Noise sensitivity questions in percolation-like models

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Noise sensitivity of Boolean functions

$f : \{-1,1\}^N \longrightarrow \{-1,1\}$ a Boolean function, usually monotone.
Input is i.i.d. Bernoulli(p).

Take critical density $p = p_c(N)$, where $\mathbf{P}_p[\, f(\omega) = 1\,] = 1/2$.

Resample each input bit with probability ϵ, independently, get ω^ϵ.

Given a typical ω, can we predict what $f(\omega^\epsilon)$ will be? What is the correlation between $f(\omega^\epsilon)$ and $f(\omega)$? Three simple examples:

Dictator$_n(\omega_1, \dots, \omega_n) := \omega_1$. Here $p_c(n) = 1/2$.
Here Corr$\big[\,\text{Dic}_n(\omega), \text{Dic}_n(\omega^\epsilon)\,\big] = 1 - \epsilon$, hence noise-stable.

Majority$_n(\omega_1, \dots, \omega_n) := \operatorname{sgn}(\omega_1 + \cdots + \omega_n)$. Again, $p_c(n) = 1/2$.
Here Corr$\big[\,\text{Maj}_n(\omega), \text{Maj}_n(\omega^\epsilon)\,\big] = 1 - O(\sqrt{\epsilon})$, hence noise-stable.

Parity$_n(\omega_1, \dots, \omega_n) := \omega_1 \cdots \omega_n$. Again, $p_c(n) = 1/2$.
Here Corr$\big[\,\text{Par}_n(\omega), \text{Par}_n(\omega^\epsilon)\,\big] = (1 - \epsilon)^n$, very sensitive to noise.

Noise sensitivity of Boolean functions

A sequence of Boolean functions $f_k : \{-1,1\}^{N_k} \longrightarrow \{-1,1\}$ is called noise sensitive at density p if

$$\forall\, \epsilon > 0 : \quad \text{Corr}\big[\, f_k(\omega^\epsilon), f_k(\omega)\,\big] \to 0 \quad \text{as} \quad k \to \infty\,,$$

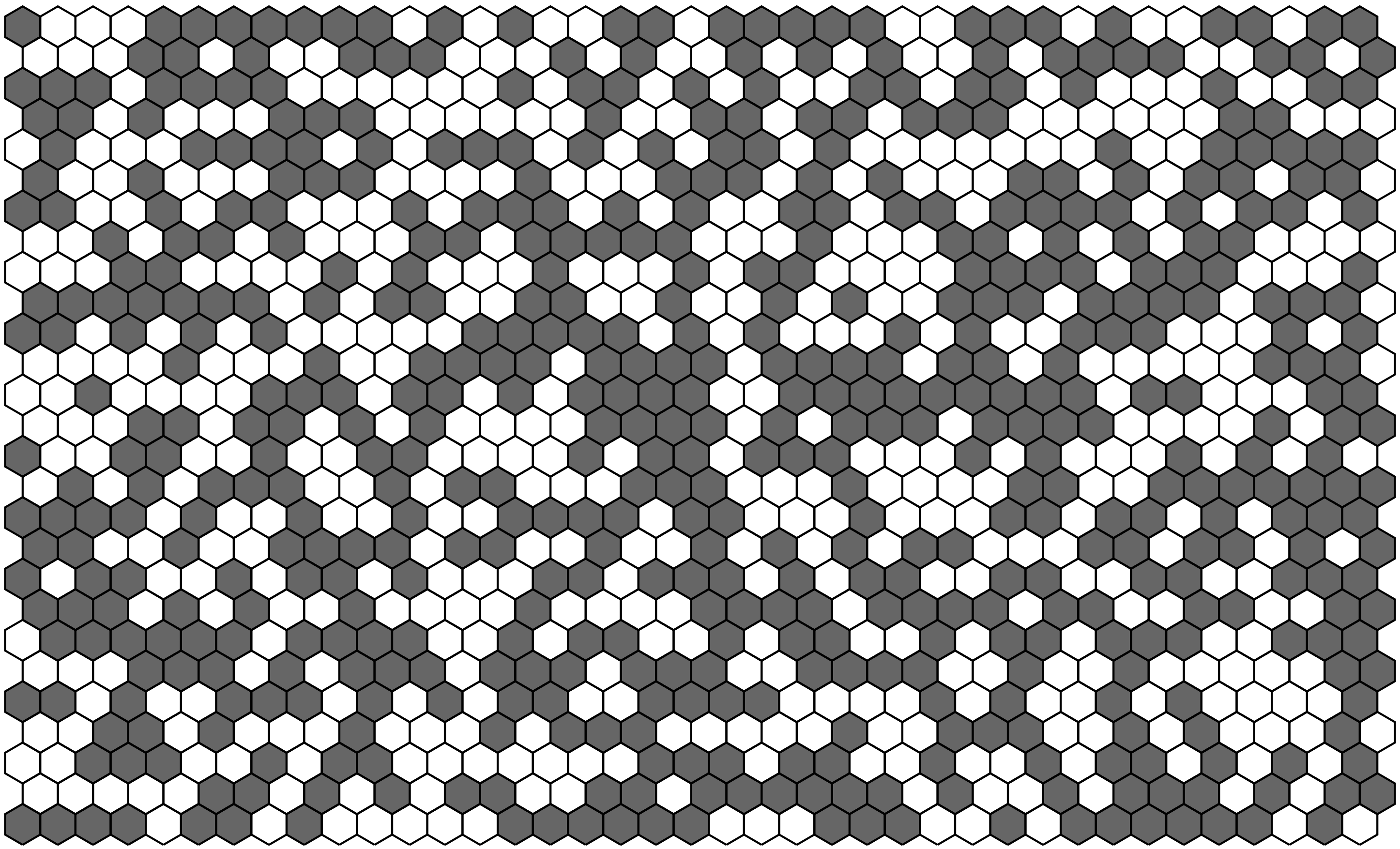
and noise stable if

$$\lim_{\epsilon\to 0} \sup_k \mathbf{P}_p\big[\, f_k(\omega^\epsilon) \neq f_k(\omega)\,\big] = 0\,.$$

Could be insensitive but not stable.

