The scaling limits of dynamical and near-critical planar percolation and the Minimal Spanning Tree

Gábor Pete
http://www.math.toronto.edu/∼gabor

Joint work with Christophe Garban (Université Paris-Sud and ENS)
and Oded Schramm (Microsoft Research)

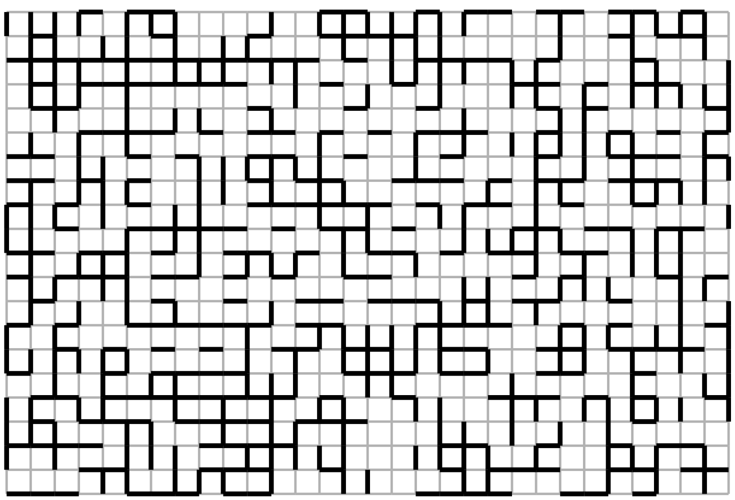
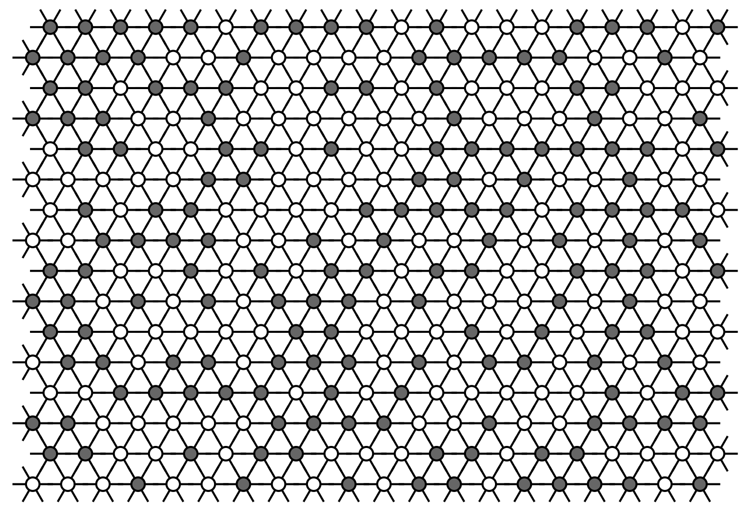
Outline of talk

- Critical percolation in the plane
- Dynamical Percolation, Near-Critical Ensemble, Minimal Spanning Tree
- Existence and conformal covariance of DPSL and NCESL
- The two main ingredients of the proof:
  1. Scaling limits of counting measures on special points of percolation
  2. Stability inside critical window
- This proves existence and scaling and rotational invariance of MSTSL, and some topological results. *Suggests* conformal non-invariance.

Bernoulli(p) site and bond percolation

Graph $G(V,E)$ and $p \in [0,1]$. Each site (or bond) is open with probability p, closed with $1-p$, independently. Consider open connected clusters.

$$p_c(G) := \inf \left\{ p : \mathbf{P}_p[0 \longleftrightarrow \infty] > 0 \right\} = \inf \left\{ p : \mathbf{P}_p[\exists\ \infty \text{ cluster}] = 1 \right\}$$

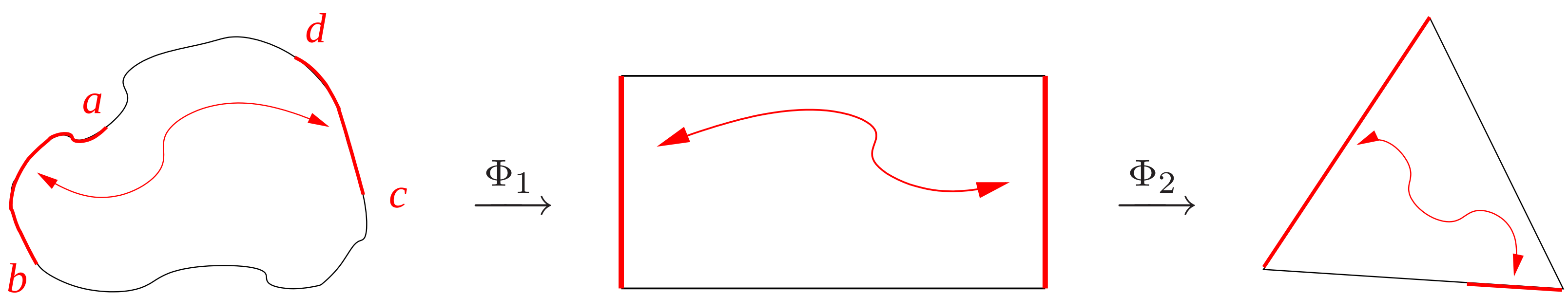
Theorem (Harris 1960 and Kesten 1980).
$p_c(\mathbb{Z}^2, \text{bond}) = p_c(\Delta, \text{site}) = 1/2$, and $\mathbf{P}_{p_c}[0 \longleftrightarrow \infty] = 0$.
For $p > 1/2$, there is a.s. one infinite cluster.

Conformal invariance on Δ

Theorem (Smirnov 2001). For $p = 1/2$ site percolation on Δ_η, and $\mathcal{Q} \subset \mathbb{C}$ a piecewise smooth quad (simply connected domain with four boundary points $\{a, b, c, d\}$),

$$\lim_{\eta \to 0} \mathbf{P}\Big[ab \longleftrightarrow cd \text{ inside } \mathcal{Q}, \text{ in percolation on } \Delta_\eta \Big]$$

exists, is strictly between 0 and 1, and conformally invariant.

