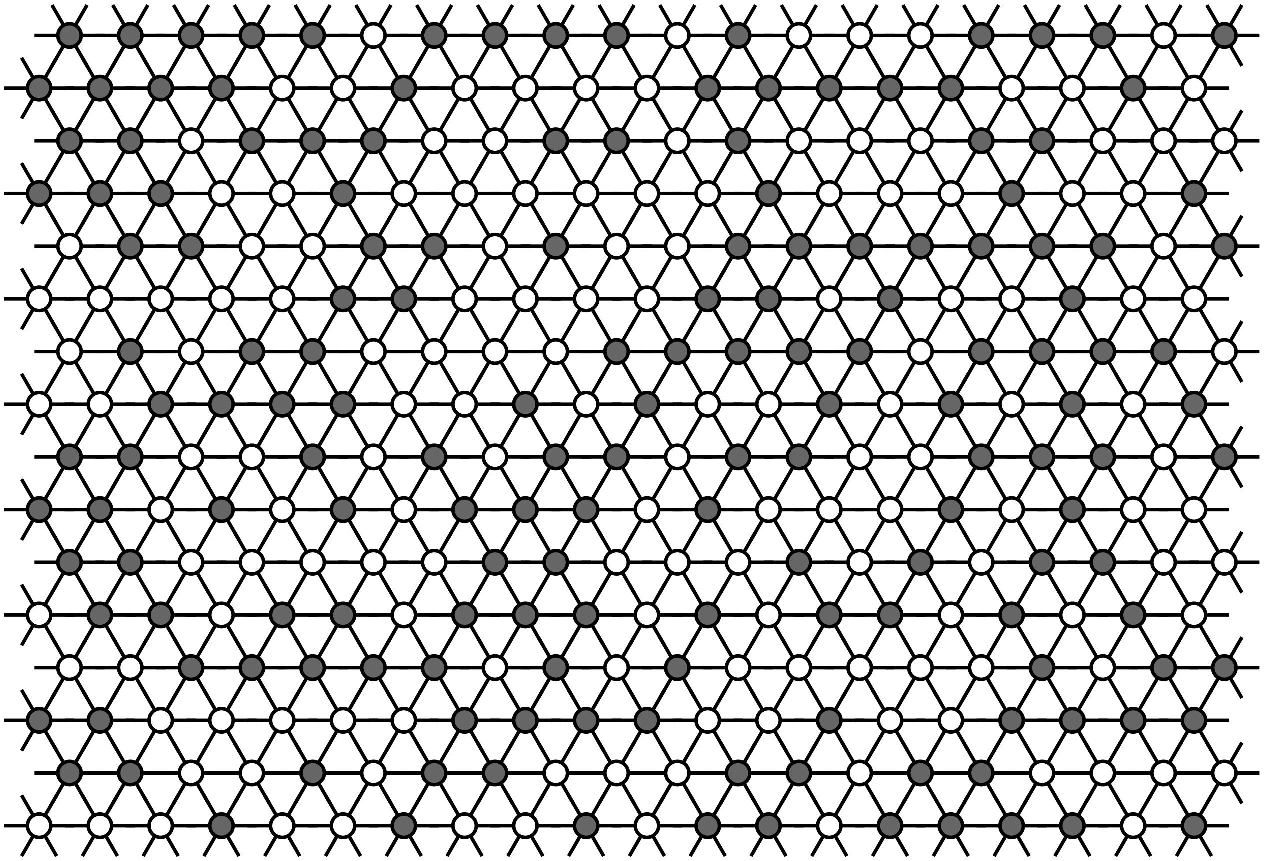




Theorem (Harris 1960 and Kesten 1980).
$p_c(\mathbb{Z}^2, \text{bond}) = p_c(\Delta, \text{site}) = 1/2$, and $\mathbf{P}_{p_c}[0 \longleftrightarrow \partial B_n(0)] = n^{-\Theta(1)}$.
For $p > 1/2$, there is almost surely one infinite cluster.

Site percolation on triangular grid Δ
= face percolation on hexagonal grid:



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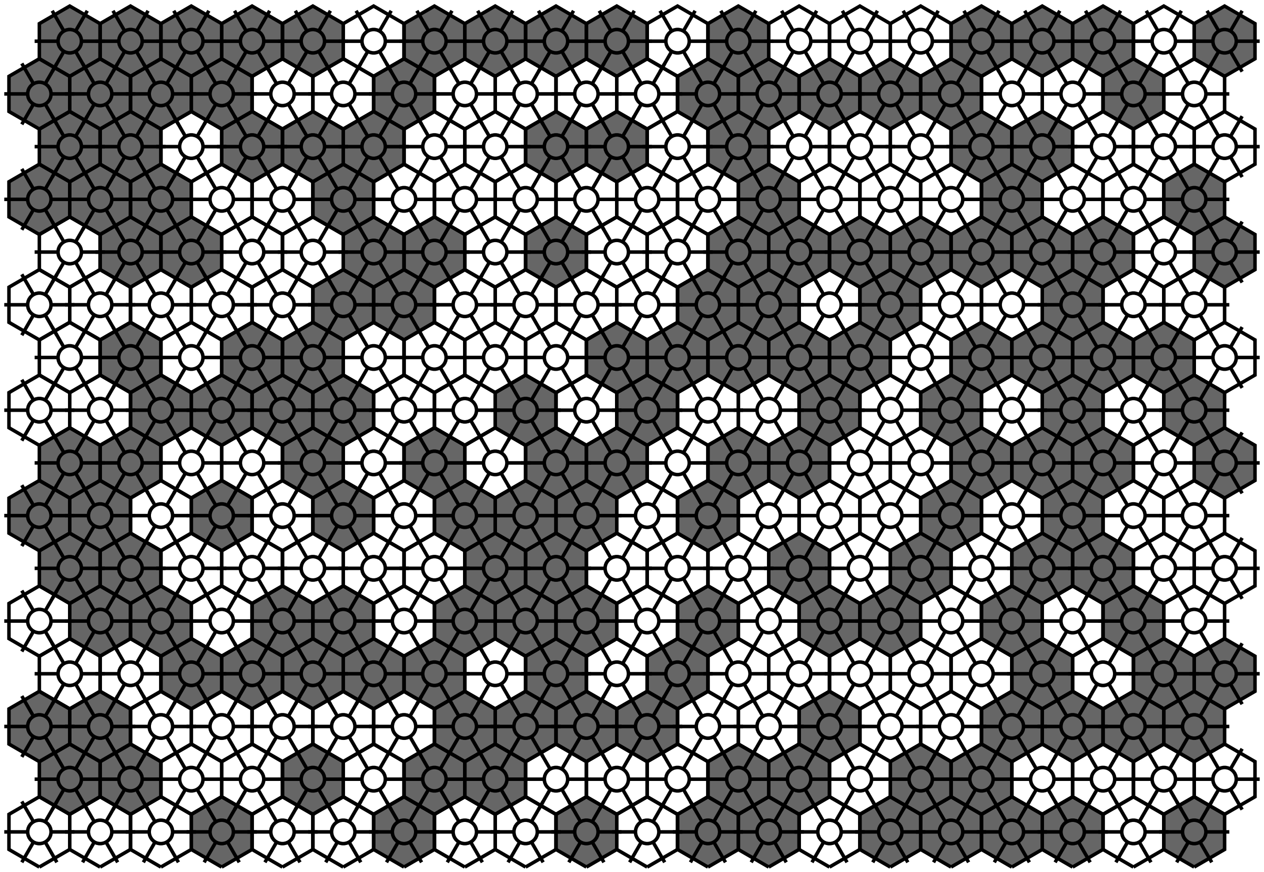

