Critical phenomena and conformal invariance in the plane — Homework problems

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Constantly updated.

▷ **Exercise 1.** To clarify what the measurability of having an infinite cluster means:
Let $G(V,E)$ be any bounded degree infinite graph, and $S_n \nearrow V$ an exhaustion by finite connected subsets. Is it true that, for $p > p_c(G)$, we have

$$\lim_{n\to\infty} \mathbf{P}_p[\,\text{largest cluster for percolation inside } S_n \text{ is the subset of an infinite cluster}\,] = 1\,?$$

▷ **Exercise 2.** Show that if in a graph G the number of minimal edge cutsets (a subset of edges whose removal disconnects a given vertex from infinity, minimal w.r.t. containment) of size n is at most $\exp(Cn)$ for some $C < \infty$, then $p_c(G) \le 1 - \epsilon(C) < 1$. Show that $\mathbb{Z}^d$, $d \ge 2$, has such an exponential bound. In particular, $p_c(\mathbb{Z}^d) < 1$, although we know that already from $\mathbb{Z}^2 \subseteq \mathbb{Z}^d$.

▷ **Exercise 3.** Recall that the energy in the Ising model is defined by

$$H(\sigma, h) := -h \sum_{x\in V(G)} \sigma(x) - \sum_{(x,y)\in E(G)} \sigma(x)\sigma(y)\,.$$

(a) Show that the **expected total energy** is

$$\mathbf{E}_{\beta,h}[\,H\,] = -\frac{\partial}{\partial\beta} \ln Z_{\beta,h}\,, \text{ with variance } \mathrm{Var}_{\beta,h}[H] = -\frac{\partial}{\partial\beta}\mathbf{E}_{\beta,h}[\,H\,]\,.$$

(b) The **average free energy** is defined by $f(\beta,h) := -(\beta|V|)^{-1}\ln Z_{\beta,h}$. Show that for the **total average magnetization** $M(\sigma) := |V|^{-1}\sum_{x\in V}\sigma(x)$, we have $\mathbf{E}_{\beta,h}[\,M\,] = -\frac{\partial}{\partial h} f(\beta,h)$.

▷ **Exercise 4.**

(a) Show that if μ and ν are probability measures on a finite poset $(\mathcal{P},\ge)$, and both $\mu \ge \nu$ and $\mu \le \nu$, then $\mu = \nu$. Conclude that if two infinite volume limits of Ising measures on an infinite graph dominate each other, then they are equal.

(b) Let $\mathcal{P} = \{-1,+1\}^V$ with coordinatewise ordering. Show that if $\mu \le \nu$ on $\mathcal{P}$, and $\mu|_x = \nu|_x$ for all $x \in V$ (i.e., all the marginals coincide), then $\mu = \nu$. (Hint: use Strassen's coupling.)

(c) On any transitive infinite graph, the limit measures $\mathbf{P}^+_{\beta,h}$ and $\mathbf{P}^-_{\beta,h}$ are translation invariant. They are equal iff $\mathbf{E}^+_{\beta,h}\sigma(x) = \mathbf{E}^-_{\beta,h}\sigma(x)$ for one or any $x \in V(G)$.

▷ **Exercise 5.** On any transitive infinite graph, the limit measures $\mathbf{P}^+_{\beta,h}$ and $\mathbf{P}^-_{\beta,h}$ are ergodic.

▷ **Exercise 6.** For the FK(p,q) model on any finite graph $G(V,E)$ with boundary $\partial V \subset V$, show the following two types of stochastic domination:

(a) If $\pi \le \pi'$ on ∂V, then $\mathbf{P}^{\pi}_{\mathrm{FK}(p,q)} \le \mathbf{P}^{\pi'}_{\mathrm{FK}(p,q)}$ on V.

(b) Given any π on ∂V, if $p \le p'$, then $\mathbf{P}^{\pi}_{\mathrm{FK}(p,q)} \le \mathbf{P}^{\pi}_{\mathrm{FK}(p',q)}$ on V.

(c) Conclude for the + limit Ising measure on any infinite graph $G(V,E)$ that if $\mathbf{E}^+_{\beta,h}[\sigma(x)] > 0$ for some $x \in V$, then the same holds for any $\beta' > \beta$. Consequently, the uniqueness of the Ising limit measures is monotone in β.

▷ **Exercise 7.** Assuming the fact that at least one type of crossing is present in any two-colouring of the $n \times n$ rhombus in the hexagonal grid, prove Brouwer's fixed point theorem in two dimensions.