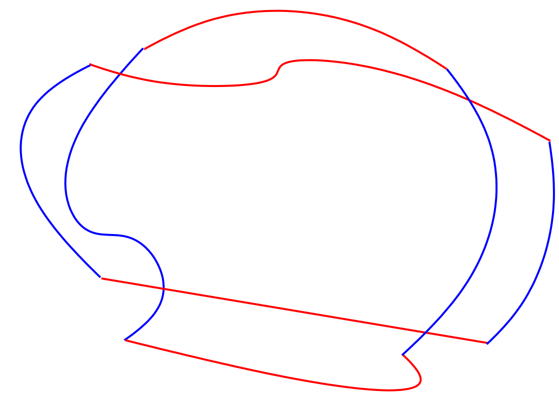


This result calls for finding a scaling limit.

What kind of limit?

A good definition: A configuration in the scaling limit is the collection of all pw-smooth quads that are crossed.

A better one (Schramm):



The set of crossed quads is closed and monotone.

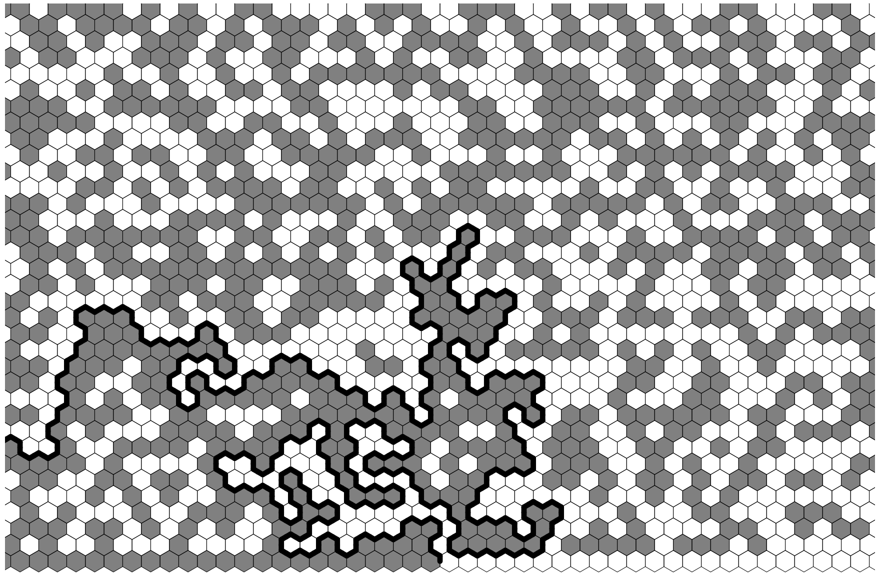
The collection $\mathcal{S}$ of all closed monotone sets of quads is a compact Hausdorff space in an appropriate topology. “Dedekind cuts” in a poset.

For each mesh η, percolation is a probability measure on $\mathcal{S}$.

Take convergence in law (weak convergence).

Other definitions by Aizenman (1995) and Camia-Newman (2006).

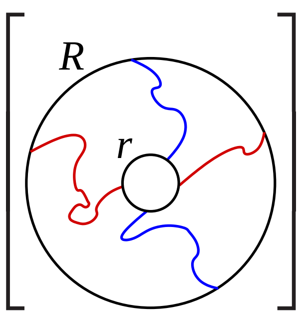
SLE_6 exponents



Given the conformal invariance, the exploration path converges to the Stochastic Loewner Evolution with $\kappa = 6$ (Schramm 2000).

Using the SLE_6 curve, several critical exponents can be computed (Lawler-Schramm-Werner, Smirnov-Werner 2001, plus Kesten 1987), e.g.:

$$\alpha_4(r, R) := \mathbf{P}$$



$$= (r/R)^{5/4+o(1)},$$

and $\alpha_1(r, R) = (r/R)^{5/48+o(1)}$, and $\mathbf{P}_{p_c+\epsilon}[0 \longleftrightarrow \infty] = \epsilon^{5/36+o(1)}$.

Dynamical percolation

Triangular lattice Δ_η with mesh η, each site is resampled according to an independent exponential clock with some rate $r(\eta)$.
What is the right time scaling to get a meaningful scaling limit?

For RW $\to$ BM, as you shrink space $(1/n)$, also shrink time $(1/n^2)$.

For dynamical percolation, need to expand (slow) time, because of noise sensitivity (BKS 1998, SchSt 2005, GPS 2008).

To change a quad crossing, must hit a pivotal point.

Take a macroscopic quad $\mathcal{Q}$. At least for pw-smooth quads, there are not many pivotals close to $\partial\mathcal{Q}$, hence $\mathbf{E}|\mathrm{Piv}_\eta| \asymp \eta^{-2}\,\alpha_4(\eta,1) \stackrel{\Delta}{=} \eta^{-3/4+o(1)}$.
(Note that $\eta^{-1}\alpha_3^+(\eta,1) = \eta$ is much smaller.)

Hence, with rate $r(\eta) := \eta^2\alpha_4(\eta,1)^{-1} \stackrel{\Delta}{=} \eta^{3/4+o(1)}$, the expected number of pivotal switches in unit time is $\Theta_{\mathcal{Q}}(1)$.

The near-critical ensemble

Whenever a clock rings, switch to **black**. So, at time t, each site is black with probability $\sim 1/2 + t\,r(\eta)$. May also take $t < 0$, bias towards $white$.

Or, to each site $x \in \Delta_\eta$, assign $V(x)$ i.i.d. Unif$[0,1]$, and let x be **black at level** p if $V(x) \leqslant p$. Big changes during $p = 1/2 + \lambda\,r(\eta)$, $\lambda \in (-\infty, \infty)$.

This is the near-critical ensemble, or percolation in the critical window.

