2. Random disease and bootstrap percolation

Let us colour each square of the $n \times n$ board black independently with probability $p = p(n)$. We say that the configuration we obtain in this way is *p-random*. Consider the 2-neighbour paiting rule, as defined in the *Introduction*; an initial configuration is *contagious*, if the process determined by this initial configuration and our spreading rule has the all-black board as the final configuration.

"Being contagious" is a monotone set property if the configurations are considered as sets of black squares, and it is clear that the probability

$$Q(p, n) = \text{Prob}[\text{a } p\text{-random initial configuration is contagious}]$$

is a strictly monotone increasing function of p. So it makes sense to define the critical probability $P(n)$ as $Q(P(n), n) = 1/2$, and the classical result of [BoT] says that this $P(n)$ is a real threshold function: if $p(n)/P(n) \to 0$, then $Q(p(n), n) \to 0$, and if $p(n)/P(n) \to \infty$, then $Q(p(n), n) \to 1$. We call the problem of determining $P(n)$ and other properties of $Q(p, n)$ the *Random Disease Problem*. The corresponding functions for the cylinder and the torus board are $P^C(n)$ and $P^T(n)$.

It would be interesting to decide whether $P(n)$ is a *sharp threshold* or not. This notion was introduced in [F] and [FK] as follows. Let $P_t(n)$ be the probability for which $Q(P_t(n), n) = t$, and define the *threshold interval* as $\delta_\epsilon(n) = P_{1-\epsilon}(n) - P_\epsilon(n)$. If $\delta_\epsilon(n)/P(n) \to 0$ as $n \to \infty$ for all $\epsilon > 0$, then we call $P(n)$ a sharp threshold; if there exists a constant $c > 0$ such that $\delta_\epsilon(n)/P(n) > c$ for all n and ϵ, we call it a *coarse threshold*. From [FK] it follows that the thresholds $P^T(n)$ and $P^C(n)$ are sharp: the main reason is that there is a large permutation group acting on the n^2 squares of the board under which the set of the contagious initial configurations is invariant. This method does not work for the case of $P(n)$, as the desired large permutation group does not exist. (For example, one can show that such an invariant permutation group cannot be transitive.) So far we have found two possible ways to prove the existence of the sharp threshold for this case. The first one could be an application of the main result of [Bg], which asserts that if a property is stable in some sense with respect to small local perturbations of the configuration, i.e. it can be regarded as a *global* property, then the corresponding threshold is sharp. The second possibility is to prove that $P_t(n)$ is very close to the sharp threshold $P_t^T(n)$.

Of course one can easily extend the problem to the k-dimensional board with an l-neighbour expansion rule ($1 \le l \le 2k$). The corresponding threshold function is denoted by $P_{k,l}(n)$, so $P(n) = P_{2,2}(n)$.

The main result of [BP] gave almost exact bounds on $P(n)$, namely, we proved that

Theorem 2.1. *If $\epsilon > 0$ is arbitrary and n is large enough, then*

$$\frac{1}{200e^2 \ln n} < P(n) < \frac{(\log^* n)^{1+\epsilon}}{\ln n}, \tag{2}$$

where $\log^* n$ is a rather slowly growing function, i.e. it denotes the minimum number k such that for the sequence $a_1 = 2$ and $a_{i+1} = 2^{a_i}$ the inequality $a_k \geq n$ holds.

Our notions can be easily extended for the infinite board. Configurations and different expansion rules are the same in this case. A configuration is *contagious* iff every square becomes black at some point in the disease process, and *strongly contagious* iff the sequence of the configurations contains the all-black plane. The p-random initial configuration is the same as above: each square is painted black with probability p, independently of the others. We can speak about a $p(x, y)$-random configuration, as well: we fix a coordinate system with axes x and y ($x, y \in \mathbf{Z}$), and the square with coordinates (x, y) is chosen to be black with probability $p(x, y)$.

The fact that $P(n) \to 0$ as $n \to \infty$ means that our disease depends not only on the local properties of a configuration, on a bounded neighbourhood of the squares. So an easy corollary of our main theorem can be formulated as follows:

Theorem 2.2.