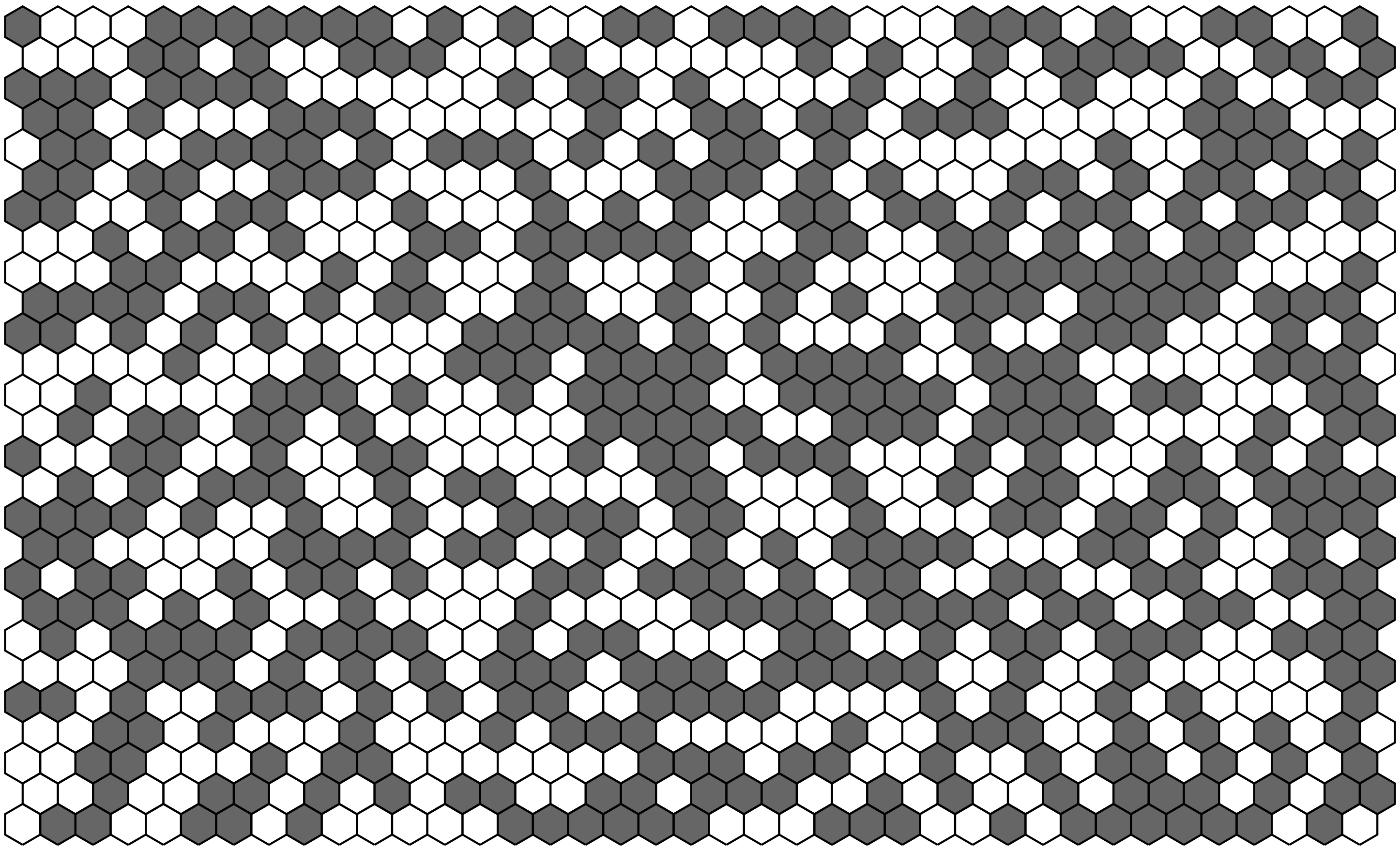
Give a monotone noise sensitive example!

Percolation and noise



At $p_c = 1/2$, left-right crossing has non-trivial probability.

Percolation and noise



At $p_c = 1/2$, left-right crossing has non-trivial probability.

Naive idea: how many pivotals are there?

A bit is pivotal for f in ω if flipping it changes the output.

Do the ϵ-noise by switching bits one-by-one. In order to change the output, need at least one pivotal switch; in fact, need an odd number of them.

Complete decorrelation $\Leftrightarrow$ so many pivotal switches that you don't know their parity.

Naively, "the more pivotals there are, the more noise sensitive the function should be".