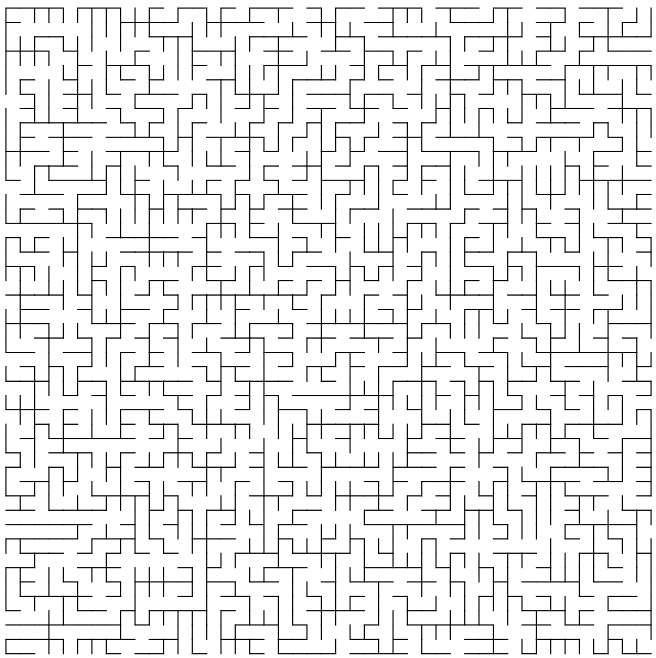
Kesten (1987): Multi-arm probabilities stay comparable inside window.

Borgs-Chayes-Kesten-Spencer (2001): Finite size versions of previous. Above window, already supercritical.

Nolin-Werner (2008): Subsequential limits of the near-critical interface exist, and are singular w.r.t. the critical interface SLE_6.

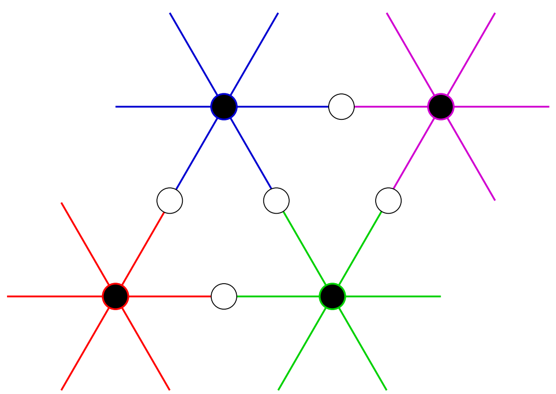
Minimal spanning tree

For each edge of a finite graph, say $e \in E(\mathbb{Z}_n^2)$, let $U(e)$ be i.i.d. Unif$[0,1]$. The Minimal Spanning Tree is the tree T for which $\sum_{e \in T} U(e)$ is minimal.



Same as deleting from each cycle the edge with highest U. Or the collection of lowest level paths between all pairs of vertices.

Hence T depends only on the ordering.

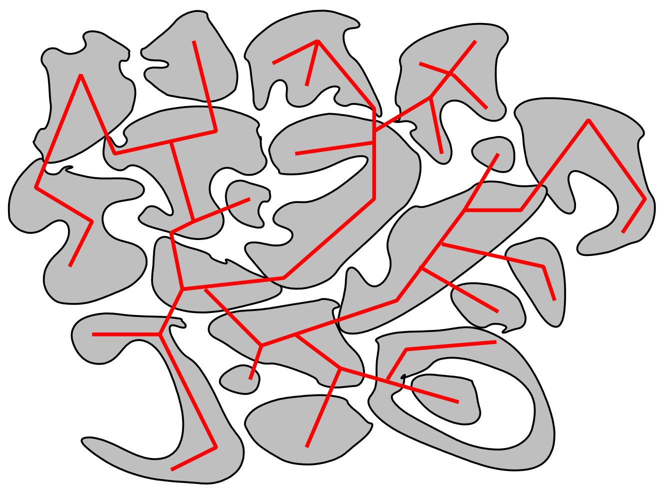


Version adapted to site percolation on Δ: replace each edge by two in series, and for each such edge e, let $U(e) := V(e^*)$, the old vertex endpoint.

MST coupled with NCE

Connection to NCE: macroscopic structure is determined by the cluster tree $T_{\geqslant\lambda}$ between the level λ clusters, $p = 1/2 + \lambda\, r(\eta)$, as $\lambda \to -\infty$.

And the collection of cluster trees $T_{\geqslant\lambda}$ is determined by the collection of λ-clusters over all $\lambda \in (-\infty, \infty)$.



Also, T is the union of the invasion trees of Invasion Percolation. Alexander 1995, Aizenman-Burchard-Newman-Wilson 1999, Häggström-Peres-Schonmann 1999, Lyons-Peres-Schramm 2006.

The results

Theorem (GPS 2009). On Δ_η, with rate $r(\eta)$ clocks,
$*$ $\exists$ DPSL
$*$ $\exists$ NCESL
$*$ both are Markov
$*$ both are conformally covariant: if the domain is changed by $\phi(z)$, then time is scaled locally by $|\phi'(z)|^{3/4}$
$*$ DPSL is ergodic (by GPS 2008)
$*$ SL of MST and IP also exist, rotationally and scaling invariant.

In either case, the process is a random map $\gamma_\eta : \mathbb{R} \mapsto \mathcal{S}$. Not continuous. For the scaling limit, we take locally uniform convergence.

For MST and IP we do **not** expect conformal invariance. For example, simulations by D. Wilson (2002).
DPSL question was asked by Schramm, ICM lecture (2006).
Results were conjectured by Camia-Newman-Fontes (2006).
NCESL results refine Bo-Ch-Ke-Sp (2001) and Nolin-Werner (2008).

The first main ingredient

Pivotal switches govern the dynamics. If we know the number of pivotal sites for each quad at any given moment, then know the rates at which pivotal switches occur. However, no pivotal sites in scaling limit any more!