(a) *Let $\widehat{P}(n)$ be arbitrary with $\widehat{P}(n)/P(n) \to \infty$. Then the $p(x, y)$-random initial configuration is contagious with probability 1, where $p(x, y) = \widehat{P}(\|(x, y)\|)$ and $\|(x, y)\| = \max\{|x|, |y|\}$. Thus the p-random configuration is almost surely contagious for any $p > 0$ fixed.*

(b) *Starting with a p-random initial configuration ($p > 0$ fixed), the time $t(p)$ needed for the complete painting of the plane is almost surely infinite, i.e. the probability that a p-random configuration is strongly contagious is 0.*

Just after publishing [BP] it turned out that our random disease problem for $l = 2$ had been already known in statistical physics as *bootstrap percolation*. Similarly to usual percolation models (see [G]), this problem was first introduced for infinite underlying graphs in 1979, see [CLR]. The first mathematically correct result was the proof of the weaker result of our *Theorem 2.2* by *van Enter* in [E]. Here the first step was to show that the probability $Q(p, \infty)$ of the event that the p-random configuration is contagious in the plane is strictly positive for $p > 0$, and then a standard ergodicity argument (see [W] or [G]) applies, as both the property 'being contagious in the plane' and the probability measure 'p-random' are translation invariant, and translation is ergodic.

The finite case was considered by *M. Aizenman* and *J. L. Lebowitz* in [AiL] in 1988, where they gave a detailed analysis of our process. Actually, they proved the sharp form of our *Theorem 2.1* and the statement of *Conjecture 2.1* for $l = 2$. Moreover, they showed the existence of the sharp threshold by direct calculations. Their proofs contain more technical analytical arguments than those of [BP], and here, in *Section 2.1*, we give only a brief discussion of their results and methods. Some of these ideas appear also in [BP], and we will try to point out the most important connections and alterations in the two papers.

In the past two decades a large variety of connected problems have been investigated. A basic paper is [S 92] by *R. H. Schonmann*. Here a larger family of nearest neighbourhood disease processes was considered on the infinite grid $\mathbf{Z}^k$, for instance the *modified* and the *oriented bootstrap percolation*, among others. The main theorem was the analogue of *van Enter*'s result for higher dimensions, which was also conjectured in [BP]; see *Section 2.4*. This theorem was extended for various models, and in the proof the occupation time problem was also analysed. In connection with this problem, which had first appeared in [AiL] (see *Theorem 2.4*), there were certain *critical exponents* defined; for the best known results see [M 92] and [An 93]. (For a related problem see *Question 2* in *Section 2.5*.) A strong relationship between oriented models and *oriented site percolation* was also pointed out. A very good description of the *semi-oriented* model for the finite square and torus board was done by *T. S. Mountford* [M 95]. Oriented models can be considered as *growth models*, as well; a continous time version was described in [KS].

In [AiL] the appearance of the sharp threshold was considered as a *metastability effect*, and an analogous phenomenon was conjectured in the classical *Ising-model* (see [G]). This conjecture about the demise of the metastable phase under the *Glauber dynamics* was settled by *R. H. Schonmann*, see [S 98].

Lots of other questions about our original model and a conjecture generalizing *Theorem 2.1* and *Theorem 2.3* are stated in *Section 2.5*.

2.1. Metastability effects in bootstrap percolation

First of all we give the definitions of [AiL], then formulate the main results and outline their proofs.

We consider a p-random initial configuration in the k-dimensional cube of side n, and the 2-neighbour spreading rule; for the sake of simplicity we use the term p-random in this section for the $(1-\mathrm{e}^{-p})$-random configuration, which are almost the same for small p. The probability of the complete painting is $Q(p,n)$, this is the *internal spanning probability* for the cube of side n. We define $\sigma(p,n)$ by

$$Q(p,n) \equiv \exp(-\sigma(p,n)p^{-1/(k-1)}).$$

We examine the behaviour of $Q(p,n)$ while $p>0$ is small but fixed, and the scaled density parameter λ changes together with n:

$$\lambda = p^{1/(k-1)} \ln n.$$

The main results of [AiL] are the following:

Theorem 2.3.

(a) *For small n the function $Q(p,n)$ decreases rapidly with n. Explicitly, there exists an absolute constant C such that, for each p,*

$$\max\{Q(p,n)\,|\,1 \le n \le C(2p)^{-1/(k-1)}\} \le p\,2^k. \tag{3}$$

Furthermore, for p small enough the maximum is at $n = 1$.

(b) *After the initial "transient regime" described in (a) there is a wide "plateau" where the function $\sigma(p,n)$ is approximately constant, i.e. for each p, and pair of positive numbers $A \gg 1 \gg B$ such that $A > 3\ln p^{-1}$, for the regime*

$$Ap^{-1} \le n \le \exp(Bp^{-1/(k-1)})$$

we have

$$|\sigma(p,n) - \hat{\sigma}(p)| \le 2kB + 2kp^{-(k-2)/(k-1)}\exp(-A/3), \tag{4}$$

where this $\hat{\sigma}(p)$ is between two absolute constants.