First issue: Maj_{2k+1} typically has no pivotal bits at all; with probability $\asymp 1/\sqrt{k}$, it has $k+1$, hence $\mathbf{E}|\mathsf{Piv}_{\mathsf{Maj}_{2k+1}}| \asymp \sqrt{k}$. This matters for sharp thresholds by the Russo-Margulis formula,

$$\frac{d}{dp}\mathbf{P}_p\big[\, f(\omega) = 1 \,\big] = \mathbf{E}_p\big[\, |\mathsf{Piv}_f| \,\big]\,,$$

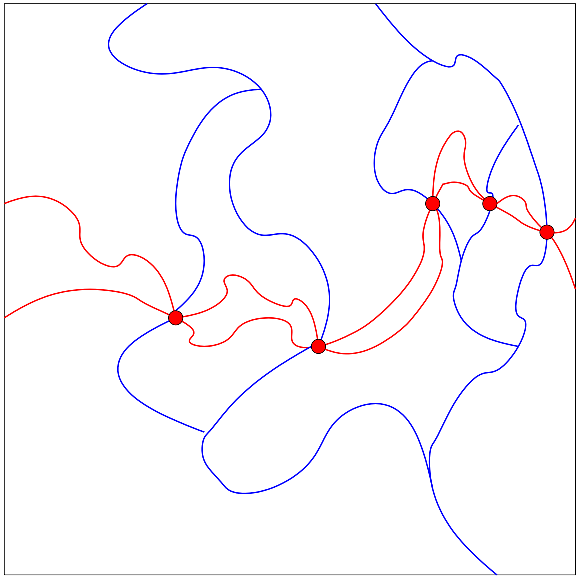
but apparently not for noise sensitivity.

Naive idea: how many pivotals are there?

A site is pivotal for left-right crossing in ω if it has the alternating 4-arm event to the sides. $\mathbf{E}|\mathsf{Piv}_n| \asymp n^2\,\alpha_4(n) \quad (= n^{3/4+o(1)})$.

Furthermore, $\mathbf{E}[\,|\mathsf{Piv}_n|^2\,] \leqslant C\,(\mathbf{E}|\mathsf{Piv}_n|)^2$. So, $\mathbf{P}\big[\,|\mathsf{Piv}_n| > \lambda \mathbf{E}|\mathsf{Piv}_n|\,\big] < C/\lambda^2$, any λ.

And not only $\exists\epsilon\ \mathbf{P}\big[\,|\mathsf{Piv}_n| > \epsilon\mathbf{E}|\mathsf{Piv}_n|\,\big] > \epsilon$, but $\mathbf{P}\big[\,0 < |\mathsf{Piv}_n| < \epsilon\mathbf{E}|\mathsf{Piv}_n|\,\big] \asymp \epsilon^{11/9+o(1)}$, as $\epsilon \to 0$ (exponent only for Δ).



If $\epsilon_n\,\mathbf{E}[\,|\mathsf{Piv}_n|\,] \to 0$, then $\mathbf{E}[\,\text{number of pivotal switches}\,] \to 0)$
$\Longrightarrow$ asymptotically full correlation

If $\epsilon_n\,\mathbf{E}[\,|\mathsf{Piv}_n|\,] \to \infty$, then $\mathbf{E}[\,\text{number of pivotal switches}\,] \to \infty$
$\not\Longrightarrow$ $\mathbf{P}[\,\text{hit (many) pivotals}\,] \to 1$ $\not\Longrightarrow$ asymptotic independence!!

Dynamical 2nd Moment Method for pivotal switches

Make $m = t/\alpha_4(n)$ switches, $\omega = \omega_0, \ldots, \omega_m$, so that

$$\mathbf{E}[\, S_t \,] = \mathbf{E}[\,\text{number of pivotal switches}\,] \asymp t\,.$$

Want to show $\mathbf{P}[S_t = \text{odd}] > c_t > 0$, uniformly in n, because then $\text{Corr}\big[\, f(\omega), f(\omega_m)\,\big] < 1 - \tilde{c}_t$.

Will prove $\mathbf{E}[\, S_t^2 - S_t \,] = O(t^2)$.

Then note $\mathbf{E}[\, S_t^2 \,] \geqslant \mathbf{P}[\, S_t = 1 \,] + 2\big(\mathbf{E}S_t - \mathbf{P}[\, S_t = 1 \,]\big)$. Rearranging gives $\mathbf{P}[\, S_t = 1 \,] \geqslant 2\mathbf{E}[\, S_t \,] - \mathbf{E}[\, S_t^2 \,] \asymp t - O(t^2) > 0$ for $t > 0$ small enough. Also done for all t, since correlation is monotone decreasing in t.

And the second moment calculation:

$$
\begin{aligned}
\mathbf{E}[\, S_t^2 - S_t \,] &\asymp \sum_{\substack{i,j=1\\ i\neq j}}^{m} \sum_{\substack{x,y\in V_n\\ x\neq y}} n^{-4}\, \mathbf{P}\big[\, x \in \mathsf{Piv}(\omega_i);\ \ y \in \mathsf{Piv}(\omega_j)\,\big] \\
&\leqslant n^{-4} \sum_{i,j=1}^{m} \sum_{r=0}^{\lceil \log_2 n\rceil} \sum_{\substack{x,y\\ 2^r \leqslant d(x,y) < 2^{r+1}}} \mathbf{P}\big[\, A_x^r(\omega_i)\,\big] \mathbf{P}\big[\, B_y^r(\omega_j) \mid A_x^r(\omega_i)\,\big] \\
&\leqslant n^{-4}\, m^2\, n^2 \sum_{r=0}^{\lceil \log_2 n\rceil} O(1)\, 2^{2r} \alpha_4(1,2^r)^2\, \alpha_4(2^r,n) \\
&\leqslant n^{-2}\, m^2\, \alpha_4(n) \sum_{r=0}^{\lceil \log_2 n\rceil} O(1)\, 2^{2r} \alpha_4(2^r)\,,\ \ \text{recall } k^2\alpha_4(k) = k^{3/4+o(1)}, \\
&\asymp O(1)\, \alpha_4(n)^2\, m^2 = O(t^2)\,,
\end{aligned}
$$