Quantity of microscopic pivotals can be seen from macroscopic information:

Theorem 1 (Measurability). For any pw-smooth quad $\mathcal{Q}$, let $\mu_\eta^{\mathcal{Q}}$ be the number of $\mathcal{Q}$-pivotal sites normalized by $\eta^{-2}\alpha_4(\eta,1)$. Then there is a limit of the joint law $(\mu_\eta^{\mathcal{Q}}, \omega_\eta) \to (\mu^{\mathcal{Q}}, \omega)$, where $\mu^{\mathcal{Q}}$ is a function of ω. Similar statement for μ_η^ρ, the normalized number of ρ-important sites.

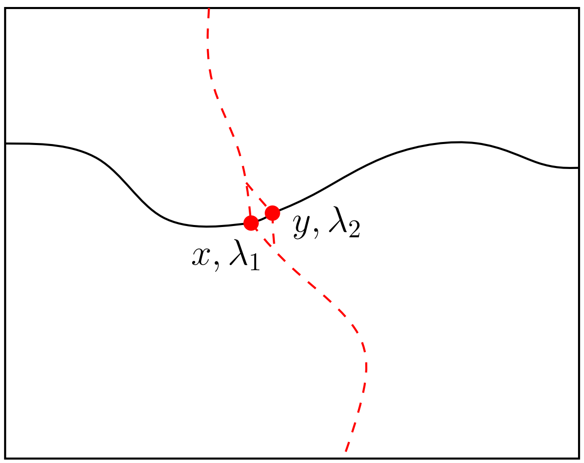
A similar proof gives e.g. natural time-parametrizations for SLE_6 and $SLE_{8/3}$: questions studied for general κ by Lawler, Sheffield, Alberts.

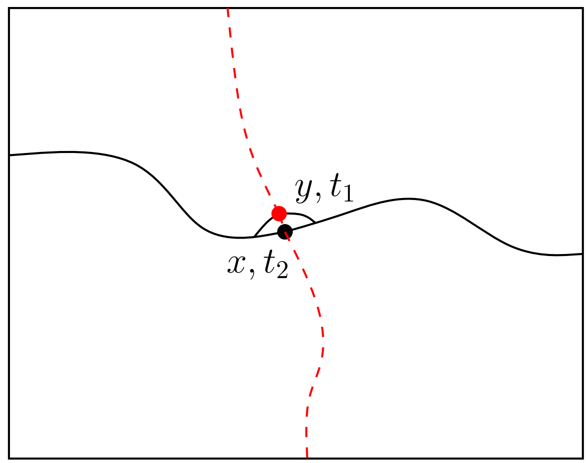
So, can hope that scaling limit of dynamics is given by $\omega_{t=0}$ plus a "filtered" Poisson point process $(\mathscr{P}^\rho)_{\rho>0}$ of flips from μ^ρ(domain) $\times$ Lebesgue(time). This was suggested by Camia-Fontes-Newman.

The second main ingredient

But if we follow all these changes, will we know later what quads are crossed? During the dynamics, no new macroscopic information appears:

Theorem 2 (Stability). Quad $\mathcal{Q}$. Set of sites switched in $[0,t]$ is W_t. The probability that a configuration ω can be changed on W_t into ω', ω'' such that they agree on any site that is at least ϵ-important in ω, but $\mathcal{Q}$ is crossed by ω' while not crossed by ω'', is small if ϵ is small.





Such scenarios of "cascade of importance" do not happen.

Strengthening and simplifying Kesten (1987), saying that in the near-critical window the 4-arm probabilities remain comparable.

Measure on pivotals is measurable in scaling limit

$X = X_\eta^\rho$ is the number of ρ-important sites in D, with mesh η.

Intermediate scale: $Y = Y_\eta^{\rho,\epsilon}$ is number of ρ-important ϵ-boxes in a lattice.

$\beta = \beta_\eta^{\rho,\epsilon} := \mathbf{E}\big[\,\rho\text{-important sites in } \epsilon\text{-box } B \;\big|\; B \text{ is } \rho\text{-important}\,\big]$.

Hence $\mathbf{E}[\,X\,] \sim \beta\,\mathbf{E}[\,Y\,]$.

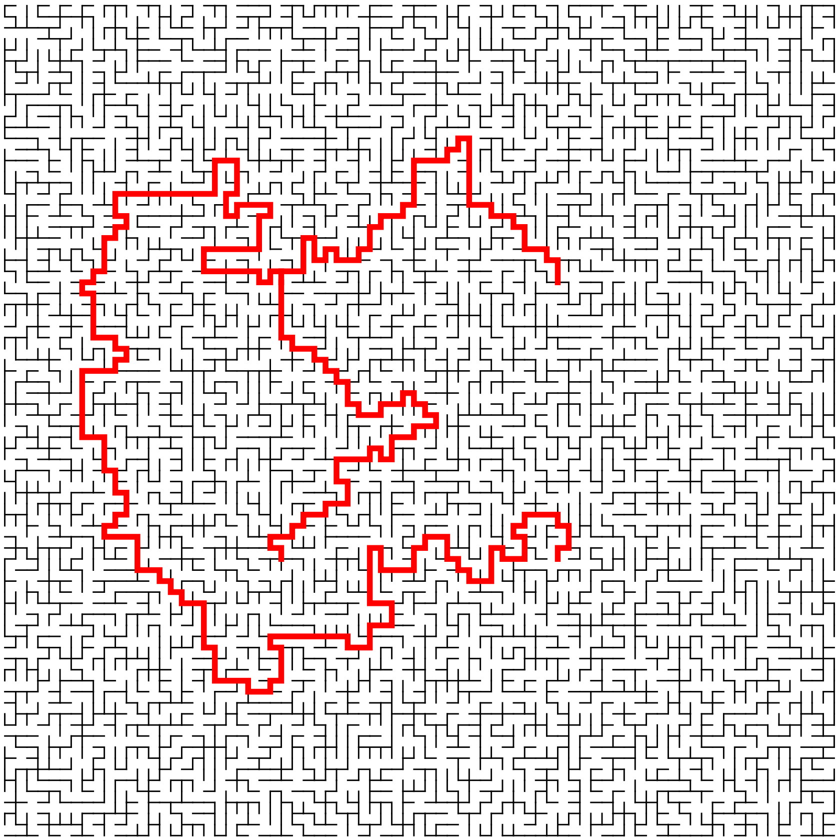
Want that $\lim_{\eta\to 0} \frac{X_\eta^\rho}{\eta^{-2}\alpha_4(\eta,1)}$ exists, and the limit can be read off from macroscopic information (measurable w.r.t. the percolation scaling limit).

