PERCOLATION IN THE PLANE, ON $\mathbb{Z}^d$, AND BEYOND

Homework problems

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To clarify what the measurability of having an infinite cluster means:

▷ **Exercise 1.** *Let $\Gamma(V,E)$ be any bounded degree infinite graph, and $S_n \nearrow V$ an exhaustion by finite connected subsets. Is it true that, for $p > p_c(\Gamma)$, we have*

$$\lim_{n\to\infty} \mathbf{P}_p[\,\textit{largest cluster for percolation inside } S_n \textit{ is the subset of an infinite cluster}\,] = 1\,?$$

▷ **Exercise 2.**

(a) *Find the edge Cheeger constant of the infinite binary tree.*

(b) *Show that a bounded degree tree is amenable iff there is no bound on the length of "hanging chains", i.e., chains of vertices with degree 2.*

(c) *Construct a bounded degree infinite tree with exponential growth that does not satisfy $IP_{1+\epsilon}$ for any $\epsilon > 0$, moreover, it is recurrent for simple random walk and has $p_c = 1$.*

The archetypical examples for the difference between amenable and non-amenable graphs are the Euclidean versus hyperbolic lattices, e.g., tilings in the Euclidean versus hyperbolic plane. The notions "non-amenable", "hyperbolic", "negative curvature" are very much related to each other, but there are also important differences. Here is a down-to-earth exercise to practice these notions; it might not be obvious at first sight, but part (a) is a special case of part (b).

▷ **Exercise 3.**

(a) *Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.*

(b) *In a locally finite planar graph Γ, define the* **combinatorial curvature** *at a vertex x by*

$$\mathsf{curv}_\Gamma(x) := 2\pi - \sum_i \frac{(L_i - 2)\,\pi}{L_i}\,,$$

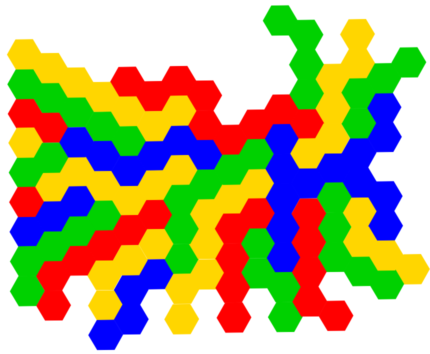

Figure 1: Trying to create at least 7 neighbours for each country.

where the sum runs over the faces adjacent to x, and L_i is the number of sides of the i^{th} face. Show that if there exists some $\delta > 0$ such that curvature is less than $-\delta\pi$ at each vertex, then it is not possible that both Γ and its planar dual Γ^* are edge-amenable.

The following exercise is used in my PGG notes to prove that the Følner nonamenability of a Cayley graph implies that the group is nonamenable in von Neumann's sense:

▷ **Exercise 4.*** *Let X be a metric space. $\varphi : X \to X$ is wobbling if $\sup_x d(X, \varphi(x)) < K$. Further, if $\Gamma = (V, E)$ is a graph, then the maps α and β are a paradoxical decomposition of Γ if they are wobbling injections such that $\alpha(V) \sqcup \beta(V) = V$.*

Show that a bounded degree graph is nonamenable if and only if there exists a wobbling paradoxical decomposition. (Hint: State, prove and use the locally finite infinite bipartite graph version of the Hall marriage theorem, called the Hall-Rado theorem.)

Recall that there is a topological **space of ends** of a graph, and that any quasi-isometry of graphs induces naturally a homeomorphism of their spaces of ends. In particular, the number of ends is a quasi-isometry invariant of the graph, and we can define the space of ends of a finitely generated group to be the space of ends of any of its Cayley graphs.