▷ **Exercise 8.** Consider any sequence of non-trivial monotone events $\mathcal{A} = \mathcal{A}_n$. For $t \in [0,1]$, let $p^t_{\mathcal{A}}(n)$ be the p for which $\mathbf{P}_p[\,\mathcal{A}_n\,] = t$, and call $p_{\mathcal{A}}(n) := p^{1/2}_{\mathcal{A}}(n)$ the **critical probability** for $\mathcal{A}$. Show that for any $\epsilon > 0$ there is $C_\epsilon < \infty$ such that $\left|p^{1-\epsilon}_{\mathcal{A}}(n) - p^{\epsilon}_{\mathcal{A}}(n)\right| < C_\epsilon\, p^{\epsilon}_{\mathcal{A}}(n) \wedge (1 - p^{1-\epsilon}_{\mathcal{A}}(n))$. Conclude that every sequence of monotone events has a threshold. (Hint: take many independent copies of a low density percolation to get success with good probability at a larger density.)

The **Margulis-Russo formula** for a general (non-monotone) event $\mathcal{A}$ is that

$$\frac{d}{dp}\mathbf{P}_p[\,\mathcal{A}\,] = \sum_{i\in[n]} \bar{I}^{\mathcal{A}}_p(i)\,,$$

where $\bar{I}^{\mathcal{A}}_p(i) := \mathbf{P}_p[\,\Psi_i\mathcal{A}\,] - \mathbf{P}_p[\,\Psi_{\neg i}\mathcal{A}\,]$ is the **signed influence** of the variable i on $\mathcal{A}$, with $\Psi_i\mathcal{A} := \{\omega : \omega \cup \{i\} \in \mathcal{A}\}$ and $\Psi_{\neg i}\mathcal{A} := \{\omega : \omega \setminus \{i\} \in \mathcal{A}\}$. Of course, if $\mathcal{A}$ is increasing, then $\bar{I}^{\mathcal{A}}_p(i)$ equals $I^{\mathcal{A}}_p(i) := \mathbf{P}_p[\,i \text{ is pivotal for } \mathcal{A}\,] := \mathbf{P}_p[\,\Psi_i\mathcal{A} \bigtriangleup \Psi_{\neg i}\mathcal{A}\,]$.

▷ **Exercise 9.**

(a) Prove the identity $I^{\mathcal{A}}_{1/2} = |\partial_E\mathcal{A}|/2^{n-1}$ for the **total influence** $I^{\mathcal{A}}_p := \sum_i I^{\mathcal{A}}_p(i)$ of any event $\mathcal{A} \subseteq \{0,1\}^{[n]}$, with the edge boundary ∂_E understood in the hypercube $\{0,1\}^{[n]}$.

(b) Show that, among all monotone events $\mathcal{A}$ on $[n]$, the total influence $I^{\mathcal{A}}_{1/2}$ is maximized by the majority Maj_n, and find the value. (Therefore, Maj_n has the sharpest possible threshold at $p = 1/2$.)

▷ **Exercise 10.** Prove the Poincaré inequality

$$I^{\mathcal{A}}_{1/2} \ge \mathbf{P}_{1/2}[\,\mathcal{A}\,]\,(1 - \mathbf{P}_{1/2}[\,\mathcal{A}\,])\,.$$

(Hint: Define a random map from the set of pairs $(\omega,\omega') \in \mathcal{A} \times \mathcal{A}^c$ into $\partial_E\mathcal{A}$.)

▷ **Exercise 11.** *

(a) Show that the "conditional FKG-inequality" does not hold: find three increasing events A, B, C in some $\mathsf{Ber}(p)$ product measure space such that $\mathbf{P}_p[\,AB \mid C\,] < \mathbf{P}_p[\,A \mid C\,]\,\mathbf{P}_p[\,B \mid C\,]$.

(b) Show that conditional FKG would imply that $\mathbf{P}_p[\,\cdot \mid 0 \longleftrightarrow \partial B_{n+1}(o)\,]$ stochastically dominates $\mathbf{P}_p[\,\cdot \mid 0 \longleftrightarrow \partial B_n(o)\,]$ restricted to any box $B_m(0)$ with $m < n$. (However, this monotonicity is not known and might be false, and hence it was proved without relying on it that, for $p = p_c(\mathbb{Z}^2)$, these measures have a weak limit as $n \to \infty$, called Kesten's Incipient Infinite Cluster.)

▷ **Exercise 12** (Density of the infinite cluster)**.** * Our goal here is to show that $\theta(p) = \mathbf{P}_p[\,0 \longleftrightarrow \infty\,]$ can be viewed as the density of the infinite cluster $\mathscr{C}$, in the sense that $|\mathscr{C} \cap B_n|/|B_n| \to \theta(p)$ in L^2 (and in fact almost surely) as $n \to \infty$.