(c) *When the length gets to be of the order of $n = \exp(\lambda p^{-1/(k-1)})$ with $\lambda = O(1)$, then $\sigma(p,n)$ drops down linearly in λ, i.e. we have a critical value $\lambda_c(p)$ for which*

$$Q(p,n) = \exp\{-p^{-1/(k-1)}[\lambda_c(p) - \lambda + o(1)]\} \qquad \text{for } o(1) < \lambda < \lambda_c(p). \tag{5}$$

(d) *Beyond the critical value we have*

$$Q(p,n) \ge 1 - \exp(-\text{const} \times n^{k-1}) \qquad \text{for } \lambda > \lambda_c(p). \tag{6}$$

The critical value is $\lambda_c(p) = \hat{\sigma}(p)/k$, though [AiL] claims it without the division by k, but it makes no crucial difference anywhere.

(e) *The function $Q(p,n)$ has a sharp threshold, i.e. the threshold interval $\delta_\epsilon(n)$ defined in the previous section satisfies for each p*

$$\delta_\epsilon(n) \le C\ln\epsilon^{-1}/\ln^2 n. \tag{7}$$

Sketch of the proof. There are two key steps in the proof. The first one is to prove that the height $\hat{\sigma}(p)$ of the plateau in (4) is bounded by two constants, that is,

$$\exp\left(-C_1 p^{-1/(k-1)}\right) \le Q(p,n) \le \exp\left(-C_2 p^{-1/(k-1)}\right).$$

An upper bound (i.e. a lower bound for $Q(p,n)$) comes from the sufficient condition that if each of the cubes of size $1 \le 2m+1 \le n$ centered at a given point has at least one initially black particle on each of its $2k$ faces, then the whole cube of size n will become black.

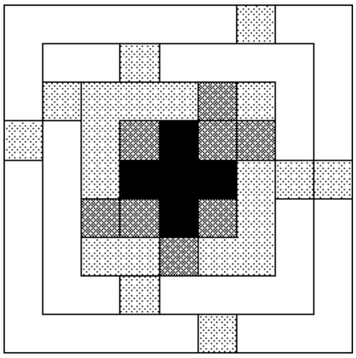

Figure 2.1

This yields

$$
\begin{aligned}
Q(p,n) &\geq \prod_{m=0}^{n/2} \{1-\exp[-p(2m+1)^{k-1}]\}^{2k} \\
&= \exp\left(-2k\sum_{m=0}^{n/2} \ln\{1-\exp[-p(2m+1)^{k-1}]\}^{-1}\right) \\
&\geq \exp\left(-k\int_0^{n+1} \ln[1-\exp(-ps^{k-1})]^{-1}\,\mathrm{d}s\right) \\
&\geq \exp\left(-kp^{-1/(k-1)}\int_0^{\infty} g(z)/z\,\mathrm{d}z\right),
\end{aligned} \tag{8}
$$

for

$$g(z) = z\ln[1-\exp(-z^{k-1})]^{-1},$$

and some simple analysis shows that the last integral in the inequality is finite.

For a lower bound on $\sigma(p,n)$ we need first the easy estimation

$$Q(p,n) \leq [1-\exp(-p\,2n^{k-1})]^{n/2}$$

coming from the fact that for a complete painting we need at least one initial black particle in each of $n/2$ disjoint slabs of width 2:

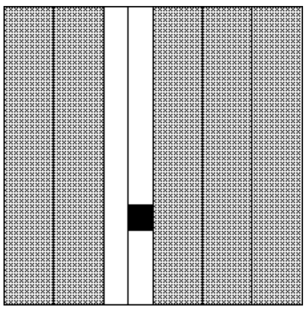

Figure 2.2

This gives the statement (3) of *(a)* almost immediately. Now we have to prove (4), which says that $\sigma(p,n)$ varies only very slowly in the regime of the plateau in *(b)*. For this we need the following entropy-type inequality:

$$Q(p,n) \leq m^k n^k Q_1(p,m),$$

where $Q_1(p,m)$ is the maximum of the spanning probabilities of rectangular regions whose longest side length falls in the interval $[(m-2)/2, m]$. This can be shown by the simple combinatorial observation that if a region of size n is internally spanned, then, for all $m \leq n$, it contains at least one subregion of size in the interval $[m, 2m+2]$ which is also internally spanned. (This argument is closely related to the diameter method used in *Fact 1.2.*) For an opposite bound on the ratio of two spanning probabilities we may use the "seed construction" in *Figure 2.1* and a computation similar to (8) again.