▷ **Exercise 5.** *Show that a group has two ends iff it has $\mathbb{Z}$ as a finite index subgroup.*

▷ **Exercise 6.**

(a) *Show that if Γ_1, Γ_2 are two infinite graphs, then the direct product graph $\Gamma_1 \times \Gamma_2$ has one end.*

(b) *Show that if $|G_1| \geq 2$ and $|G_2| \geq 3$ are two finitely generated groups, then the free product $G_1 * G_2$ has a continuum number of ends.*

▷ **Exercise 7.** *Show that a finitely generated group with at least 3 ends must in fact have a continuum number of ends, and it must be non-amenable.*

▷ **Exercise 8.**

(a) *Assume that for some $H \triangleleft G$, both H and G/H are finitely presented. Show that G is also finitely presented.*

(b) *Show that any finitely generated almost-nilpotent group is finitely presented.*

▷ **Exercise 9.** * *Show that the Cayley graph of the lamplighter group* $G = \mathbb{Z}_2 \wr \mathbb{Z}$ *with generating set* $S = \{\mathsf{R}, \mathsf{Rs}, \mathsf{L}, \mathsf{sL}\}$ *is isomorphic to the Diestel-Leader graph* $\mathsf{DL}(2,2)$*, see Example 7.2 in the Lyons-Peres book, and satisfies* $\mathsf{CutCon}_{\Gamma(G,S)} = \infty$*.*

(There used to be an exercise here saying that still $p_c(\Gamma) < 1$ *with some generating set, by finding a Fibonacci subtree, but I've realized it's also in Lyons-Peres, sorry.)*

▷ **Exercise 10.** *

(a) *Let* T *be the infinite binary tree. Show that* $\mathrm{Aut}(T)$ *is not finitely generated.*

(b) *Find a finitely generated non-amenable subgroup of* $\mathrm{Aut}(T)$*.*

That transitivity is needed for $\theta(p_c)$=0 can be seen from the case of general trees:

▷ **Exercise 11.** *Consider a spherically symmetric tree* T *where each vertex on the* n^{th} *level* T_n *has* $d_n \in \{k, k+1\}$ *children, such that* $\lim_{n\to\infty} |T_n|^{1/n} = k$*, but* $\sum_{n=0}^{\infty} k^n/|T_n| < \infty$*. Using the second moment method, show that* $p_c = 1/k$ *and* $\theta(p_c) > 0$*.*

▷ **Exercise 12.** *Still with the first and second moment method, show that for* $\mathsf{Ber}(p)$ *percolation on a* d*-regular tree* T *with* $d \geq 3$*, the percolation probability is* $\theta(p_c + \epsilon) \asymp \epsilon$ *as* $\epsilon \searrow 0$*.*

Two exercises about unimodularity:

▷ **Exercise 13.**

(a) *Give an example of a unimodular transitive graph* Γ *such that there exist neighbours* $x, y \in V(\Gamma)$ *such that there is no graph-automorphism interchanging them.*

(b)* *Can you give an example with a Cayley graph?*

▷ **Exercise 14.** *Give an invariant percolation on a non-unimodular transitive graph with infinitely many trifurcation points a.s., but only finitely many in each infinite cluster.*

We had a characterization of amenability using high-marginal invariant percolation processes without infinite clusters. The following exercise shows that the bound we had is tight:

▷ **Exercise 15.** *Show that for the set of invariant bond percolations on the 3-regular tree* $\mathbb{T}_3$ *without an infinite cluster, the supremum of edge-marginals is 2/3. (Hint: the complement of a perfect matching has density 2/3 and consists of* $\mathbb{Z}$ *components.)*

A similar characterization of amenability is the following:

▷ **Exercise 16.** *Show that a Cayley graph* $\Gamma(G,S)$ *is amenable iff it has a* G*-invariant random spanning* $\mathbb{Z}$ *subgraph. (Hint: for one direction, produce the invariant* $\mathbb{Z}$ *using the high-marginal construction; for the other direction, produce an invariant mean from the invariant* $\mathbb{Z}$*.)*