(a) Show that $\mathbf{E}|\mathscr{C} \cap B_n| = \theta(p)$.

An important property of subcritical percolation: for any $p < 1/2$, there exist constants $C, C' > 0$ depending on p such that for all N,

$$\mathbf{P}_p[\,0 \longleftrightarrow \partial B_N\,] \le C' e^{-CN}.$$

(b) Prove that this implies the following for the supercritical regime: for any $p > 1/2$, for all N,

$$\mathbf{P}_p\big[\,0 \longleftrightarrow \partial B_N \;\big|\; 0 \not\longleftrightarrow \infty\,\big] \le C' e^{-CN}.$$

(c) Show that there exists a constant $C > 0$ (depending only on p) such that for all N,

$$\mathrm{Var}|\mathscr{C} \cap B_N| \le CN^2.$$

(d) Show that the convergence takes place almost surely. Can one replace B_n by any nested increasing family of domains?

▷ **Exercise 13.** The previous exercise showed a certain duality between super- and subcritical percolation, which is of course unsurprising, given the planar duality. However, this duality between the two regimes is a more general phenomenon. Here is an example where this duality holds exactly:

Let $\mathsf{PGW}(\lambda)$ be the Galton-Watson branching process tree with offspring distribution $\mathsf{Poisson}(\lambda)$. Note that for any supercritical $\lambda > 1$ there is a unique $\mu \in (0,1)$ satisfying $\lambda e^{-\lambda} = \mu e^{-\mu}$. Show that $\mathsf{PGW}(\lambda)$, conditioned on being finite, has the distribution of $\mathsf{PGW}(\mu)$. (Hint: one way of doing this is via generating functions. Another way is to draw the tree in the plane, consider the contour walk around it (which is basically the depth-first search), encode the walk by the distance from the root, and understand this nearest-neighbour walk on $\mathbb{N}$ in terms of the offspring distribution.)

▷ **Exercise 14** (Two-arm exponent in the half-plane)**.** We consider the half-box $H_n = \{z \in B_n(0) : \mathrm{Im}(z) \ge 0\}$. Its boundary can be decomposed into two parts: the segment $[-n, n]$ on the real axis, and h_n. Let $\alpha_2^+(n)$ be the probability of having an open path from 0 to h_n and a closed path from 1 to h_n, both staying in H_n.

(a) Consider critical percolation in H_{2n}. Prove that with a probability bounded from below independently of n, there exist an open cluster and a closed cluster, that both intersect the segment $[-n/2, n/2]$ and h_{2n}, such that the closed cluster is "to the right" of the open one. Looking at the right-most point of the open cluster on the real line, deduce that $\alpha_2^+(n) \ge c/n$ for some absolute constant c.

The **BK inequality**, due to van den Berg and Kesten, says that for $\mathsf{Ber}(p)$ product measure on any set S, for any increasing events A and B, we have $\mathbf{P}_p[\,A \square B\,] \le \mathbf{P}_p[\,A\,]\,\mathbf{P}_p[\,B\,]$, where

$$A \square B := \big\{\omega \in \{0,1\}^S : \exists \text{ disjoint } S_A, S_B \subseteq S \text{ such that } \omega|_{S_A} \text{ implies } A \text{ and } \omega|_{S_B} \text{ implies } B\big\},$$

called the “disjoint occurrence” of A and B.

(b) Consider critical percolation in H_n, and let K be the number of open clusters that join h_n to $[-n/2, n/2]$. Using the BK inequality, show that there is some absolute constant $\lambda < 1$ such that $\mathbf{P}[\, K \geq k \,] \leq \lambda^k$. Deduce that $(n+1)\alpha_2^+(2n) \leq \mathbf{E}(K) \leq C$ for some absolute constant $C < \infty$.

(c) Conclude that for some positive absolute constants c_1 and c_2, $c_1/n \leq \alpha_2^+(n) \leq c_2/n$.

▷ **Exercise 15** (Three-arm exponent in the half-plane)**.** We say that a point x is n-good if it is the *unique* lowest point in $x + H_n$ of an open cluster C such that $C \not\subseteq x + H_n$. Note that the probability that a point is n-good does not depend on x.

(a) Show that this event corresponds to the existence of three arms originating from x in the half-box $x + H_n$. Therefore, denote its probability by $\alpha_3^+(n)$.

(b) Using the BK inequality, show that the expected number of clusters that join $h_{n/2}$ to h_n is uniformly bounded. Compare this number of clusters with the number of n-good points in $H_{n/2}$ and deduce from this that for some constant c_1, $\alpha_3^+(n) \leq c_1/n^2$.

