Book review for Acta Sci. Math. (Szeged) on

G. Grimmett: *Percolation, Second edition,* Springer-Verlag, Berlin, 1999,

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Percolation theory has its origin in statistical physics, but by now it has become a cornerstone of the mathematics of large disordered systems, and it is one of the most important and lively subjects of probability theory. The basic question is from the 1950's: if you immerse a large porous stone in a bucket of water, what is the probability that the centre of the stone is wetted? The mathematical model is the following:

Choose randomly each edge of $\mathbb{Z}^d$ to be *open* with probability p, and to be *closed* with $1-p$, independently of each other. Consider the *open cluster* C of the origin: the vertices of $\mathbb{Z}^d$ that can be reached from the origin through open edges. The question is whether C is bounded or not, depending on the parameter p. For this, one introduces the following two functions: $\theta_d(p) = \mathbf{Pr}_p\{|C| = \infty\}$ and $\chi_d(p) = \mathbf{Exp}_p|C|$. As usual in statistical physics, we have a *phase transition phenomenon*: for the above functions there are critical values $p_H(d) = \sup\{p : \theta_d(p) = 0\}$ and $p_T(d) = \sup\{p : \chi_d(p) < \infty\}$, where the behaviour of these functions changes abruptly. Now there are statements which are clear for a physicist: $\theta_d(p)$ is a monotone non-decreasing continuous function of p, with a singularity at the critical point $p_H(d)$. There is exactly one non-trivial phase transition, so we have $0 < p_H(d) = p_T(d) = p_c(d) < 1$ for $d \geq 2$.

However, it took decades to prove some of these results mathematically, and still a number of challenging conjectures are left. In fact, it was only in 1980 when *H. Kesten* proved the long-standing conjecture $p_c(2) = 1/2$, and in 1986 when *M.V. Menshikov* established $p_T(d) = p_H(d)$ for all d. We know the continuity of $\theta_d(p)$ *apart from the critical point*, and the central open problem of the subject is the continuity at p_c. This is known for $d = 2$ using the topology of the plane (which is also the reason for knowing the exact value of $p_c(2)$), and for $d \geq 19$, where the graph $\mathbb{Z}^d$ can already be approximated locally by a $2d$-regular tree.

G. Grimmett, who is a well-known researcher in the subject, devotes his book almost completely to the above-mentioned questions of bond percolation on $\mathbb{Z}^d$. After a nice chapter of introductary definitions, results and questions, Chapter 2 is devoted to basic techniques, such as the FKG-inequality. Chapter 3 describes a very general method of the author to establish strict inequalities like $p_c(\mathbb{Z}^d, \text{site}) > p_c(\mathbb{Z}^d, \text{bond})$. The short Chapter 4 is about the average number of open clusters per vertex. Chapters 5 and 6 form a systematic study of the subcritical phase $p < p_c$, including e.g. the results of Menshikov, while Chapters 7 and 8 are devoted to the supercritical phase $p > p_c$. Chapters 9 and 10 describe the model near the critical point, and contain e.g. a sketch of the non-rigorous renormalization method of physics and of the rigorous results for $d \geq 19$. Chapter 11 is devoted to percolation in two dimensions, including the result $p_c(2) = 1/2$. The book terminates with two chapters of pencil sketches of related random processes, e.g. continuum percolation, fractal percolation, and the random-cluster model, which is the common generalization of percolation and the famous *Ising-model*.

The style of the book is clear and enjoyable, the proofs are the nicest available at this time. The arrangement of the chapters results in a very clear structure, which makes the book very handy for a graduate course, for self-study, or for researchers as a reference book. However, the strict direction of developing the material hides the motivation and origin of methods, hides the interplay of ideas with different models. For a beginner, for instance, it will be clear why the tools of Chapter 2 are so basic only after reading the whole book, and I am afraid that an average student in probability theory will not find out quickly what is his or her point in studying percolation. G. Grimmett himself writes: "I have tried to stay reasonably close to the core material. No critical reader will agree entirely with my selection, and physicists may sometime feel that my intuition is crooked." I think that percolation is a deep and interesting subject because of its very strong connections with random walks, potential theory, and geometry. For example, one of the most important open problems is the so-called universality and conformal invariance of two-dimensional percolation, which claims that percolation actually sees only the large-scale geometry of the underlying graph, i.e. it describes the geometry of the ambient space through combinatorics and probability. Fortunately, G. Grimmett is evidently aware of these recent and extremely lively directions, and the careful and splendid historical remarks and the almost complete list of references help a lot to cure these problems. Nevertheless, for an interested reader I would definitely suggest e.g. the homepage of *R. Lyons*, `php.indiana.edu/~rdlyons` for these related topics.

The second edition differs from the first one (published in 1989) through the reorganization of certain material, and through the inclusion of numerous fundamental results and techniques. With these substantial improvements this second edition is a very useful and nice book, which has been an object of desire till now.