The key step is that $\mathbf{E}\Big[\Big(\frac{X}{\mathbf{E}[\,X\,]} - \frac{Y}{\mathbf{E}[\,Y\,]}\Big)^2\Big] = o(1)$ as ϵ and $\eta/\epsilon \to 0$.

This will follow from $\mathbf{E}\big[\,(X - \beta\,Y)^2\,\big] = o(1)\,\mathbf{E}\big[\,X^2\,\big]$ as ϵ and $\eta/\epsilon \to 0$.

Want loss of information when zooming in to smaller scales, proved by a coupling argument that uses a Strong Separation Lemma.

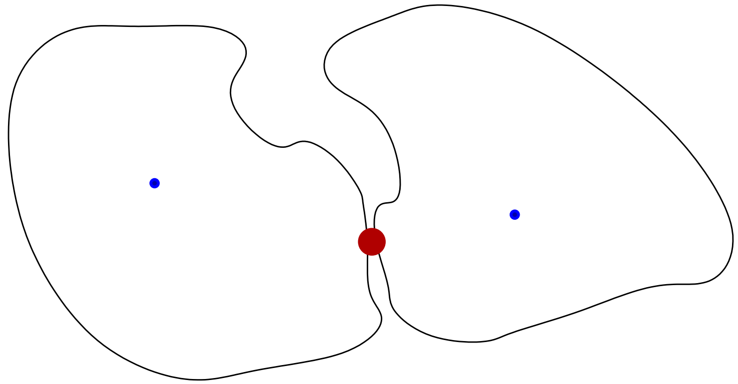
Minimal spanning tree



How do we see the MST?

Look at path joining two fixed points $x, y \in \mathbb{R}^2$, and NCESL.

Naive idea: They will become connected at large level $\lambda_\emptyset$, at point $z_\emptyset$.



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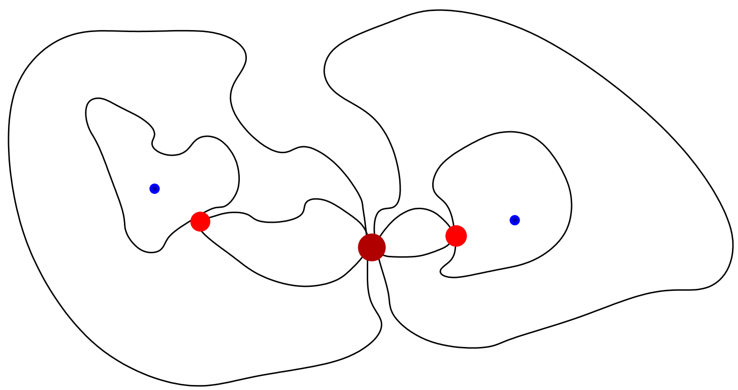
Repeat between x and $z_\emptyset$ to get $\lambda_0 < \lambda_\emptyset$ at point z_0, and between $z_\emptyset$ and y to get $\lambda_1 < \lambda_\emptyset$ at point z_1.



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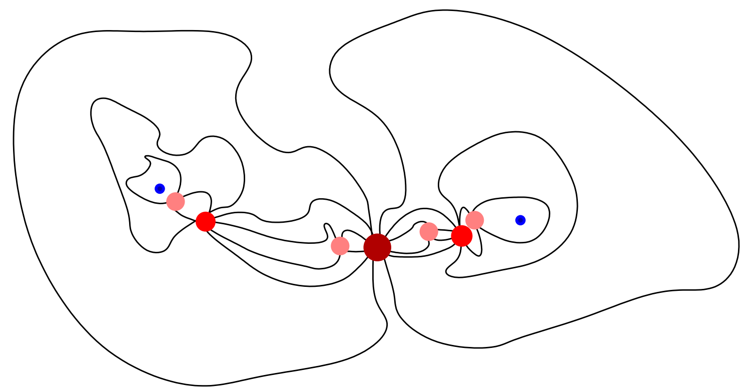
And so on, dyadic recursion.



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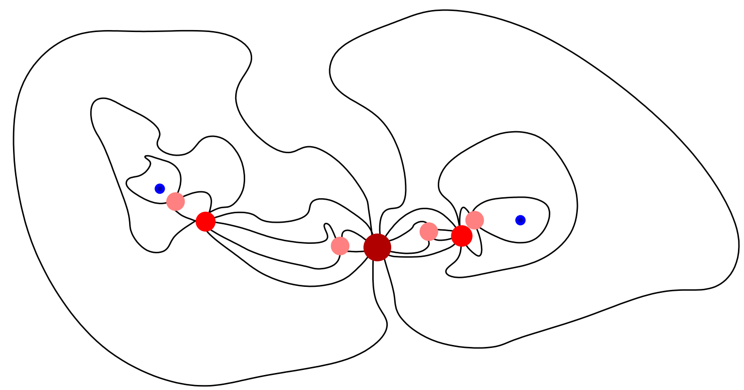

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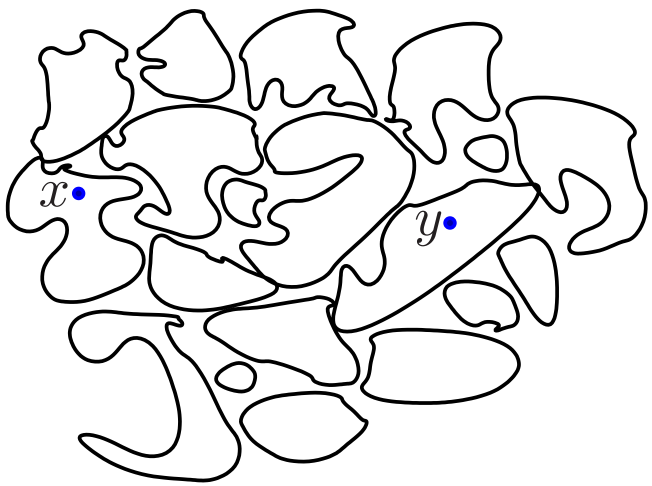
But in the SL, the maximum $z_\emptyset$ is at either x or at y! So need some cutoff: take ϵ-ball around x and y. Then $z_\emptyset$ is close to one of the endpoints, but doesn't coincide.