where

$$A_x^r(\omega_i) := \{\text{alternating 4 arms in } A_x(1,2^{r-1}) \text{ and in } A_x(2^{r+2},n) \text{ in } \omega_i\},$$

$$B_y^r(\omega_j) := \{\text{alternating 4 arms in } A_y(1,2^{r-1}) \text{ in } \omega_j\}.$$

Same for weakly dependent input?

This was a robust argument! Critical FK-Ising and critical spin-Ising on $\mathbb{Z}^2$ also satisfy

$$\mathbf{P}\big[\, B_y^r(\omega_j) \,\big|\, A_x^r(\omega_i) \,\big] \asymp \alpha_4(1, 2^r)\,,$$

for their natural Glauber / heat-bath / Gibbs sampler dynamics.

Moreoever, the exponents $\alpha_4^{\text{FK-Ising}}(n) = n^{-35/24+o(1)}$ and $\alpha_4^{\text{spin-Ising}}(n) = n^{-21/8+o(1)}$ are known (Chelkak, Duminil-Copin, Hongler, Garban).

For FK-Ising, because of $35/24 < 2$, which means many pivotal points in the discrete world and self-touches of SLE$(16/3)$ in the continuum, the previous argument works fine. Hence $t\, n^{-13/24+o(1)}$ is the good space-time scaling to watch macroscopic connections start changing, and Garban-P (2015+) proves that there is a Markovian scaling limit of the dynamics.

For spin-Ising Glauber dynamics, because of $21/8 > 2$, we do not know anything. Btw, the mixing time of the *entire* system is known to be polynomial (Lubetzky-Sly 2010), but the exponent is not known.

Noise sensitivity of percolation

All results use Fourier analysis of Boolean functions:

Theorem (Benjamini, Kalai & Schramm 1998). If $\epsilon > 0$ is fixed, and f_n is the indicator function for a left-right percolation crossing in an $n \times n$ square, then as $n \to \infty$

$$\mathbf{E}\big[\, f_n(\omega)\, f_n(\omega^\epsilon)\,\big] - \mathbf{E}\big[\, f_n(\omega)\,\big]^2 \to 0.$$

This holds for all $\epsilon = \epsilon_n > c/\log n$.

Theorem (Schramm & Steif 2005). Same if $\epsilon_n > n^{-a}$ for some positive $a > 0$. If triangular lattice, may take any $a < 1/8$.

Theorem (Garban, P & Schramm 2008). Same holds if and only if $\epsilon_n\, \mathbf{E}\big[\,|\mathsf{Piv}_n|\,\big] \to \infty$. For triangular lattice, this threshold is $\epsilon_n = n^{-3/4+o(1)}$.

What is the Fourier spectrum and why is it useful?

$f_n : \{\pm 1\}^{V_n} \longrightarrow \{\pm 1\}$ indicator of left-right crossing, $V = V_n$ vertices.

$(N_\epsilon f)(\omega) := \mathbf{E}[\, f(\omega^\epsilon) \mid \omega \,]$ is the noise operator, acting on the space $L^2(\Omega, \mu)$, where $\Omega = \{\pm 1\}^V$, μ uniform measure, inner product $\mathbf{E}[\, fg \,]$.

Correlation: $\mathbf{E}[\, f(\omega^\epsilon) f(\omega) \,] - \mathbf{E}[\, f(\omega) \,]\mathbf{E}[\, f(\omega^\epsilon) \,] = \mathbf{E}[\, f(\omega) N_\epsilon f(\omega) \,] - \mathbf{E}[\, f(\omega) \,]^2$. So, we would like to diagonalize the noise operator N_ϵ.

Let χ_i be the function $\chi_i(\omega) = \omega(i)$, $\omega \in \Omega$.

For $S \subset V$, let $\chi_S := \prod_{i \in S} \chi_i$, the parity inside S. Then

$$N_\epsilon \chi_i = (1-\epsilon)\, \chi_i \,; \qquad N_\epsilon \chi_S = (1-\epsilon)^{|S|}\, \chi_S .$$

Moreover, the family $\{\chi_S,\ S \subseteq V\}$ is an orthonormal basis of $L^2(\Omega, \mu)$.