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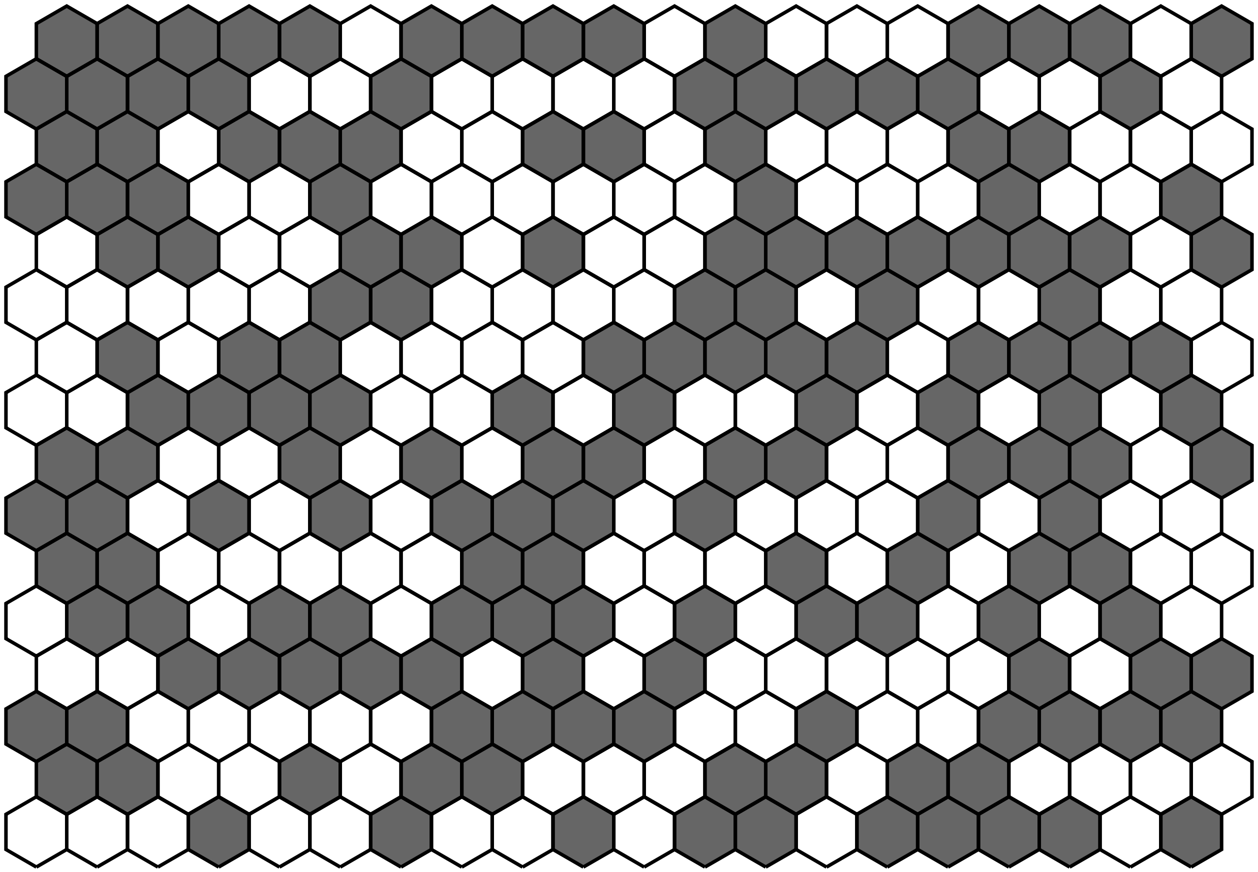
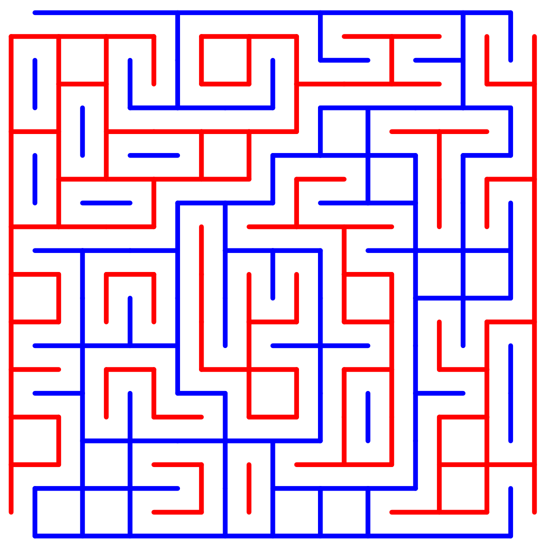