But, when iterating, do we have convergence as $\epsilon \downarrow 0$?

How do we see the MST?

Instead: Fix some very negative λ_1. Look at outermost λ_1-clusters.



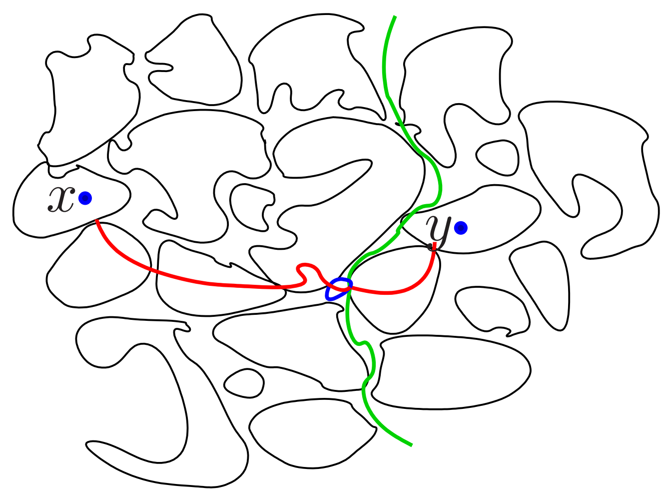
Since the outermost λ_1-clusters are small, only need to watch connections.

Difficulty: There are infinitely many outermost clusters in a bounded region, so it's not clear that the MST on this cluster-graph is even well-defined.

Take only outermost λ_1-clusters with diameter $\geqslant \epsilon$.

If ϵ is small enough, then there is a path between x and y in this finite cluster-graph. The resulting MST path uses labels $\leqslant \lambda_2$.

Now take $\delta \ll \epsilon$, and the corresponding cluster graph. Can the new MST path go through a small λ_1-cluster? Still has labels $\leqslant \lambda_2$.

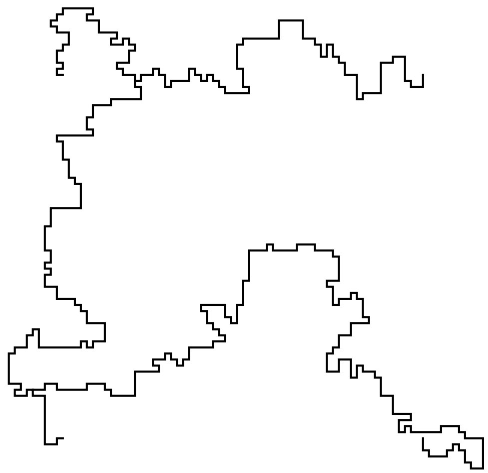


Moving cutpoint labels from λ_2 to λ_1, small cutpoint becomes very important at level λ_1, although it was very little important before. Contradicts stability result.

So, don't need to watch very small clusters, and $\epsilon \downarrow 0$ convergence is fine.

Topology of MST SL

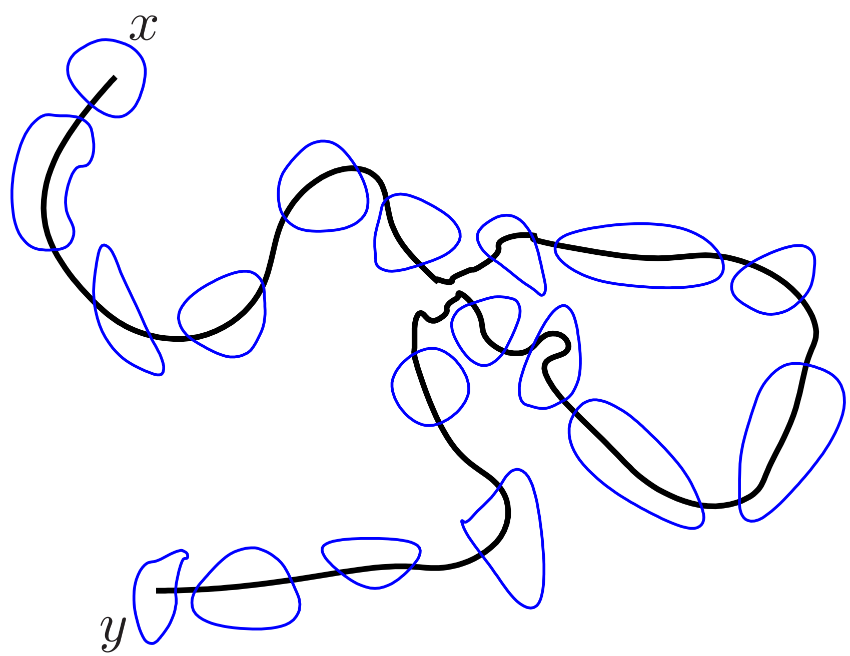
Theorem. In either lattice, there are no degree $\geqslant 5$ points. If you fix two points in the plane, a.s. the MST scaling limit path joining them will be a simple path. (And there are no figures of 6.)



Questions. Are there random pairs with non-simple path or figures of 6? Are there degree 4 points?