Summing up we have just learned about $\hat{\sigma}(p)$ it may seem to us that the behaviour of $Q(p,\cdot\,)$ in the regimes *(a)* and *(b)* is dominated by the occurance of a fairly local "bottleneck event", whose probability is close to $\hat{Q}(p) = \inf\{Q(p,n) \,|\, n \geq 1\}$. The following key lemma states explicitly what this local event is.

Critical Droplet Lemma. *If $A_{p,n}$ is the event that whole board is internally spanned, and $B_{p,n}$ is that the board contains a subboard of size $m(p) = p^{-3}$ internally spanned, then*

$$\mathrm{Prob}[A_{p,n} \Delta B_{p,n}] \leq \exp(-Cp^{-2})$$

with some absolute constant C. Here $A \Delta B = (A \backslash B) \cup (B \backslash A)$.

The necessity of the droplet $B_{p,n}$ follows from the combinatorial observation above about the lack of gaps in the size of the internally spanned subregions, or from the argument of *Fact 1.2.* The sufficientness can be verified by the seed construction.

Let us note that this value for $m(p)$ is much too large, a more natural choice would be $m(p) = p^{-1/(k-1)}/o(1)$.

Now the statements of *(c)* and *(d)* are the immediate consequences of the previous ingredients, since we have

$$[n/m(p)]^k Q(p, m(p)) \geq Q(p,n) \geq 1 - [1 - Q(p, m(p))]^{[n/m(p)]^k}. \tag{9}$$

The result we get for the regime *(d)* is improved by a *renormalization method*, occuring also in [S 92].

The sharp threshold (7) in *(e)* follows almost directly from the estimations (5) and (6) above. ∎

Now it is worth comparing this proof to that of [BP], described in the next two sections. The *Critical Droplet Lemma* is implicit in [BP], even in a stronger form: the proof of the upper bound in *Section 2.3* yields that a droplet of size $O(p^{-1/(k-1)})$ already suffices for the complete painting. But, on the other hand, this method fails in proving the sharp upper bound, because we were able to show the constant lower bound of (8) on $\sigma(p,n)$ only for $n \le O\left(p^{-1/(k-1)} / (\log^* 1/p)^{1+o(1)}\right)$. This difference is somewhat strange, since the underlying combinatorial idea, the seed construction, was almost the same in both methods.

The lower bound in [BP] is a nice combinatorial reformulation of the critical droplet idea.

Finally, let us define $M(p,n)$ to be the probability that the center point of our board is painted black in the final configuration. It is related to the occupation time parameter for the infinite board: if $M_T(p,t)$ denotes the probability that the origin is already black after time t, then

$$M(p, t^{1/k}/2) \le M_T(p,t) \le M(p,t). \tag{10}$$

With the help of the *Critical Droplet Lemma* it can be proved that

Theorem 2.4.

$$\lim_{p\to 0} \sup_n \{|M(p,n) - Q(p,n)|\} = 0. \tag{11}$$

This result implies the threshold property of $M(p,n)$ and because of (10) a weaker result for $M_T(p,n)$. This phenomenon was considered in [AiL] as a metastability effect, see the introduction to this chapter.

The occupation time problem will also occur in the proof of *Theorem 2.2*, in *Question 2*, and it is related to the *critical phase problem* of *Section 2.5*.

2.2. Proof of the lower bound on *P(n)* in [BP]

In the disease process a black square remains black forever. Hence in the case of a finite board our sequence of the configurations will be constant after a certain time. We call this configuration the *final configuration.* What can the final configuration be? Obviously, the black squares of the final configuration can be partitioned into groups, such that the squares in each group form a rectangle, and the rectangles of different groups are *far* from each other in the sense that no square is bordered by two black rectangles. In particular, two rectangles will be far if they can be separated by a strip consisting of two neighbouring columns or rows of the board. We call this type of strips *width 2 strips.*

In some sense the reverse is also true: if the initial black squares can be covered by rectangles that are pairwise separated by width 2 strips, then during the disease process all of the black squares stay in the covering rectangles. This can easily be proved by induction on the time of the process.

So for the lower bound we will define a $p(n)$ such that we will be able to show that with high probability the black squares can be covered by rectangles, in such a way that any two rectangles can be separated from each other by width 2 strips.