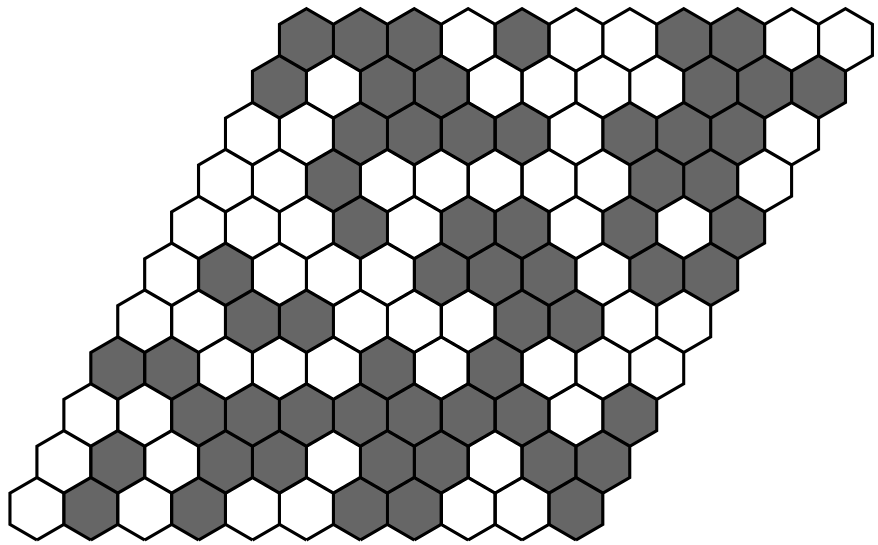
Why is $p_c = 1/2$? Duality!

$\mathbb{Z}^2$ bond percolation at $p = 1/2$: in an $n \times (n+1)$ rectangle, left-right crossing has probability exactly $1/2$, because:

$\mathbf{P}[\,\text{LeftRight}(n, n+1)\,] + \mathbf{P}[\,\text{TopBottom}(n+1, n)\,] = 1$, and they are equal.

For site percolation on Δ, same on an $n \times n$ rhombus.

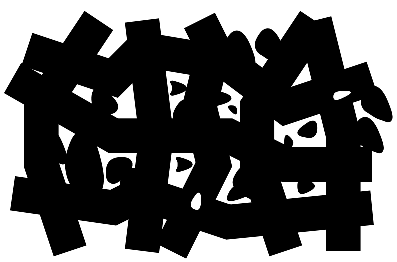




Crossing probabilities and criticality

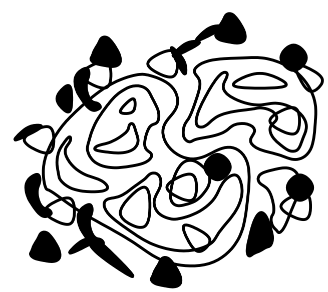
Theorem (Russo 1978 and Seymour-Welsh 1978). For $p = 1/2$ bond percolation on $\mathbb{Z}^2$ or site percolation on Δ, for $L, n > 0$,

$$0 < a_L < \mathbf{P}\big[\,\text{left-right crossing in } n \times Ln\,\big] < b_L < 1.$$



