Book review for Acta Sci. Math. (Szeged) on

Ernst Kleinert: *Units in Skew Fields,* (Progress in Mathematics Vol. 186), viii+80 pages, Birkhäuser Verlag, Basel, 2000.

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Let D be a skew field with center $\mathbb{Q}$, $\dim_{\mathbb{Q}} D = d^2$, and let us suppose that D splits over $\mathbb{R}$, i.e. $\mathbb{R}\otimes_{\mathbb{Q}} D = M_d(\mathbb{R})$, the matrix ring. So we can consider the homomorphism $\det : \mathbb{R}\otimes_{\mathbb{Q}} D \longrightarrow \mathbb{R}$, which has a surjective restriction $D^{\times} \longrightarrow \mathbb{Q}^{\times}$. The kernel $G(\mathbb{Q})$ of this map is an algebraic group, and $G(\mathbb{R}) = SL_d(\mathbb{R})$. Now let $\{\alpha_1, \dots, \alpha_{d^2}\}$ be a $\mathbb{Q}$-base of D consisting of integral elements; the free $\mathbb{Z}$-module Λ generated by them is called a maximal order. Using this base we can define G also over $\mathbb{Z}$, and it is not difficult to see that $|\Lambda^{\times} : G(\mathbb{Z})| = 2$. The book is a survey of results and methods concerning arithmetic, group theoretic and geometric properties of this group $\Gamma = G(\mathbb{Z})$, which are independent of the choice of Λ, since all maximal orders are conjugate in D.

The most important example is clearly the modular group $|SL_2(\mathbb{Z}) : \Gamma| = 2$, in the case of the quaternionic skew field. However, our group Γ can be defined even in much more general situations than the one considered above, and it is a central object in noncommutative arithmetics. It can be examined from a lot of different points of view, but these directions have been almost orthogonal so far. The intention of the author in writing this short monograph was to stimulate communication between these different approaches. To make the leading ideas more visible, the author restricted himself to the special case we described above. This keeps technicalities at the absolute minimum, and makes it possible to present full proofs of the most important results.

§0 is a collection of basic facts about the skew fields and their maximal orders in question. §1 contains Hey's theorem (1929) that $SL_d(\mathbb{R})/\Gamma$ is compact, and its immediate consequence that Γ is finitely generated. §2 presents the simplest form of the classical Siegel-Weyl reduction theory, which yields that Γ is also finitely presented. §3 brings Weil's calculation of the Tamagawa number $\tau(G) = \mathrm{Vol}\,(G(\mathbb{A})/G(\mathbb{Q})) = 1$, where $\mathbb{A}$ is the adèle ring of $\mathbb{Q}$. This produces a formula for $\mathrm{Vol}\,(SL_d(\mathbb{R})/\Gamma)$ in terms of the zeta function of D. In §4 it is shown that Γ is a large subgroup of $SL_d(\mathbb{R})$ in various respects: it is Zariski-dense; it is an almost maximal discrete subgroup; reduction modulo primes is surjective almost everywhere. §5 is devoted to a theorem of Margulis: if $d \geq 3$, then every nontrivial normal subgroup of Γ has finite index. The two main ingredients of the proof are that Γ has Kazhdan's property (T), and ergodic theory. The author considers this result as the most difficult part of his book, hence this section is the most elaborated one. For example, it contains a short but nice introduction to amenability and property (T), and an appendix on measure theory. In §6 there is an explicit construction of a Zariski-dense, and a semi-explicit construction of a nonabelian free subgroup of Γ. This material is new. §7 is devoted to an example with $d = 3$, presumably the simplest one apart from the quaternion case. The final §8 briefly discusses three major open problems: finding generators for Γ, the congruence subgroup problem, and the computation of Betti numbers.

The subject is very difficult, since each of the different approaches is truly deep in itself, and there is only little visible interconnection between them. The book is addressed to researchers in number theory and arithmetic groups, and it requires a good deal of background knowledge to read. Nevertheless, anyone who works even tangentially only in these fields may find refreshing ideas in the book.