Recall that for a transitive d-regular graph $\Gamma(V,E)$ and any $o \in V$, we defined

$$\delta^{\mathrm{erg}}(\Gamma) := \sup \left\{ \frac{\mathbf{E}_\mu \big| \{(o,x) \in E : \sigma(x) = \sigma(o)\} \big|}{d_\Gamma} : \begin{array}{l} \text{ergodic invariant measures } \mu \text{ on } \sigma \in \{\pm 1\}^V \\ \text{with } \mathbf{E}_\mu \sigma(o) = 0 \end{array} \right\},$$

and mentioned the theorem of Glasner and Weiss that a f.g. group G is Kazhdan iff any (or one) of its Cayley graphs Γ has $\delta^{\mathrm{erg}}(\Gamma) < 1$. We had a similar definition for tail-trivial measures, giving $\delta^{\mathrm{tt}}(\Gamma)$, and for factors of i.i.d. processes, giving $\delta^{\mathrm{fiid}}(\Gamma)$. Clearly, $\delta^{\mathrm{erg}}(\Gamma) \geq \delta^{\mathrm{tt}}(\Gamma) \geq \delta^{\mathrm{fiid}}(\Gamma)$.

▷ **Exercise 17.** *Show that $\delta^{\mathrm{erg}}(\mathbb{T}_3) = 1$. (Hint: free groups are not Kazhdan e.g. because they surject onto $\mathbb{Z}$.)*

▷ **Exercise 18.** ***
(a) *Find the value of $\delta^{\mathrm{fiid}}(\mathbb{T}_3)$.*
(b) *Show that $\delta^{\mathrm{tt}}(\mathbb{T}_3) < 1$.*

Pak and Smirnova-Nagnibeda constructed a Cayley graph $\Gamma(G,S)$ with $p_c(\Gamma) < p_u(\Gamma)$ for each nonamenable group G using that $\iota_E(\Gamma(G,S^k)) \to 1$ as $k \to \infty$, where S^k is the "k-ball with multiplicities". Can this be done without multiple edges? Consider the outer vertex Cheeger constant $h_V := \inf |\partial_V^{\mathrm{out}} S|/|S|$ and note that $|\partial_V^{\mathrm{out}} S| \leq |\partial_E S| \leq (d-1)|S|$ in a d-regular graph.

▷ **Exercise 19.** *Show that for any d-regular non-amenable graph Γ and any $\epsilon > 0$, there exists $K < \infty$ such that we can add edges connecting vertices at distance at most K, such that the new graph Γ^* will be d^*-regular, no multiple edges, and $\iota_V(\Gamma^*) := h_V(\Gamma^*)/d^*$ will be larger than $1-\epsilon$. (Hint: use the wobbling paradoxical decomposition from Exercise 4. The Mass Transport Principle shows that this proof cannot work in a group-invariant way.)*

▷ **Exercise 20.** ***
(a) *Is it true that $\iota_E(\Gamma(G,B_k^S))/|B_k^S| \to 1$ as $k \to \infty$ for any nonamenable group G and the ball of radius k in any finite generating set S?*
(b) *Is it true that $\iota_V(\Gamma(G,B_k^S))/|B_k^S| \not\to 1$ for any group G and any finite generating set S?*

Regarding the characterization of uniqueness of infinite clusters in an invariant percolation ω via $\inf_{x,y\in V(\Gamma)} \mathbf{P}\big[\, x \stackrel{\omega}{\longleftrightarrow} y \,\big] > 0$, here are two exercises:

▷ **Exercise 21.** *Give an example of a* $\mathsf{Ber}(p)$ *percolation on a Cayley graph Γ that has non-uniqueness, but there is a sequence $x_n \in V(\Gamma)$ with $\mathrm{dist}(x_0,x_n) \to \infty$ and $\inf_n \mathbf{P}_p[\, x_0 \longleftrightarrow x_n \,] > 0$.*

▷ **Exercise 22.** * *Give an example of an ergodic uniformly insertion tolerant invariant percolation on $\mathbb{Z}^2$ with a unique infinite cluster but $\inf_{x,y\in\mathbb{Z}^2} \mathbf{P}\big[\, x \stackrel{\omega}{\longleftrightarrow} y \,\big] = 0$. (Hint: you can use the ideas of* `arXiv:1011.2872 [math.PR]`*.)*