Any function $f \in L^2(\Omega, \mu)$ in this basis (Fourier-Walsh series):

$$\hat{f}(S) := \mathbf{E}[\, f \chi_S \,]\, ; \qquad f = \sum_{S \subseteq V} \hat{f}(S)\, \chi_S\, .$$

The correlation:

$$\mathbf{E}[\, f N_\epsilon f \,] - \mathbf{E}[\, f \,]^2 = \sum_{S} \sum_{S'} \hat{f}(S)\, \hat{f}(S')\, \mathbf{E}\big[\, \chi_S\, N_\epsilon \chi_{S'} \,\big] - \mathbf{E}[\, f \chi_\emptyset \,]^2$$

$$= \sum_{\emptyset \neq S \subseteq V} \hat{f}(S)^2\, (1-\epsilon)^{|S|} = \sum_{k=1}^{|V_n|} (1-\epsilon)^k \sum_{|S|=k} \hat{f}(S)^2.$$

By Parseval, $\sum_S \hat{f}(S)^2 = \mathbf{E}[\, f^2 \,] = 1$. So can define probability measure $\mathbf{P}\big[\, \mathscr{S}_f = S \,\big] := \hat{f}(S)^2 / \mathbf{E}[\, f^2 \,]$, the spectral sample $\mathscr{S}_f \subseteq V$.

If, for some functions f_n and numbers k_n, we have $\mathbf{P}\big[0 < |\mathscr{S}_n| < t k_n \big] \to 0$ as $t \to 0$, uniformly in n, then $(1-\epsilon)^k \sim \exp(-\epsilon k)$ implies that for $\epsilon_n \gg 1/k_n$ we have asymptotic independence. Maybe with $k_n = \mathbf{E}|\mathscr{S}_n|$?

Pivotals versus spectral sample

$\nabla_i f(\omega) := f(\sigma_i(\omega)) - f(\omega) \in \{-2, 0, +2\}$ gradient.

$\nabla_i f(\omega) = \sum_S \hat{f}(S)[\chi_S(\sigma_i(\omega)) - \chi_S(\omega)]$, hence $\widehat{\nabla_i f}(S) = -2\hat{f}(S)\mathbf{1}_{i\in S}$.

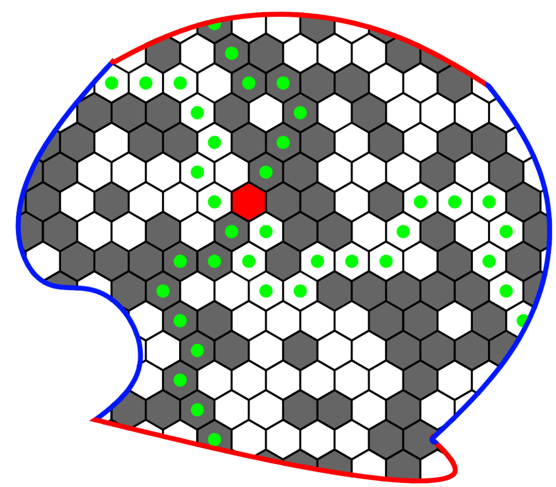
$\mathbf{P}[\, i \in \mathsf{Piv}_f \,] = \frac{1}{4}\|\nabla_i f\|_2^2 = \frac{1}{4}\sum_S \widehat{\nabla_i f}(S)^2 = \sum_{S\ni i} \hat{f}(S)^2 = \mathbf{P}[\, i \in \mathscr{S}_f \,]$.

Thus, $\mathbf{E}|\mathscr{S}_f| = \mathbf{E}|\mathsf{Piv}_f|$. So, the pivotal upper bound for noise sensitivity is sharp if there is tightness around $\mathbf{E}|\mathscr{S}|$.

Alos, $\mathbf{P}[\, i, j \in \mathsf{Piv}_f \,] = \mathbf{P}[\, i, j \in \mathscr{S}_f \,]$, hence $\mathbf{E}|\mathscr{S}_f|^2 = \mathbf{E}|\mathsf{Piv}_f|^2$.

Not for more points and higher moments! Both random subsets measure the "influence" or "relevance" of bits, but in different ways.

For percolation, $\mathbf{E}[\, |\mathsf{Piv}_n|^2 \,] \leqslant C\,(\mathbf{E}|\mathsf{Piv}_n|)^2$, hence $\exists c > 0$ s.t. $\mathbf{P}\big[\, |\mathscr{S}_n| > c\,\mathbf{E}|\mathscr{S}_n| \,\big] > c$. That's why one hopes for tightness around mean.



The earlier simple examples

Dictator$_n(\omega_1, \ldots, \omega_n) := \omega_1$. Here $p_c(n) = 1/2$.
Here Corr$\big[$ Dic$_n(\omega)$, Dic$_n(\omega^\epsilon)\big] = 1 - \epsilon$, hence noise-stable. And $\mathbf{P}\big[\mathscr{S}_n = \{x_1\}\big] = 1$.

Majority$_n(\omega_1, \ldots, \omega_n) := \mathrm{sgn}\,(\omega_1 + \cdots + \omega_n)$. Again, $p_c(n) = 1/2$.
Here Corr$\big[$ Maj$_n(\omega)$, Maj$_n(\omega^\epsilon)\big] = 1 - O(\sqrt{\epsilon})$, hence noise-stable.
And $\mathbf{P}\big[\mathscr{S}_n = \{x_i\}\big] \asymp 1/n$, most of the weight is on singletons.
On the other hand, $\mathbf{E}|\mathscr{S}_n| = \mathbf{E}|\mathrm{Piv}_n| \asymp \frac{1}{\sqrt{n}}\, n \asymp \sqrt{n}$.

Parity$_n(\omega_1, \ldots, \omega_n) := \omega_1 \cdots \omega_n$. Again, $p_c(n) = 1/2$.
Here Corr$\big[$ Par$_n(\omega)$, Par$_n(\omega^\epsilon)\big] = (1 - \epsilon)^n$, very sensitive to noise.
And $\mathbf{P}\big[\mathscr{S}_n = \{x_1, \ldots, x_n\}\big] = 1$.