$p \approx 0.9$

$p \approx 0.55$

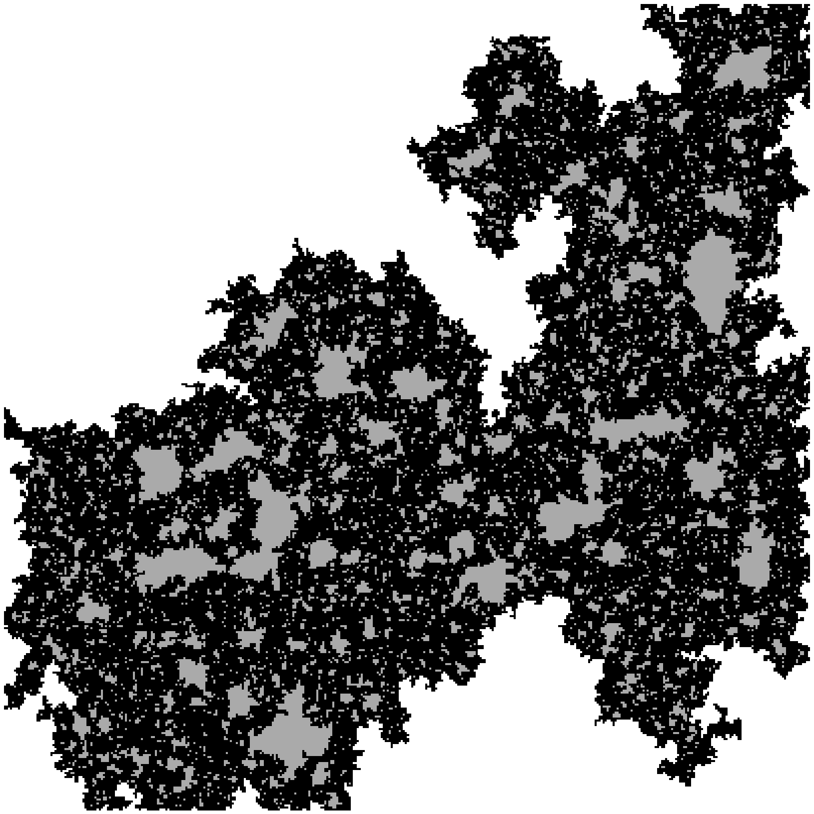


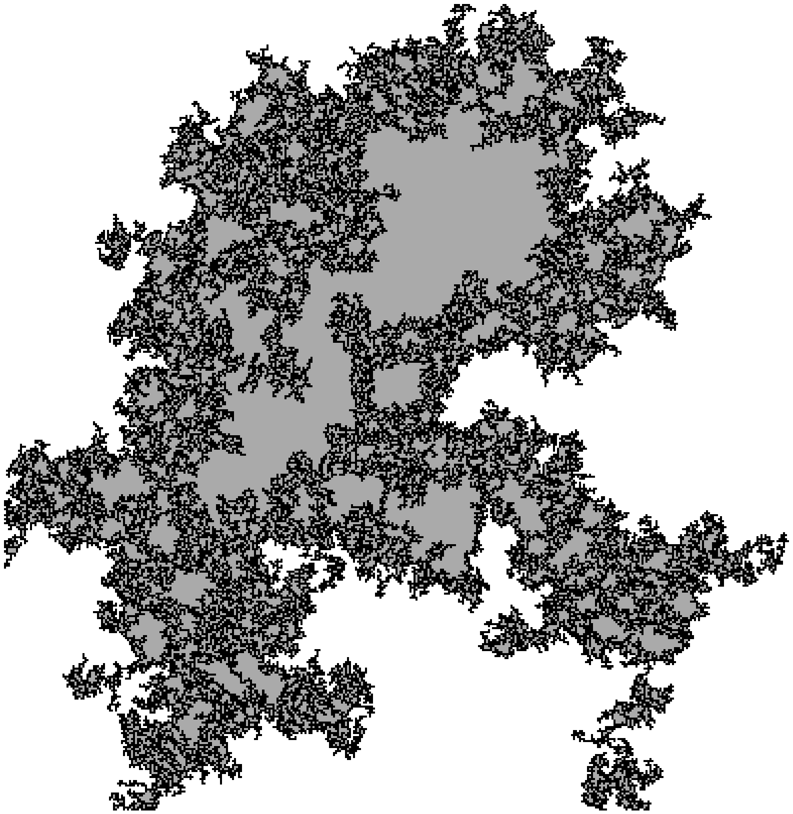
$p = 0.5$

$p \approx 0.45$

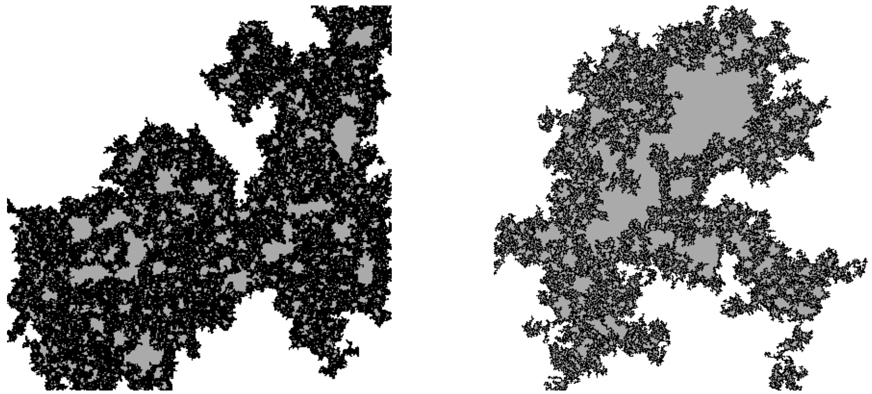
For $p > 1/2$, correlation length $L_\delta(p) := \min\big\{n : \mathbf{P}_p\big[\,\mathsf{LR}(n)\,\big] > 1 - \delta\big\}$. This is roughly the size of holes in the infinite cluster.

Critical percolation on different lattices





Universality Conjecture



Although p_c depends on the lattice, behavior at p_c should be the same!

E.g., “dimension” of large cluster boundaries should always be 7/4.

Or, $\mathbf{P}_{p_c}[0 \longleftrightarrow \partial B_n] = n^{-5/48+o(1)}$.

Or, off-critical exponent $\mathbf{P}_{p_c+\epsilon}[0 \longleftrightarrow \infty] = \epsilon^{5/36+o(1)}$.

Analogy: Simple random walk on any planar lattice has the same scaling limit: planar Brownian Motion.

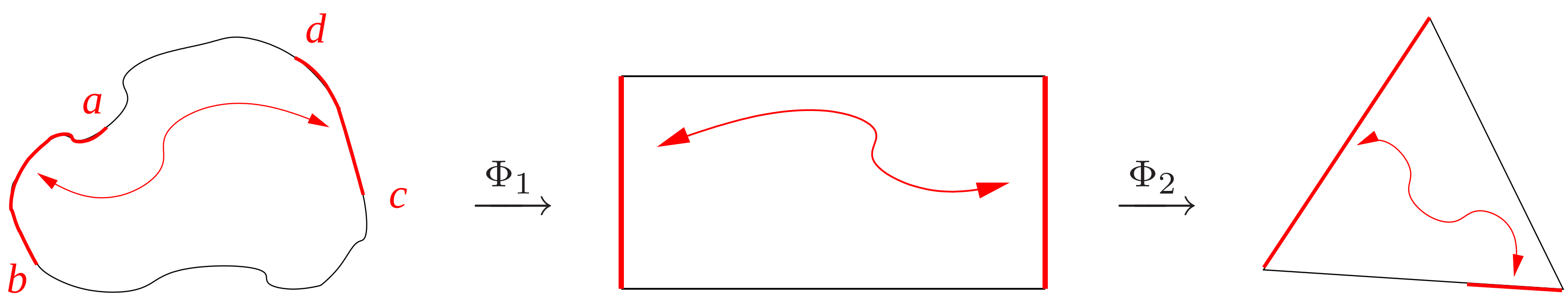
Conformal invariance

Theorem (Lévy '48). Planar Brownian motion is invariant under not only scalings and rotations, but also general conformal maps.