Let us divide the $n \times n$ chessboard into smaller rectangular pieces, *subboards*, of size $(L \text{ or } L+1) \times (L \text{ or } L+1)$, where $L = \lfloor c_1 \ln n \rfloor$, c_1 will be determined later. We will have $\lfloor n/L \rfloor^2$ subboards.

First we show that with an appropriate choice for $p = p(n)$ the p-random initial configuration contains only $L/10$ black squares in each subboard with high probability.

Lemma 2.1. *Let $p = c_2(\ln n)^{-1}$, $L = \lfloor c_1 \ln n \rfloor$. Then with a probability tending to 1 there will be no subboard with more than $L/10$ black initial squares, if $c_1 = 20$ and $c_2 = 1/(200\mathrm{e}^2)$.*

Proof. It is easy to give an upper bound on the probability that an $L \times L$ chessboard contains more than $L/10$ black squares of a p-random configuration:

$$\binom{L^2}{L/10} p^{L/10}.$$

Thus the probability that there is no subboard with more than $L/10$ initial black squares is at most:

$$\left(\frac{n}{L}\right)^2 \binom{L^2}{L/10} p^{L/10} < 2\frac{n^2}{L^{2.5}}(10eLp)^{L/10}, \tag{12}$$

where we used *Stirling's formula.*

Our choice for c_1 and c_2 is nearly optimal for getting $\overline{\lim}_{n\to\infty} n^2(10eLp)^{L/10} < \infty$, so by (12) we have obtained w.h.p. that there is no subboard of size $L \times L$ with more than $L/10$ initial black squares.

A similar calculation handles the cases of subboards of size $L \times (L+1)$, $(L+1) \times L$ and $(L+1) \times (L+1)$. ∎

We take two subboards, B_1 and B_2, sharing a common vertical side. Let B be the rectangle we obtain by gluing together the two subboards: $B = B_1 \cup B_2$. We define the following property of the initial configuration:

$\mathcal{P}(B) =$ "there are two horizontal width 2 strips in $B_1 \cup B_2$ which contain only white squares, one in the upper half of B and one in the lower half of B"

If $B_1 \cup B_2$ has property $\mathcal{P}$, we fix the two strips which proves this, and call them *lower and upper channels.*

One can easily define an analogous $\mathcal{P}^*$ property for subboards B_1 and B_2 sharing a horizontal side:

$\mathcal{P}^*(B) =$ "there are two vertical width 2 strips in $B_1 \cup B_2$ which contain only white squares, one in the left half of B and one in the right half of B"

If $B_1 \cup B_2$ has property $\mathcal{P}^*$, we fix the two strips which proves this, and call them *left and right channels.*

We will prove that with the choice of *Lemma 1* for p the initial p-random configuration will have the property $\mathcal{P}(B)$ or $\mathcal{P}^*(B)$ for any pair B of two neighbouring subboards (depending on whether their common side is vertical or horizontal), i.e. it has property $\widehat{\mathcal{P}}$.

Lemma 2.2. *If in a configuration C each subboard contains at most $L/10$ initial black squares, then C has property $\widehat{\mathcal{P}}$.*

Proof. By symmetry it is enough to prove $\mathcal{P}(B)$. There are at least $(9/10)L$ white rows in each subboard, hence in each half of a subboard there are at least $(4/10)L - 1$ white rows. So if we have two neighbouring subboards, then in each half of them there will exist at least $(3/10)L - 2$ common white rows, thus two of them will be neighbouring, if L is large enough. ∎

From now on we assume that our initial configuration has property $\widehat{\mathcal{P}}$. We are going to prove that the configuration is not contagious by providing a cover of the initial black squares by rectangles as promised.

Let B be an "inner subboard". We define two partitions of B into five rectangles with a border between them. B has four neighbouring subboards: the upper, the right, the lower and the left. The left neighbour with B together have property $\mathcal{P}$, hence we have an upper and a lower channel. We will call these the left-upper and left-lower channels. Similarly, we can define right-upper, right-lower, upper-left, upper-right, lower-left and lower-right channels. Using four of these channels we can get a desired partition, called leftist, and using the other four channels we get the other partition, called rightist. The construction of a leftist partition of B is shown in the self-explanatory *Figure 2.3*, the borders between the five rectangles are the four dark channels.