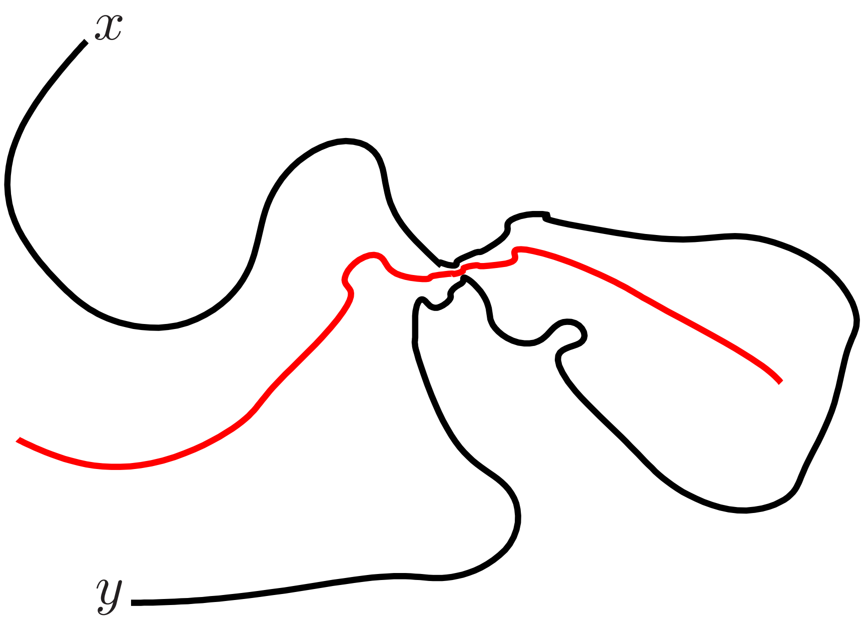
Earlier results by Aizenman, Burchard, Newman, Wilson (1999).

Simple path proof



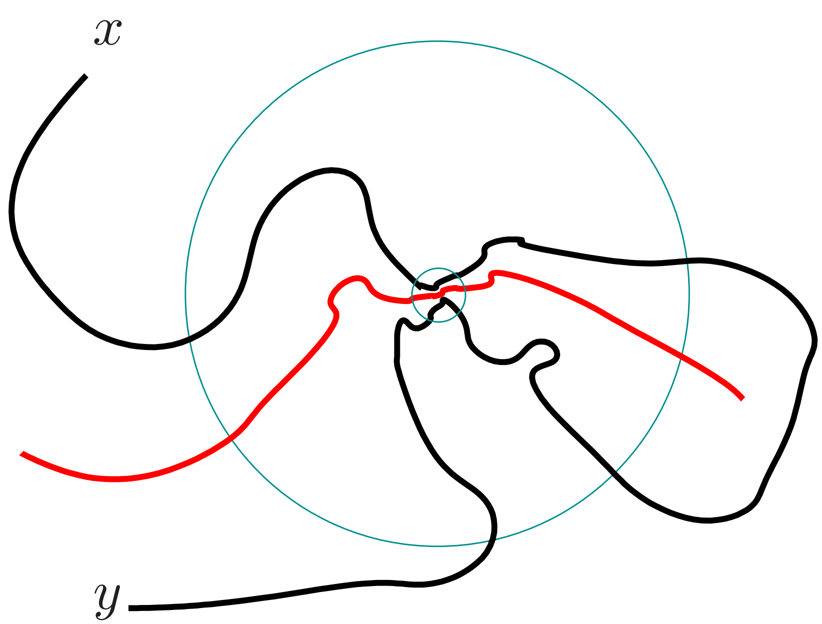
Generic nearly non-simple path. For λ_1 very negative and λ_2 very positive, the part separated from x and y is entirely below level λ_2, and has labels above λ_1 "all over the place".

Simple path proof



Dual arms with labels all above λ_1 exist.

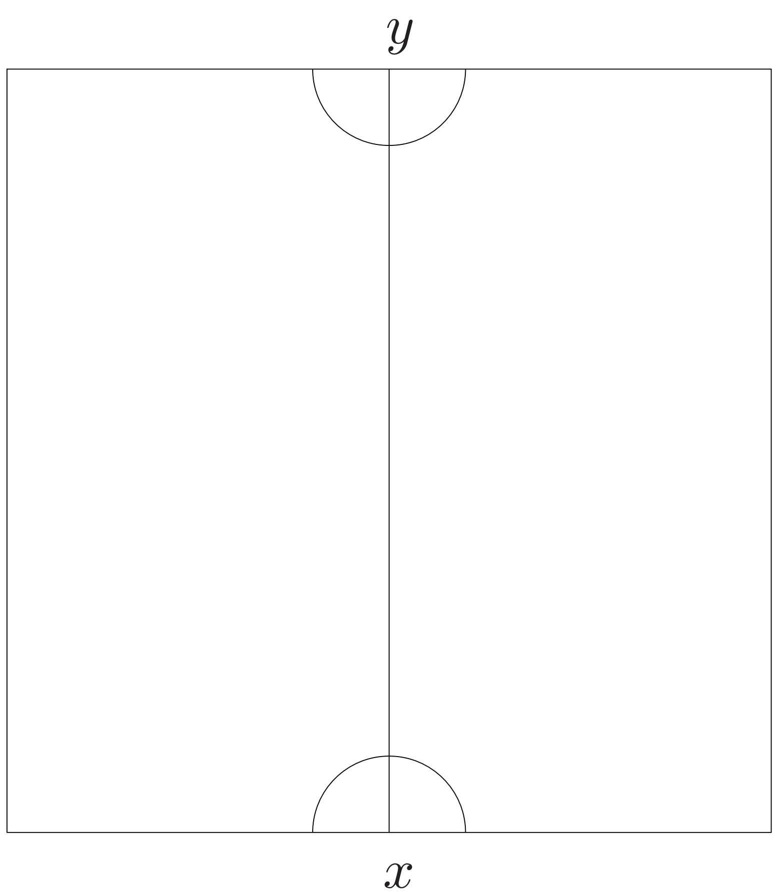
Simple path proof



Six arm event within W_λ-modification — contradicts stability.

Conformal non-invariance?

Consider conformal map onto unit square from some domain, with very different $|\phi'(z)|$ at different places. Say, extremely large on right side, extremely small on left. So, by changing λ, rapid change on the right, almost no change at left.