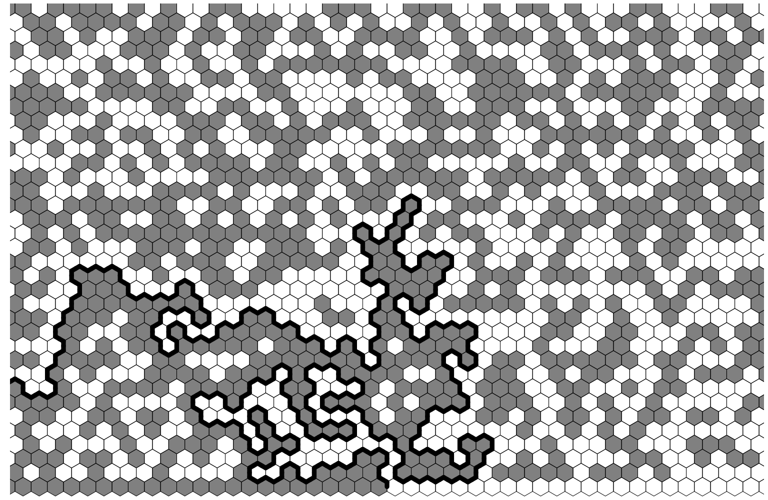
Theorem (Smirnov '01). For critical site percolation on $\Delta_{1/n}$, if $\mathcal{Q} \subset \mathbb{C}$ is a piecewise smooth quad, then

$$\lim_{n\to\infty} \mathbf{P}\Big[ab \longleftrightarrow cd \text{ inside } \mathcal{Q} \cap \Delta_{1/n} \Big]$$

exists, is strictly between 0 and 1, and conformally invariant.

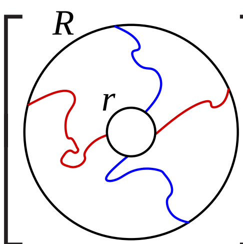


Schramm-Loewner Evolution



Given the conformal invariance, the exploration path converges to the Stochastic Loewner Evolution with $\kappa = 6$ (Schramm 2000).

Using the SLE_6 curve, critical exponents mentioned above can be computed (Lawler-Schramm-Werner, Smirnov-Werner ‘01, Kesten ‘87). E.g.:

$\alpha_4(r, R) := \mathbf{P}$ [

] $= (r/R)^{5/4+o(1)}$,

Lawler-Schramm-Werner ‘04: the scaling limit of Loop-Erased Random Walk on nice lattices is SLE_2. The scaling limit of the Peano curve around the Uniform Spanning Tree is SLE_8. Exponents can be computed again.

The Ising and q-Potts models

Spin configuration $\sigma : V \longrightarrow \{1, \ldots, q\}$. For $q = 2$, usually $\{-1, +1\}$.

Hamiltonian: $H(\sigma) := \sum_{(x,y) \in E(G)} \mathbb{1}_{\{\sigma(x) \neq \sigma(y)\}}$. Disagreements between neighbors increase energy.

We prefer lower energy configurations. For $\beta = 1/T \geqslant 0$ inverse temperature, Gibbs measure on configurations:

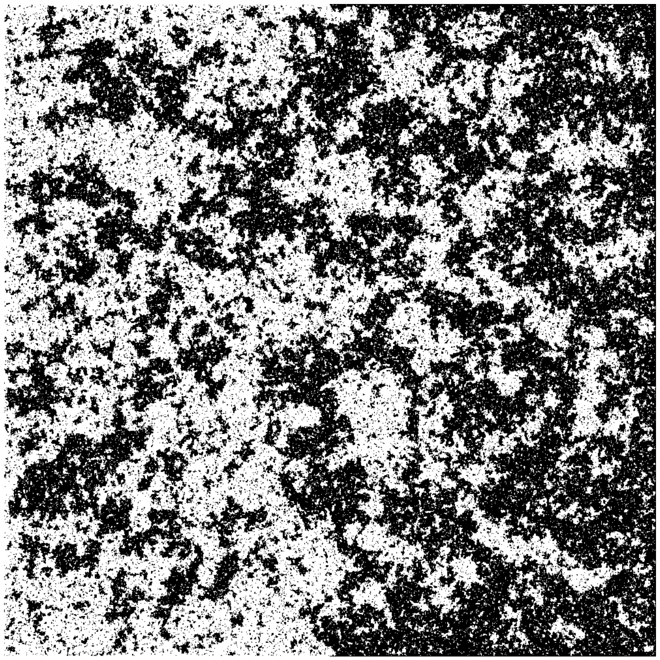
$$\mathbf{P}_\beta[\sigma] := \frac{\exp(-\beta H(\sigma))}{Z_\beta}, \quad \text{where} \quad Z_\beta := \sum_\sigma \exp(-\beta H(\sigma)) \, .$$

This Z_β is called the partition function.

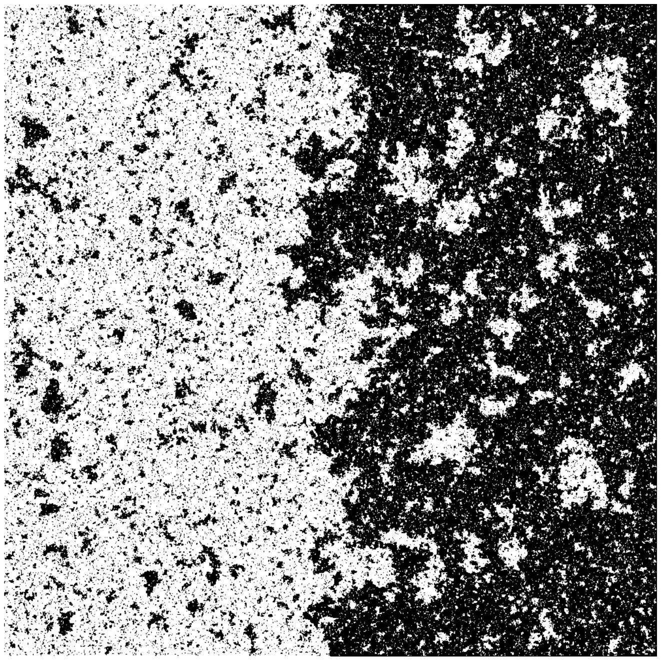
Sometimes external field, favoring one kind of spin.