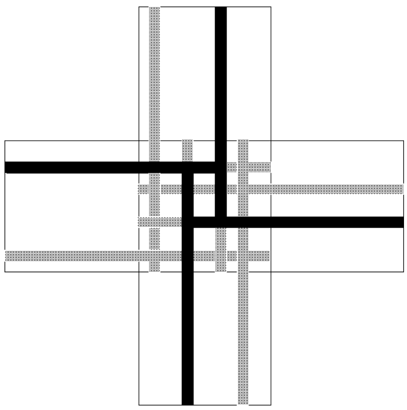

Figure 2.3

Now we combine the leftist and rightist partitions of the subboards alternating in a chessboard manner. The result is a suitable cover: rectangles and all-white borders providing a width 2 strip for any two rectangles to separate them (see *Figure 2.4*).

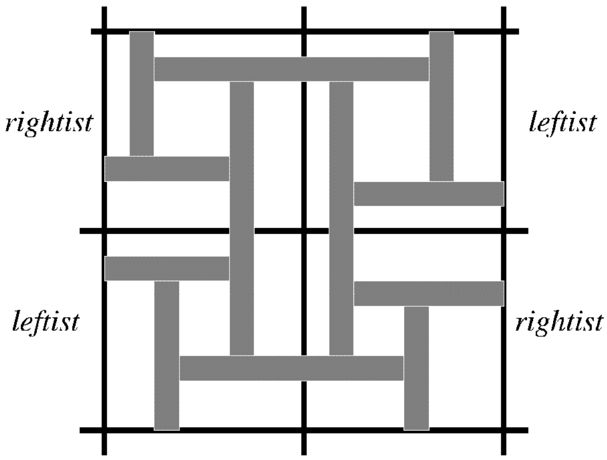

Figure 2.4

Side and corner subboards can be covered naturally by rectangles between the channels constructed in partitioning the inner subboards that are the neighbours of our outer ones.

Summarizing our construction:

Lemma 2.3. *A configuration with property $\widehat{\mathcal{P}}$ is not contagious.* ∎

Now the combination of *Lemma 2.1, 2.2* and *2.3* gives us the lower bound on $P(n)$. ∎

2.3. Proofs of the upper bound and the whole plane case

For the upper bound first, of all we observe that in the initial configuration there are quite long horizontal runs of pure black squares in each row of the chessboard. The proof that the initial configuration is contagious with high probability then can be based on the existence of these initial runs. The method we use is a clever and efficient modification of the one in a previous version of this paper; this improvement is due to *N. Alon*, [A].

Throughout this section we will use that

$$(1-a_n)^{b_n} \sim \exp(-a_n b_n),$$

if $a_n < 1$ and $b_n \to \infty$ such that $a_n^2 b_n \to 0$. We need this simple analytic fact in almost every estimation of our probabilities, but we will not refer to it if the computations are clear.

Split the first row into, say, $n^{0.99}$ pairwise disjoint segments of equal size. (From now on we can afford not to bother with the integer parts, this simplification causes no problem.) In a $p(n)$-random initial configuration the probability that such a segment starts with a run of $f(n)$ black squares is $p(n)^{f(n)}$. If we plug in $p(n) \geq \frac{1}{\ln n}$ and $f(n) = \frac{\ln n}{100 \ln \ln n}$, we get that this probability is at least $n^{-1/100}$. Now we can easily estimate the probability that a bunch of $n^{0.04}$ consequetive segments does not contain a run of $f(n)$ black squares: this is less than $\exp(-n^{0.02})$. We can make $n^{0.95}$ pairwise disjoint bunches, and so the probability that there exists such a bunch without a long horizontal black run is less than $1 - \exp\left(-\mathrm{e}^{-n^{0.02}} n^{0.95})\right) \to 0$. Thus we have proved the following

Many Black-runs Lemma. *For $p(n) \geq \frac{1}{\ln n}$ the initial configuration contains at least $n^{0.95}$ horizontal black runs of length $\frac{\ln n}{100 \ln \ln n}$ each, where each such run starts from the leftmost point of one of our disjoint segments of length $n^{0.01}$, and the probability that this does not hold is exponentially small.* ∎

Remark 1. More sophisticated methods give that the length of the longest black run in a row of length n is $\mu(n) \sim \frac{\ln n}{\ln 1/p(n)}$ w.h.p. For $p(n) = 1/2$ it was stated first by *P. Erdős* and *A. Rényi* in [ErRn] from a slightly different point of view, and more precise results can be found in [ErRv].

Another way to handle the lots of not necessarily disjoint long runs is the usage of the *Janson inequalities*, see [AS] or [J].

Remark 2. Actually, we need only "almost pure" black-runs instead of the pure ones, namely we can allow single white squares between the black ones, as they will change into black by the next day. But this relaxation does not help much, the expected length of the longest run would increase only by a constant factor, and in the proof of the upper bound this improvement means nothing.