Simplification: $n \times n$ square, Unif$([0, 1/5] \cap [4/5, 1])$ on left, Unif$[2/5, 3/5]$ on right.

Does the MST path from x to y feel the asymmetry?

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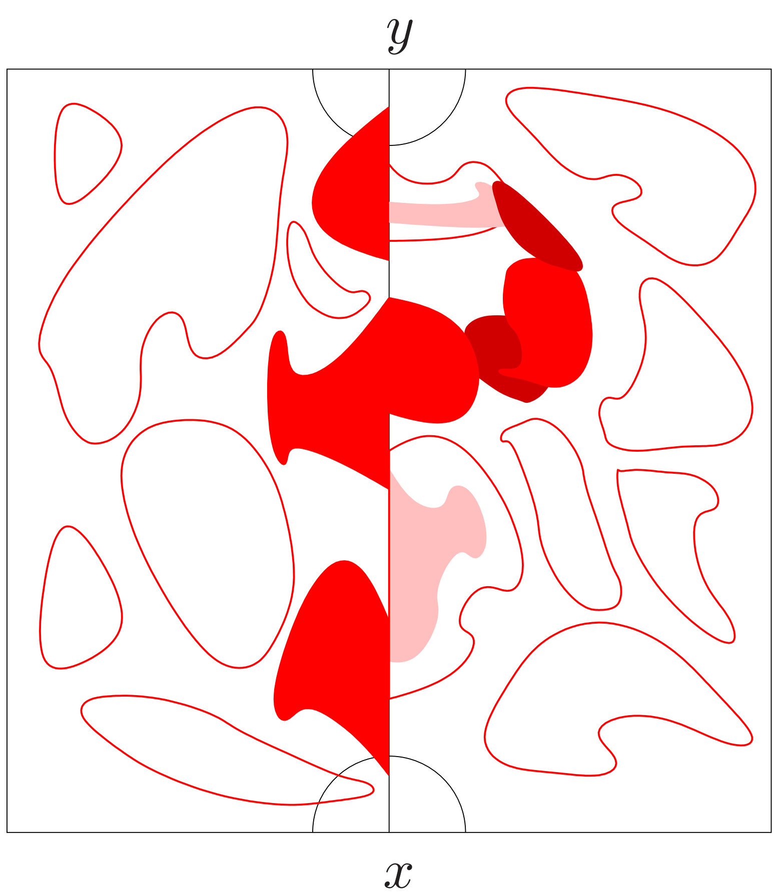

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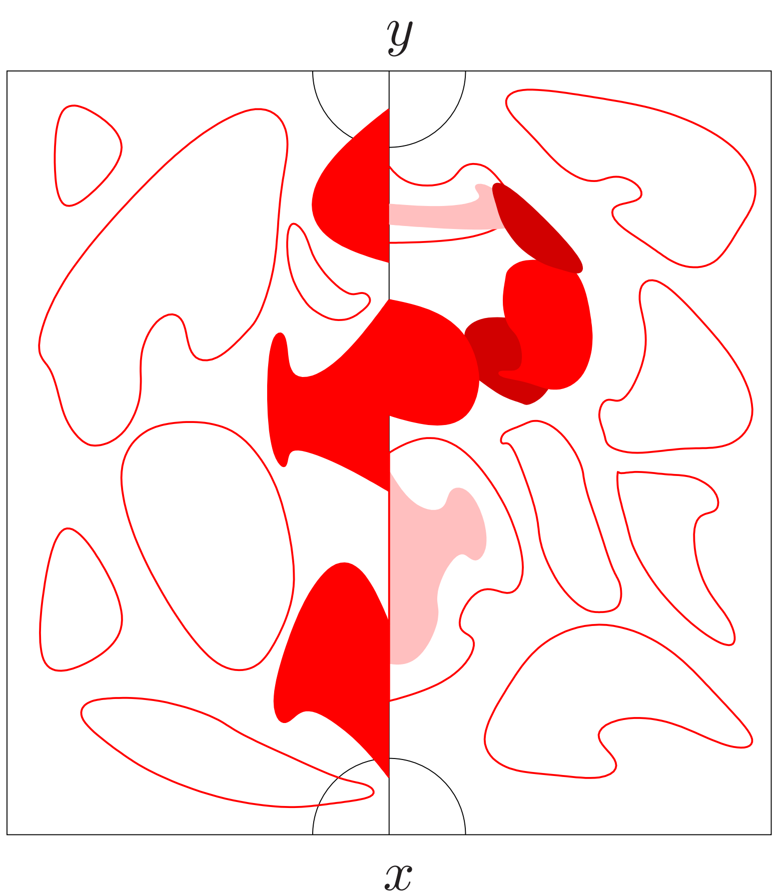

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Maybe for Invasion Percolation instead of MST is easier, using Damron-Sapozhnikov-Vágvölgyi (2008).

Further results and questions

Hope to describe near-critical interfaces with an equation similar to SLE_6 but involving a self-interacting drift term. Refining Nolin-Werner (2008). $dW_t = \sqrt{6}\, dB_t + c\,\lambda\,|d\gamma_t|^{3/4}\,dt^{1/2}$. But is this useful? Near-critical Cardy?

The Fourier spectrum of critical percolation (GPS 2008) implies that DPSL is ergodic; correlation of LR crossing between times 0 and t is $t^{-2/3}$. What is the probability of having the crossing all along $[0, t]$?
(Some thoughts with A. Hammond, E. Mossel, O. Schramm)

Is the expected number of pivotals in any quad $\mathcal{Q}$ at most $O_{\mathcal{Q}}(1)\,\eta^{-2}\alpha_4(\eta, 1)$? ($\Longrightarrow$ finite expected crossing changes in DPSL)

"Kaufman-type dimension 7/4-ing" for SLE_6? (with C. Garban)

Dynamical and near-critical FK-cluster and Potts models? (with C. Garban)

Asymptotic circularity of Wulff shapes as $p \downarrow p_c$? (with A. Hammond)