Benjamini, Kalai & Schramm 1998

Using hypercontractivity of the noise operator N_ϵ:

Theorem 1. A sequence f_n of monotone Boolean functions is noise sensitive, iff it is asymptotically uncorrelated with all weighted majorities $\mathrm{Maj}_w(\omega_1, \ldots, \omega_n) = \mathrm{sign} \sum_{i=1}^n \omega_i w_i$.

Theorem 2. A sequence f_n of monotone Boolean functions is noise sensitive at density $p = 1/2$, iff $\sum_x \mathbf{P}[\, x \in \mathsf{Piv}_n \,]^2 \to 0$ as $n \to 0$.

Extended to $p_c(n) \asymp 1/\mathrm{poly}(\log n)$ by Keller-Kindler '10 and Bouyrie '14.

Corollary. The left-right percolation crossing in an $n \times n$ square is noise sensitive, even with $\epsilon = \epsilon_n > c/\log n$.

Schramm & Steif 2005

Theorem. If $f : \Omega \longrightarrow \mathbb{R}$ can be computed with a randomized algorithm with revealment δ (each bit is read only with probability $\leqslant \delta$), then

$$\sum_{S:|S|=k} \hat{f}(S)^2 \leqslant \delta\, k\, \|f\|_2^2 \,.$$

For left-right crossing in $n \times n$ box on the hexagonal lattice, exploration interface with random starting point gives revealment $n^{-1/4+o(1)}$ (it has length $n^{7/4+o(1)}$, given by 2-arm exponent), while $\sum_{k \leqslant m} k \asymp m^2$, thus:

Corollary. Left-right crossing on the triangular lattice is noise sensitive under $\epsilon_n > n^{-a}$, with any $a < 1/8$. Even on square lattice, can take some positive $a > 0$.

The revealment is at least $n^{-1/2+o(1)}$ for *any* algorithm computing the crossing (O'Donnell-Servedio '07, plus $n^{3/4}$ pivotals), hence this method can give only $n^{-1/4+o(1)}$-sensitivity, far from the conjectured $\epsilon_n = n^{-3/4+o(1)}$.

Garban, P. & Schramm 2008

Although goal is to understand size, Gil Kalai suggested trying to understand entire distribution of $\mathscr{S}_f$. A strange random set of bits.

[Smirnov '01] + [Tsirelson '04] + [Schramm-Smirnov '10] implies that it has a conformally invariant scaling limit.

How to prove tightness for the size of strange random fractal-like sets?

For $A \subseteq V$: $\mathbf{E}\big[\,\chi_S \mid \mathscr{F}_A\,\big] = \begin{cases} \chi_S & S \subseteq A\,, \\ 0 & \text{otherwise}\,. \end{cases}$

Therefore, $\mathbf{E}\big[\,f \mid \mathscr{F}_A\,\big] = \sum_{S \subseteq A} \widehat{f}(S)\,\chi_S$, a nice projection.

$$\mathbf{P}\big[\,\mathscr{S}_f \subset U\,\big] = \sum_{S \subset U} \hat{f}(S)^2 = \mathbf{E}\Big[\Big(\sum_{S \subset U} \hat{f}(S)\,\chi_S\Big)^2\Big] = \mathbf{E}\Big[\mathbf{E}\big[\,f \mid \mathscr{F}_U\,\big]^2\Big].$$

From this, for disjoint subsets A and B, can try to give percolation meaning to $\mathbf{P}\big[\,\mathscr{S}_f \cap B \neq \emptyset = \mathscr{S}_f \cap A\,\big]$. Very restricted independence.

Influence notions of subsets

Besides resampling a random subset U_ϵ, could also use any fixed subset U.

Influence:

$$I(U) := \mathbf{P}[\, U \text{ is pivotal}\,] = \mathbf{P}[\, U^c \text{ does not determine the value of } f\,].$$

Significance: the amount of information missing if the bits in U^c are known,

$$\text{sig}(U) := \frac{\mathbf{E}\big[\,\mathrm{Var}[f \mid \mathscr{F}_{U^c}]\,\big]}{\mathrm{Var} f} = \frac{\mathbf{P}[\,\mathscr{S} \cap U \neq \emptyset\,]}{\mathrm{Var} f}.$$

Clue: the amount of information we gain from the bits of U,

$$\text{clue}(U) := \frac{\mathrm{Var}\big[\mathbf{E}[\, f \mid \mathscr{F}_U\,]\big]}{\mathrm{Var} f} = \frac{\mathbf{P}[\,\emptyset \neq \mathscr{S} \subseteq U\,]}{\mathrm{Var} f}.$$

Clearly, $I(U) \geqslant \text{sig}(U) \geqslant \text{clue}(U)$. Also, $\text{clue}(U) = 1 - \text{sig}(U^c)$.

Influence notions of subsets

A few examples:

U_n is all the vertical edges in $\mathbb{Z}^2$ bond percolation.
Then $\mathrm{sig}(U_n) \to 1$ and $\mathrm{clue}(U_n) \to 0$.

U_n has a scaling limit of Hausdorff-dimension γ:
If $\gamma < 5/4$, then $\mathrm{sig}(U_n) \to 0$, or $\mathrm{clue}(U_n^c) \to 1$.
If $5/4 < \gamma < 2$, then usually $\mathrm{sig}(U_n) \to 1$, but also $\mathrm{sig}(U_n^c) \to 1$, hence $\mathrm{clue}(U_n) \to 0$.