▷ **Exercise 23.** *Fill in the missing details in either proof sketches in PGG for $p_u < 1$ for Kazhdan groups.*

▷ **Exercise 24.** *** *Prove $p_c < p_u$ for Kazhdan groups by finding an appropriate representation.*

▷ **Exercise 25.** *Show that in Schramm's proof of $\mathbf{P}[X_0 \stackrel{p_c}{\longleftrightarrow} X_n] \leq 2\rho^n$ for unimodular non-amenable graphs, the auxiliary percolation process ξ on the tree $\mathbb{T}_{m+1}$ indexing the branching random walk is* $\mathrm{Aut}(\mathbb{T}_{m+1})$*-invariant.*

Starting percolation in the plane:

▷ **Exercise 26.** *Assuming the fact that at least one type of crossing is present in any two-colouring of the $n \times n$ rhombus in the hexagonal grid, prove Brouwer's fixed point theorem in two dimensions. (Show, by the way, that Brouwer's theorem needs the ball to be closed.)*

▷ **Exercise 27.** *Assuming the main RSW inequality (12.13) in PGG, complete the proof of the full RSW proposition, and explain what to change so that the proof works for bond percolation on $\mathbb{Z}^2$.*

▷ **Exercise 28.** *Assuming quasi-multiplicativity and the bounds*

$$c\,(r/R)^{2-\epsilon} < \alpha_4(r,R) < C\,(r/R)^{1+\epsilon}$$

and

$$\alpha_2^+(r,R) \asymp r/R\,, \qquad \alpha_3^+(r,R) \asymp (r/R)^2\,,$$

prove the second moment estimate $\mathbf{E}\big[\,|\mathrm{Piv}(\mathcal{Q}_n)|^2\,\big] \le C\,\big(\mathbf{E}|\mathrm{Piv}(\mathcal{Q}_n)|\big)^2$ for any piecewise smooth quad $\mathcal{Q}$, with $C = C_{\mathcal{Q}}$.

▷ **Exercise 29.** *Consider recursive 3-majority with depth h, with uniform distribution on the 3^h input variables. Show that the set of pivotals is the leaves of a GW tree with offspring distribution $\mathbf{P}\big[\,\pi = 0\,\big] = 1/4$ and $\mathbf{P}\big[\,\pi = 2\,\big] = 3/4$, and the spectral sample is the leaves of a GW tree with offspring distribution $\mathbf{P}\big[\,\sigma = 1\,\big] = 3/4$ and $\mathbf{P}\big[\,\sigma = 3\,\big] = 1/4$. Note that $\mathbf{E}[\,\pi\,] = \mathbf{E}[\,\sigma\,] = 3/2$ and $\mathbf{E}[\,\pi^2\,] = \mathbf{E}[\,\sigma^2\,] = 3$, and hence $\mathbf{E}\big[\,|\mathrm{Piv}_h|^i\,\big] = \mathbf{E}\big[\,|\mathrm{Spec}_h|^i\,\big]$ for $i = 1, 2$.*

▷ **Exercise 30.** *Show that the spectral result*

$$\mathbf{P}\big[\,0 < |\mathrm{Spec}(f_n)| < \lambda\,\mathbf{E}|\mathrm{Spec}(f_n)|\,\big] = \lambda^{2/3+o(1)}$$

for the spectral sample of left-right crossing f_n in the $n \times n$ square for critical percolation on the triangular lattice implies the decorrelation

$$\mathbf{E}\big[\,f_n(\omega_0)\,f_n(\omega_{t\epsilon(n)})\,\big] - \mathbf{E}[\,f_n\,]^2 = t^{-2/3+o(1)}$$

as $t \to \infty$, uniformly in n, for $\epsilon(n) = 1/\mathbf{E}|\mathrm{Spec}(f_n)| = 1/\mathbf{E}|\mathrm{Piv}(f_n)| = n^{-3/4+o(1)}$.