If we pick a horizontal black run of length $f(n)$, i.e. a black block of size $1 \times f(n)$, we can see in the neighbouring row that the block "below" our black one will change into black in at most $f(n)$ days, even if only one square is initially black in it. So a neighbouring block becomes black with probability $q = q(n)$, where

$$1 - q(n) = (1 - p(n))^{f(n)}. \tag{13}$$

If this event with probability q does happen, then the same thing can be repeated for the $2 \times f(n)$ black block we have just obtained, and so on; we stop when we find a pure white block. So we get a run of black blocks with a random length Z, where

$$\mathrm{E}[Z] = 1q(1-q) + 2q^2(1-q) + \ldots + (n-1)q^{n-1}(1-q) > \frac{1}{2}\frac{q}{(1-q)},$$

and for $g(n) \leq n$ we have

$$\mathrm{Prob}[Z \geq g(n)] \sim q(n)^{g(n)}. \tag{14}$$

As two special cases of (14) we can state the following two lemmas. For a detailed verification one should use our analytic fact again and estimations like $1/(2k) \leq 1 - \exp(-1/k)$, if $k > 1$.

Lemma 2.4. *Suppose we have a black horizontal run of length $l = \frac{\ln n}{k}$ (with $k > 1$), and suppose each square is now becoming black, randomly and independently, with probability*

$p(n) \geq \frac{1}{\ln n}$. *Then the probability that the process described above creates a black vertical run of length* $\frac{\delta \ln n}{\ln(2k)}$ *is at least* $n^{-\delta}$. ∎

Lemma 2.5. *Suppose we have a black horizontal run of length* $l = k \ln n$ *(with* $o(\ln n) = k > 1$*), and suppose each square is now becoming black, randomly and independently, with probability* $p(n) \geq \frac{1}{\ln n}$. *Then the probability that the process creates a black vertical run of length* $\frac{e^k \delta \ln n}{2}$ *is at least* $n^{-\delta}$. ∎

Needless to say, the assertions of both lemmas hold if we replace vertical by horizontal and vice versa. In this case the longer horizontal runs are created to the right of the existing black blocks (see *Figure 2.5*). This method of enlarging the black blocks also yields that we will be able to iterate these lemmas such that the realizations of the iteration steps will be mutually independent of each other.

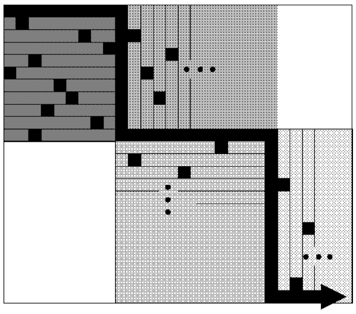

Figure 2.5

Now we are ready to describe why a $p(n)$-random initial configuration is contagious if $p(n) = \frac{(\log^* n)^{1+\epsilon}}{\ln n}$ and n is large enough. At the beginning by the *Many Black-runs Lemma* in the first row we have $n^{0.95}$ pairwise disjoint horizontal black runs of length $\frac{\ln n}{100 \ln \ln n}$ each w.h.p., and these blocks are rather far away from each other. Note that here and sometimes later, as well, we use only $p(n) \geq \frac{1}{\ln n}$. We do this only for the sake of simplicity, and the usage of the stronger condition would not produce a better result. It also simplifies matters to note that in the last few iterations we will do, it will be convenient to have each square become black again with probability $1/\ln n$, so as to obtain independence. Clearly, if every square first becomes black with probability $\frac{(\log^* n)^{1+\epsilon}}{\ln n}$ and then (if it is white) becomes black with probability $1/\ln n$, then this still corresponds to having each square become black with probability $(1 + o(1))\frac{(\log^* n)^{1+\epsilon}}{\ln n}$.

Define, now, for $i \geq 1$,

$$\delta_i = \frac{\nu}{i^{1+\epsilon/3}}, \qquad k_1 = 100 \ln \ln n,$$

$$k_{i+1} = \ln(2k_i)/\delta_i = \frac{1}{\nu}\, i^{1+\epsilon/3} \ln(2k_i),$$

where $\nu > 0$ is small enough to have $\sum_{i=1}^{\infty} \delta_i = C(\nu, \epsilon) = C < \frac{1}{100}$. If we consider one of our long black runs, then by applying *Lemma 2.4* $\log^* n + O(1)$ times repeatedly we conclude that with a probability greater than $\prod_{i=1}^{\infty} n^{-\delta_i} = n^{-C} > n^{-0.01}$ our iteration results a horizontal black run of length greater than