For Majority$_n$, any subset U of size ϵn has $\mathrm{sig}(U) \asymp \sqrt{\epsilon}$.

Benjamini: does $|U_n| = o(n^2)$ imply also for percolation that $\mathrm{clue}(U_n) \to 0$?

If we can choose the revealed bits *adaptively*, then, using the exploration interface, $n^{7/4+o(1)}$ bits suffice.

Small subsets are clueless (Galicza & P.)

If $f : \{\pm 1\}^V \longrightarrow \{\pm 1\}$ is *transitive*, and $U \subset V$, then

$$\text{clue}(U) \cdot \mathrm{Var} f = \mathbf{P}[\,\emptyset \neq \mathscr{S} \subseteq U\,] \leqslant \mathbf{P}[\,X \in U\,] = \sum_{u \in U} \mathbf{P}[\,X = u\,] = \frac{|U|}{|V|}.$$

Percolation left-right crossing is *not transitive*, but if we consider an $n \times n$ torus, then the indicator F of the event that $\{$there is a translate of the square on the torus where the left-right crossing happens$\}$ is transitive.

Now F_ϵ, the indicator of $\{$there is a good translate by some vector $(k, \ell)\epsilon n$, $k, \ell \in \{0, 1, \dots, 1/\epsilon\}\}$ is a good approximation to F, because of low probability of half-plane 3-arm events etc.

If a small subset U had a clue about left-right crossing in a square, then its $1/\epsilon \times 1/\epsilon$ translates, using FKG, would have a clue about F_ϵ, so also about F, which is impossible, since $1/\epsilon^2 |U|$ is still small.

The clue of random subsets

It Ain't over till it's over (Mossel, O'Donnell, Oleszkiewicz '10). For any density $0 < \rho < 1$ and $\epsilon > 0$ there is a δ and a τ such that for every Boolean function f satisfying $\mathbf{P}[\, i \in \mathsf{Piv}_f \,] \leqslant \tau$ for all $i \in V$, we have

$$\mathbf{P}[\, \mathsf{clue}(U_\rho) \geqslant 1 - \delta \,] < \epsilon\,.$$

This is a deep result, though obvious for noise sensitive functions. Our easy method gives the result when $\sum_i \mathbf{P}[\, i \in \mathsf{Piv}_f \,]^2 = o(1)$, via an Azuma-Hoeffding concentration argument:

$$\mathbf{E}[\, \mathsf{clue}(U_\rho) \,] \cdot \mathrm{Var} f = \mathbf{P}[\, \emptyset \neq \mathscr{S} \subseteq U_\rho \,] \leqslant \mathbf{P}[\, X \in U_\rho \,] = \rho.$$

And if U and U' differ only in bit i, then

$$|\mathsf{clue}(U) - \mathsf{clue}(U')| \leqslant \mathbf{P}[\, i \in \mathscr{S} \,] = \mathbf{P}[\, i \in \mathsf{Piv} \,]\,.$$

Hence we get concentration around the mean on the scale $\sum_i \mathbf{P}[\, i \in \mathsf{Piv} \,]^2$.

Bootstrap percolation on infinite graphs

Infinite transitive graph. Start with Bernoulli(p) percolation of occupied sites. If a site has at least k occupied neighbors, then gets occupied. Repeat ad infinitum. For what p do we occupy the entire graph, with probability 1?

van Enter '87, Schonmann '90. $p_c(\mathbb{Z}^d, k) = 0$ for $k \leqslant d$, and 1 for $k > d$.

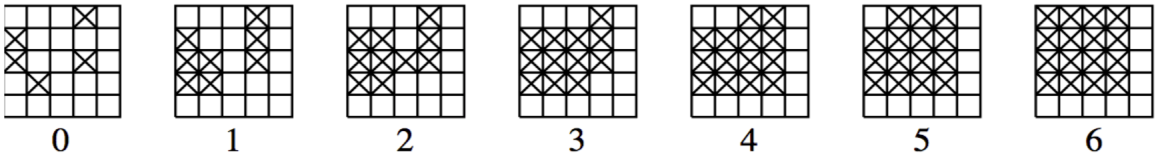
Probably also true for the Heisenberg group (observation by Rob Morris).