But it's more interesting to vary β: decay of correlations?

The critical temperature of Ising



$\beta = 0.881374$



$\beta = 0.9$

Theorem (Onsager 1944, Aizenman-Barsky-Fernández 1987, Beffara-Duminil-Copin 2010). $\beta_c(\mathbb{Z}^2) = \ln(1+\sqrt{2}) \approx 0.881374.$

Onsager also showed that $\mathbf{E}_{\beta_c}\left[\sigma(0) \,\middle|\, \sigma|_{\partial B_n(0)} = +1\right] = n^{-1/8+o(1)}$.

The random cluster model FK(p,q)

Fortuin-Kasteleyn (1969): for $\omega \in \{0,1\}^{E(G)}$,

$$\mathbf{P}_{\mathrm{FK}(p,q)}[\omega] = \frac{p^{|\omega|}\,(1-p)^{|E(G)\setminus\omega|}\,q^{|\mathrm{clusters}(\omega)|}}{Z_{\mathrm{FK}(p,q)}}.$$

$q=1$: Bernoulli(p) bond percolation. $q \to 0$, then $p \to 0$: UST

For $q \in \{2,3,\dots\}$, Edwards-Sokal coupling: color each cluster independently with one of q colors, then forget ω: get q-Potts, with $\beta = \beta(p) = -\ln(1-p)$. Partition functions are equal: $Z_{\mathrm{FK}(p,q)} = Z_{\beta(p),q}$.

Therefore, Correl$_{\beta,q}[\sigma(x),\sigma(y)] = \mathbf{P}_{\mathrm{FK}(p,q)}[x \longleftrightarrow y]$!

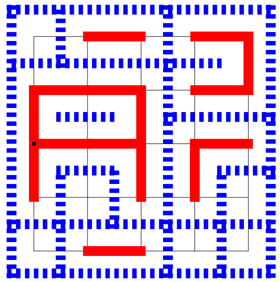
$Z_{\mathrm{FK}(p,q)}$ is a version of the Tutte polynomial.

If $q \geqslant 1$, then increasing events are positively correlated: FKG-inequality.

For $q < 1$, there should be negative correlations, proved only for UST, which is a determinantal process.

FK(p, q) on $\mathbb{Z}^2$

Infinite volume limit measure is again given by an arbitrary exhaustion $G_n \nearrow \mathbb{Z}^2$ by finite subgraphs.



Planar dual of an FK(p, q) configuration on G_n turns out to be an FK(p^*, q) configuration again (using Euler's formula), on the dual graph G_n^*. These converge to FK(p^*, q) on $\mathbb{Z}^2$.

We have $p^* = p$ at the self-dual point $p_{sd}(q) = \sqrt{q}/(1 + \sqrt{q})$. E.g., in percolation, $p_{sd}(1) = 1/2$.

Critical point $p_c(q)$: threshold for existence of infinite cluster.

Theorem (Beffara & Duminil-Copin 2010). $p_c(q) = p_{sd}(q)$ for $q \geqslant 1$.

At $p_{sd}(q = 2)$, no infinite cluster. (Maybe J. Lebowitz '72, simple proof by W. Werner '09.) Conjectured for all $q \leqslant 4$.

Edwards-Sokal coupling gives Onsager's $\beta_c = -\ln(1 - p_c(2))$.

Critical spin-Ising and FK-Ising on $\mathbb{Z}^2$

Theorem (Smirnov '06, '10, Chelkak-Smirnov '10, Kemppainen-Smirnov '12). On a large class of graphs (including $\mathbb{Z}^2$), quad-crossing probabilities are conformally invariant. Interfaces in the spin-Ising model converge to SLE_3; in FK-Ising they converge to $\text{SLE}_{16/3}$.

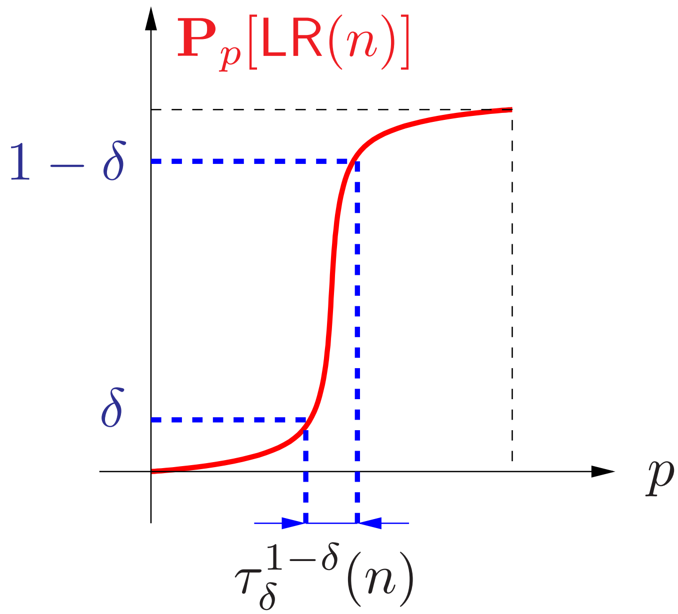
FK-Ising RSW estimates for rectangles by Duminil-Copin-Hongler-Nolin '10.