(c) Show that with probability bounded from below independently of n, there exists in $H_{n/2}$ an n-good point (note that an argument is needed to show that with positive probability, there exists a cluster with a unique lowest point). Deduce that $\alpha_3^+(n) \geq c_2/n^2$ for some absolute constant $c_2 > 0$.

▷ **Exercise 16** (Five-arm exponent)**.** Consider critical percolation in the annuli $A_j = B_{2^j} \setminus B_{2^{j-1}}$ for $j = 1, 2, \ldots, m$. Let $n = 2^m$. When x is a site of the triangular lattice, we say that $U_n(x)$ is satisfied if x is closed and from its neighbors there are five disjoint arms to $x + \partial B_n$, three of them closed, two of them open, in alternating order: the two open arms are separated by closed arms in $x + B_n$. Note that the probability $\alpha_5(n) := \mathbf{P}[\, U_n(x) \,]$ does not depend on x.

(a) Using the BK inequality, prove that the probability that there exist at least k disjoint open clusters (if one considers percolation restricted to B_{2^j}) intersecting both the outer and the inner boundary of A_j is bounded from above by λ^k, independently of j. Deduce that the number K_j of such clusters satisfies $E(K_j^2) < C$ for some absolute constant $C < \infty$.

(b) Suppose that $x \in B_{n/2}$ is such that $U_{2n}(x)$ holds. Show that it is on the boundary of two of the K_m clusters. Conversely, suppose that two of the K_m clusters are adjacent. Show that there are at most two points x in $B_{n/2}$ on their joint boundary such that $U_{2n}(x)$ holds. Conclude that $|B_{n/2}|\alpha_5(2n) \leq \mathbf{E}[\, K_m^2 \,] < C$.

(c) Prove using a Russo-Seymour-Welsh type argument that there exists at least one point in $B_{n/2}$ such that $U_{n/2}(x)$ holds with positive probability. Deduce that $|B_{n/2}|\alpha_5(n/2) \geq c$ for some absolute constant $c > 0$.

(d) Conclude that there exist two constants c_1 and c_2 such that $c_1 n^{-2} \leq \alpha_5(n) \leq c_2 n^{-2}$.

▷ **Exercise 17.*** Assume quasi-multiplicativity and the bounds

$$c\,(r/R)^{2-\epsilon} < \alpha_4(r,R) < C\,(r/R)^{1+\epsilon}$$

and

$$\alpha_2^+(r,R) \asymp r/R\,, \qquad \alpha_3^+(r,R) \asymp (r/R)^2\,,$$

known also on $\mathbb{Z}^2$. For the number of pivotals for left-right crossing in the square $\mathcal{Q}_n = [0,n]^2$, prove

(a) the first moment estimate $\mathbf{E}|\mathrm{Piv}(\mathcal{Q}_n)| \asymp n^2\alpha_4(n)$,

(b) and the second moment estimate $\mathbf{E}\big[\,|\mathrm{Piv}(\mathcal{Q}_n)|^2\,\big] \asymp \big(\mathbf{E}|\mathrm{Piv}(\mathcal{Q}_n)|\big)^2$.

(c) Deduce that there is some $c > 0$ with $\mathbf{P}\big[\,|\mathrm{Piv}(\mathcal{Q}_n)| > cn^2\alpha_4(n)\,\big] > c$.

▷ **Exercise 18** (Cayley's formula)**.** Show that the number of spanning trees of the complete graph K_n on n distinguishable vertices is n^{n-2}, via the following argument, using Wilson's algorithm.

Let $t_1 \subset t_2 \subset \cdots \subset t_{n-1}$ any sequence of subtrees of K_n such that t_i has i edges. Show that $\mathbf{P}[\,t_1 \subseteq \mathsf{UST}\,] = 2/n$ and

$$\mathbf{P}[\,t_{i+1} \subseteq \mathsf{UST} \mid t_i \subseteq \mathsf{UST}\,] = \frac{i+2}{n(i+1)},$$

then deduce Cayley's formula from the tower rule of conditional probabilities.

▷ **Exercise 19.** Let γ_n be the number of self-avoiding paths of length n in $\mathbb{Z}^2$ starting from the origin. Show that $\lim_n \gamma_n^{1/n} = \gamma$ exists and is in $(1,\infty)$.

▷ **Exercise 20** (Riemann Mapping Theorem examples)**.** Find a conformal map

(a) from $\mathbb{R} \times (0,\pi i)$ to the open upper half plane $\mathbb{H}$;

(b) from $\mathbb{R} \times (0,2i) \setminus \big((i-\infty, i-\epsilon] \cup [i+\epsilon, i+\infty)\big)$ to $\mathbb{H}$.