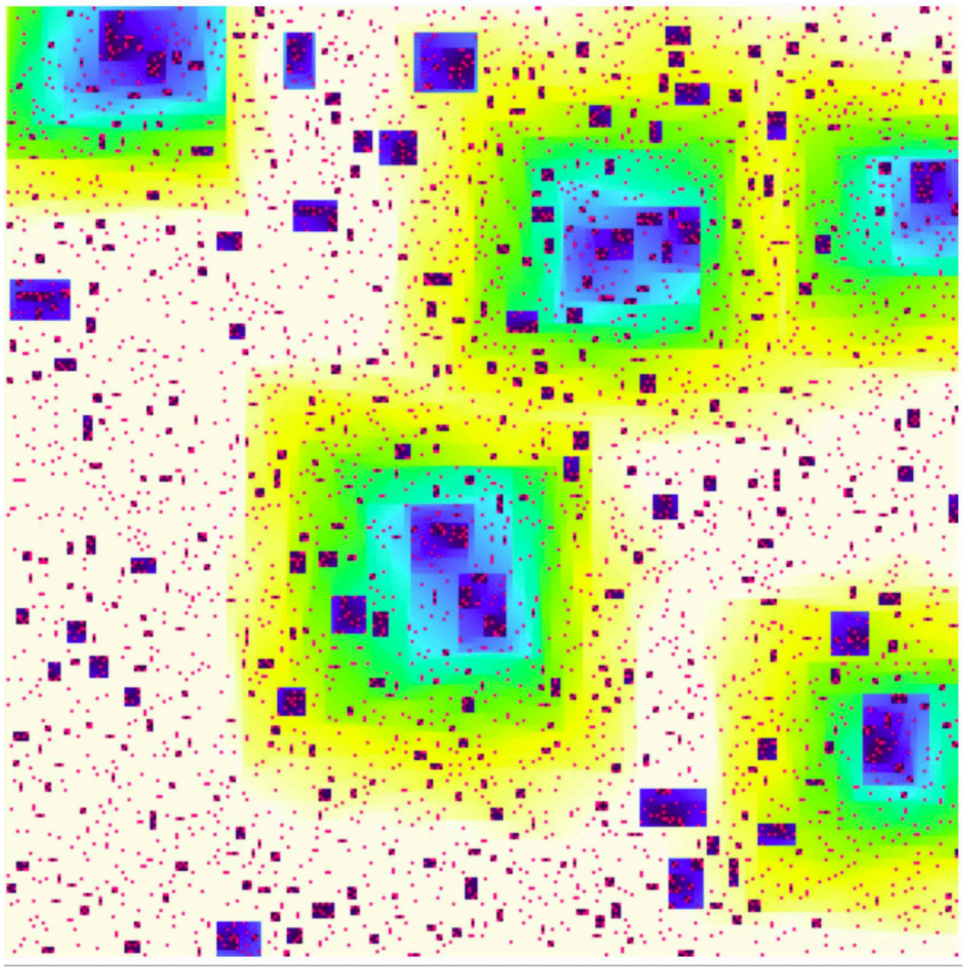
Balogh, Peres & P. '06. For any non-amenable group Γ that has a free subgroup F_2, there exists a symmetric finite generating set and a k-neighbor rule such that $G = \mathrm{Cayley}(\Gamma, S)$ has $0 < p_c(G, k) < 1$.

On d-regular tree $p_c(T_d, k)$ can be explicitly calculated, since unsuccessful occupation is equivalent to having a $d - k + 1$- regular vacant subtree in the initial configuration.

Question. What about non-amenable graph without free subgroups? Conversely, on amenable transitive graphs, does $p_c < 1$ imply $p_c = 0$?

Bootstrap percolation on infinite graphs





Bootstrap percolation on finite graphs

Aizenman, Lebowitz '88. $p_c(\mathbb{Z}_n^d, 2) \asymp \frac{1}{\log^{d-1} n}$.

More general and sharper results by Cerf, Cirillo, Manzo, Holroyd, Balogh, Bollobás, Morris, Duminil-Copin, ...

Balogh, Pittel '07. For the random d-regular graph on n vertices, $p_c(G_{n,d}, k) \to p_c(T_d, k)$ for any $2 \leqslant k \leqslant d-2$. The threshold window around $p_c(n)$ is of size $O(1/\sqrt{n})$.

Bartha & P. '14.
(1) At $p_c(\mathbb{Z}_n^d, 2)$, the process is noise sensitive.
(2) At $p_c(G_{n,d}, k)$, the process is noise insensitive.

Rough sketch of sensitivity on $\mathbb{Z}_n^2$

Will use the condition that $\sum_i \mathbf{P}[\, i \in \mathsf{Piv}\,]^2$ is small.

Complete occupation is roughly equivalent to having an internally spanned rectangular seed of side lengths $\asymp \log n$, because this will keep growing.

Hence, again using an "almost transitivity" argument,

$$\mathbf{P}[\, i \in \mathsf{Piv} \text{ and } \omega_i = 1\,] \leqslant \frac{C\,\log^2 n}{n^2}.$$

And we have $\mathbf{P}[\, i \in \mathsf{Piv} \text{ and } \omega_i = 1\,] \asymp \frac{1}{\log n}\mathbf{P}[\, i \in \mathsf{Piv} \text{ and } \omega_i = -1\,]$. So, altogether,

$$\mathbf{P}[\, i \in \mathsf{Piv}\,] \leqslant \frac{C\,\log^3 n}{n^2}\,.$$

Should hold for $k > 2$, too, but seeds are much more complicated.

Revealment does *not* work: for negative result, need to reveal a lot.

Sketch of stability on $G_{n,d}$

The very narrow threshold window $O(1/\sqrt{n})$ implies that it is correlated with a weighted majority: if majority at level $p_c(n)$ is fulfilled, then with positive probability we overshoot by density $O(1/\sqrt{n})$, but that raises the probability of complete occupation noticeably.

But this used the Balogh-Pittel explicit ODE calculation, relying on the lot of independent randomness in $G_{n,d}$.

A more intuitive reason for (2) is that it should hold for many expander graphs: for $2d$-regular expanders, with $k \geqslant d$, by the perimeter trick, all positive witnesses have size at least $c\,n$, and this is often also true for the negative witnesses, which already should imply insensitivity — that would be a converse to the revealment method.

However, there are expanders where each vertex is part of large piece of a square lattice, hence $p_c(G_n, 2) \to 0$. I do not know whether to expect noise stability or not.