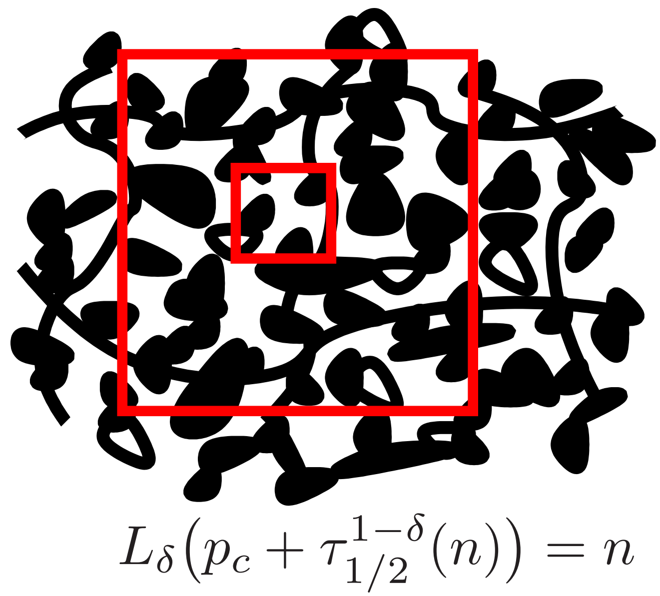
Critical exponents by Duminil-Copin & Garban '12.

Conjecture. For any $0 \leqslant q \leqslant 4$, the critical FK$(p_c(q), q)$ model on any nice planar lattice is conformally invariant, interfaces converge to some SLE_κ.

The near-critical regime

Recall the correlation length $L_\delta(p) := \min\{n : \mathbf{P}_p\big[\,\mathsf{LR}(n)\,\big] > 1-\delta\}$.





Kesten '87: Near-critical window for percolation is given by number of pivotal points at criticality: $\tau(n) = n^{-3/4+o(1)} \approx 1/\mathbf{E}_{p_c}|\mathsf{Piv}_n|$.

DC & G & P '11: In Ising-FK, this is NOT the case. Still, we can find $\tau(n) = n^{-1+o(1)}$ using conformal invariance techniques.

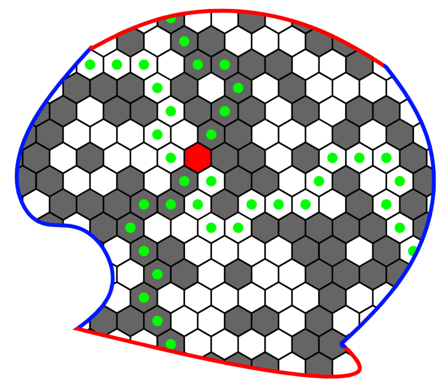
The near-critical ensemble in percolation

Standard coupling: to each site (or bond) $x \in G$, assign $V(x)$ i.i.d. Unif$[0,1]$, and let x be **open at level** p if $V(x) \leqslant p$.

In $\mathcal{Q} \cap \Delta_{1/n}$, when raising p from p_c, when does it become well-connected?

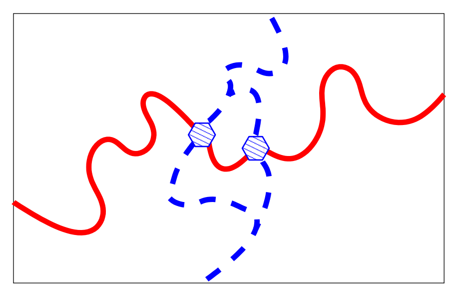
A site is pivotal in ω if flipping it changes the existence of a left-right crossing. Equivalent to having alternating 4 arms. For nice quads, there are not many pivotals close to $\partial\mathcal{Q}$, hence

$$\mathbf{E}_{p_c}|\mathrm{Piv}_n| \asymp n^2\,\alpha_4(n) = n^{3/4+o(1)} \text{ on } \Delta_{1/n}.$$



If $p - p_c \gg n^{-3/4+o(1)}$, we have opened many critical pivotals, hence already supercritical. But maybe many new pivotals appeared on the way, hence there is a pivotal switch earlier?

New pivotals do appear. But will they be switched as p is raised?



Stability by Kesten (1987): multi-arm probabilities stay comparable inside this regime, hence changes are not faster, and this $n^{-3/4+o(1)}$ is indeed the critical window.

More precise finite-size scaling results by Borgs-Chayes-Kesten-Spencer (2001). The system looks critical below the scale $L(p)$; e.g., the sizes of largest clusters are not concentrated.

Nolin-Werner (2008): Subsequential limits of the near-critical interface exist, and are singular w.r.t. the critical interface SLE_6.

Garban-P.-Schramm (2010-12): The scaling limit of near-critical ensemble with $p = p_c + \lambda n^{-3/4+o(1)}$ exists (not only subsequential limits).
It is Markovian in λ, and is conformally covariant: if domain changes by $\phi(z)$, then the change in λ scales locally by $|\phi'(z)|^{3/4}$.

Near-critical interface (the "massive SLE_6") should have a driving process involving a self-interacting drift term: $dW_t = \sqrt{6}\, dB_t + c\,\lambda\, |d\gamma_t|^{3/4}\, dt^{1/2}$.

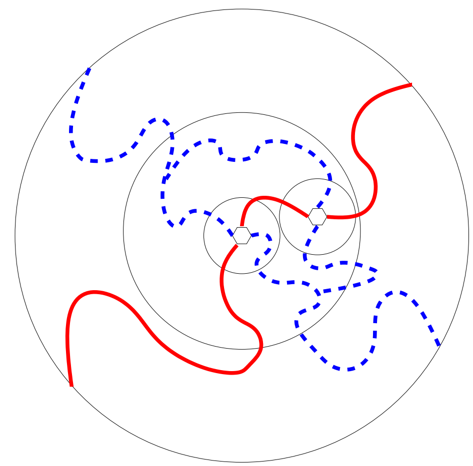
Kesten's stability and scaling relation

More elementary and more generalizable proof by Garban-P.-Schramm: careful double recursion for $b_r^R :=$

$\mathbf{P}_{p_c}\big[\mathcal{A}_4(r,R)$ holds for some $n^{-3/4+o(1)}$-perturbation of the configuration$\big]$

to show that $b_1^n \leqslant C\,\alpha_4(n)$:

The origin has the 4-arm event only after the perturbation $\Longrightarrow$ there is some site that is touched by the perturbation and is pivotal at that moment for the origin's 4-arm event.