▷ **Exercise 21.** Give an example of a simple curve $\gamma_t \subset \overline{\mathbb{H}}$ with $\gamma_0 = 0$ and $\lim_{t\to\infty}\gamma_t = \infty$ such that the half-plane capacity $a(\gamma_t)$ remains bounded.

▷ **Exercise 22.** For any interval $(a,b) \subset \mathbb{R}$, let $\nu(a,b)$ denote harmonic measure in $\mathbb{H}$ from ∞, i.e., $\lim_{y\to\infty} y\mathbf{P}_{iy}[\,B_\tau \in (a,b)\,]$, where B_t is planar BM started at iy and τ is the first hitting time of $\mathbb{R}$. Show that $\nu(a,b) = c|b-a|$ for all $a < b$, and find c.

▷ **Exercise 23.**

(a) Let $D \subset \mathbb{C}$ be a simply connected open domain, $z \in D$, and $g : D \longrightarrow \mathbb{H}$ any conformal equivalence. Show that $\mathsf{ConfRad}(z,D) = 2\,\mathrm{Im}(g(z))/|g'(z)|$.

(b) Conclude that $\mathsf{ConfRad}(-1, \mathbb{H} \setminus [0,\infty)) = 4$, showing the sharpness of the Koebe 1/4 theorem.

▷ **Exercise 24.** Improve (or at least suggest improvements) on some of the following English Wikipedia articles:

(a) `http://en.wikipedia.org/wiki/FKG_inequality`

(b) `http://en.wikipedia.org/wiki/Percolation_theory`

(c) Random cluster model exists only in `http://en.wikipedia.org/wiki/Tutte_polynomial`, and there is very little about it

(d) `http://en.wikipedia.org/wiki/Schramm-Loewner_Evolution`

(e) `http://en.wikipedia.org/wiki/Scaling_limit`

(f) anything else related to the course that you would like to edit.

▷ **Exercise 25.*** Recall the setup in the discussion of the conformal image of chordal SLE K_t under $\Phi_A : \mathbb{H} \setminus A \longrightarrow \mathbb{H}$, the usual uniformization map for a hull $A \subset \mathbb{H}$, where $g_t : \mathbb{H} \setminus K_t \longrightarrow \mathbb{H}$ is a usual chordal Loewner chain in $\mathbb{H}$ for $t \leq \tau_A$, then $\tilde{K}_t := \Phi_A(K_t)$ has the uniformizing map $\tilde{g}_t$, and finally, h_t is the uniformizing map for $A_t := g_t(A)$, so that $h_t = \tilde{g}_t \circ \Phi_A \circ g_t^{-1}$.

Recall, furthermore, that we proved a modified Loewner equation for $\tilde{K}_t$:

$$\partial_t \tilde{g}_t(z) = \frac{2h_t'(W_t)^2}{\tilde{g}_t(z) - \tilde{W}_t}.$$

(a) Show that $\partial_t h_t(z) = \frac{2h_t'(W_t)^2}{h_t(z)-\tilde{W}_t} - \frac{2\,h_t'(z)}{z-W_t}$.
(b) Deduce that $\partial_t h_t(W_t) = -3h_t''(W_t)$.
(c) Conclude using Itô's formula that $d\tilde{W}_t = h_t'(W_t)\,dW_t + (-3+\kappa/2)h_t''(W_t)\,dt$. Show that, for $\kappa = 6$, the process $\tilde{W}_t$ equals $W_t = \sqrt{6}B_t$ in distribution.
This shows that SLE_κ has the **locality property** iff $\kappa = 6$.

▷ **Exercise 26.***
(a) Differentiate the formula in part (a) of the previous exercise w.r.t. z, then take the limit $z \to W_t$, to get

$$(\partial_t h_t')(W_t) = \frac{h_t''(W_t)^2}{2h_t'(W_t)} - \frac{4}{3}h_t'''(W_t).$$

(b) Conclude using Itô's formula that, for $\kappa = 8/3$, we have

$$d[h_t'(W_t)^{5/8}] = \frac{5h_t''(W_t)}{8h_t'(W_t)^{3/8}}\,dW_t.$$

(c) Fill in the gaps in the proof of Proposition 4.2 in Wendelin Werner's SLE lecture notes [arXiv:math.PR/0303354], showing that the previous part implies that $\mathbf{P}\big[\,\forall t \geq 0,\ K_t \cap A = \emptyset\,\big] = \Phi_A'(0)^{5/8}$.
This shows that SLE_κ has the **restriction property** iff $\kappa = 8/3$.