Once 4-arm probability is stable, all other arm-probabilities follow.

$$\mathbf{P}_p\big[\,0 \leftrightarrow \infty\,\big] \asymp \mathbf{P}_p\big[\,0 \leftrightarrow L(p)\,\big] \asymp \mathbf{P}_{1/2}\big[\,0 \leftrightarrow L(p)\,\big]$$
$$\asymp \big((p-1/2)^{-4/3+o(1)}\big)^{-5/48+o(1)} = (p-1/2)^{5/36+o(1)}.$$

The near-critical ensemble in FK(p, q)

Want a monotone coupling as p varies, i.e., random $Z \in [0,1]^{E(G)}$ labeling such that $Z_{\leqslant p} \subset E(G)$ is FK(p, q). Desirably Markov in p.

Harder than in percolation. Grimmett '95 showed its existence: defined a Markov chain Z_t on labelings with the right stationary measure. (Works only for $q \geqslant 1$.)

Another difference from percolation: from specific heat computation in the Ising model, density of edges in $Z_{\leqslant p_c+\epsilon} \setminus Z_{\leqslant p_c}$ is not $\asymp \epsilon$, but $\epsilon \log(1/\epsilon)$ for $q = 2$, and polynomial blowup for $q > 2$.

Onsager vs pivotals

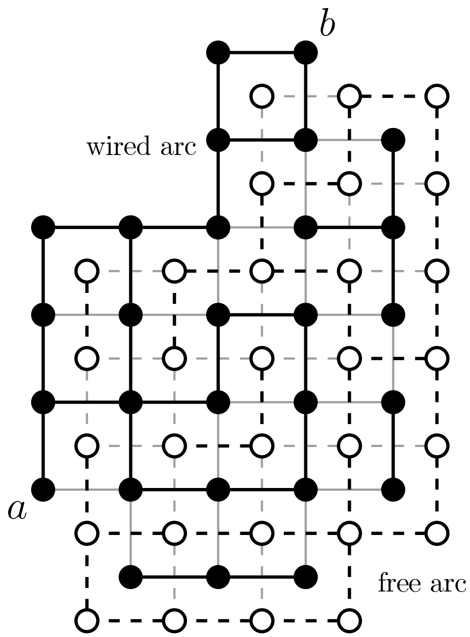
From Onsager '44 and other Ising results: correlation length $\epsilon^{-1+o(1)}$, with a related but different definition, using correlation decay. I.e., $\tau(n) = n^{-1+o(1)}$ should be the window. But DC&G computed $\mathbf{E}|\mathsf{Piv}_n| = n^{13/24+o(1)}$, too few! And specific heat doesn't help enough.

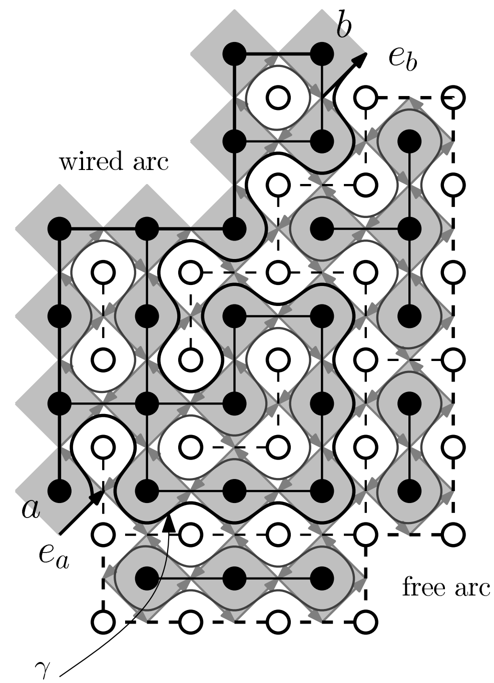
Hence, correlation length is not given by amount of pivotals at criticality. Stability in near-critical window fails, the changes are faster. How come?

Conclusion: *Any* monotone coupling must be very strange: when raising p in the monotone coupling, open bonds do not arrive in a uniform, Poissonian way, but with self-organization, to create more pivotals and build long connections. Would contradict Markov property in p, unless there are clouds of open bonds appearing together.

We don't understand geometry of clouds, but at least can see directly in Grimmett's coupling that clouds do happen. Intuitively: good to open many edges together, without lowering number of clusters.

Computing the correlation length





Smirnov's fermionic observable $F = F_p$ for any medial edge $e \in E_\diamond$:

$$F(e) := \mathbf{E}_{p,2}^{G,a,b}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}W_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}\right),$$

where γ is the exploration interface from a to b, and W_γ is the winding.

Relation to connectivity: if $u \in G$ is a site next to the free arc, and e is the appropriate medial edge next to it, then $|F(e)| = \mathbf{P}^{G,a,b}_{p,2}(u \leftrightarrow \text{wired arc})$.

Massive harmonicity (Beffara-Duminil-Copin): if X has four neighbors in $G \setminus \partial G$, then $\Delta_p F(e_X) = 0$, where the operator Δ_p is

$$\Delta_p g(e_X) := \frac{\cos[2\alpha]}{4}\left(\sum_{Y\sim X} g(e_Y)\right) - g(e_X),$$

with some $\alpha = \alpha(p)$, equalling 0 iff $p = p_c$.

Complicated boundary conditions. But, at p_c, $H(e^+) - H(e^-) := |F(e)|^2$, this H approximately solves a discrete Dirichlet boundary problem, hence $\mathbf{P}^{G,a,b}_{p_c,2}(u \leftrightarrow \text{wired arc}) \simeq (\text{harmonic measure of wired arc seen from } u)^{1/2}$, and can compute that crossing probabilities are between 0 and 1.

At $p \neq p_c$, need harmonic measure w.r.t. massive random walk, killing particle at each step with probability depending on $\cos(2\alpha)$, roughly $|p - p_c|^2$. $|p - p_c| < \frac{c}{n}$: during the roughly n^2 steps to boundary, particles dies with probability bounded away from 1, so everything is roughly the same.