$$\frac{\ln n}{(\log^* n)^{1+2\epsilon/3}}.$$

This claim can be easily verified by the following argument. If $\ln \frac{k_i}{2} > \frac{1}{\nu} i^{1+\epsilon/3}$ for all $1 \le i \le \log^* n$, then $k_{i+1} < \ln^2 k_i$, and by induction we have $k_{(\log^* n)} < c \log^* n$. Otherwise there exists a $j \le \log^* n$ with $\ln \frac{k_j}{2} < \frac{1}{\nu} j^{1+\epsilon/3}$ and so $k_{j+1} < (\log^* n)^{1+2\epsilon/3}$, supposing that n is large enough.

We have $n^{0.95}$ samples of the random iteration process described above, so with very high probability we will have in the resulting configuration at least, say, $n^{0.9}$ pairwise disjoint black horizontal runs of length at least $\frac{\ln n}{(\log^* n)^{1+2\epsilon/3}}$ each.

Given these runs, let each square become black with probability $\frac{(\log^* n)^{1+\epsilon}}{\ln n}$. Then we get w.h.p. at least $n^{0.8}$ black pairwise disjoint vertical runs of length, say, $100 \ln n$ each, these last computations are routine.

Define

$$k_1 = 100, \qquad \delta_i = \frac{\nu}{i^{1+\epsilon/3}},$$

$$k_{i+1} = \delta_i \mathrm{e}^{k_i}/2.$$

By repeatedly applying *Lemma 2.5* we can now conclude that after some $\Theta(\log^* n)$ additional iterations we get w.h.p. many (and hence at least one) horizontal black run of length at least $(\ln n)^{3/2}$, and it is easy to see that this implies, after two more additional iterations, that the whole grid becomes black with a probability tending to 1. (Note that until the last two iterations we deal with runs of length less than $n^{0.01}$, so the whole processes of the iterations for our disjoint starting runs are mutually independent of each other.) This completes the proof of the upper bound and *Theorem 2.1.* ∎

Proof of Theorem 2.2. (a) Fix $\widehat{P}(n)$ as in the statement, and let us first suppose that it is a monotone decreasing function of n, just as $P(n)$ was. Let Ω be the probability space of the $\widehat{P}(\|(x,y)\|)$-random configurations of the infinite square grid, and let S_n be the $2n \times 2n$ square with vertices $(-n,-n), (-n,n), (n,-n), (n,n)$. Define A to be the event that not every square of the plane can be painted black with an initial configuration of Ω, and let A_n be the event that there is a square in S_n remaining white forever. It is clear that $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ and $\cup_{n=1}^{\infty} A_n = A$,

$$\mathrm{Prob}[A] = \lim_{n\to\infty} \mathrm{Prob}[A_n]. \tag{15}$$

Now define the event B_n as follows: for all $n \in \mathbf{N}$ consider the subalgebra of Ω generated by S_n, i.e. let Ω_n be the probability space of the $\widehat{P}(\|(x,y)\|)$-random configurations of S_n, and let B_n be the event that the painting is not complete on S_n, forgetting about the influence of the other part of the plane. Because of losing the possible positive influence of the black squares of the plane outside the square S_n we have $\mathrm{Prob}_\Omega[A_n] \le \mathrm{Prob}_{\Omega_n}[B_n]$. But inside S_n we have $\|(x,y)\| \le n$, so

$$\widehat{P}(\|(x,y)\|) \ge \widehat{P}(n) \gg P(n),$$

since $\widehat{P}(n)$ is monotone decreasing. Thus $\mathrm{Prob}_{\Omega_n}[B_n] \to 0$ as $n \to \infty$, which proves our statement because of (15), as $\mathrm{Prob}[A] = 0$.

If $\widehat{P}(n)$ is not monotone decreasing, then define $\overline{P}(n) = \min_{1\le k\le n} \widehat{P}(n)$. Now $\widehat{P}(n) \ge \overline{P}(n)$, $\overline{P}(n)$ is already monotone decreasing, but still $\overline{P}(n)/P(n) \to \infty$, so we can repeat the argument above, and we are done.

The second part of the statement *(a)* follows from our upper bound on $P(n)$: looking at the probability field of the p-random configurations, we have $p > \widehat{P}(n)$ for $n > n_p$, so $\mathrm{Prob}_{\Omega_n}[B_n] \to 0$ again.

(b) Note that if we have a pure white $k \times k$ square in the initial configuration, then it takes at least $k-1$ days to paint it black from the outside. Now divide the plane into subboards of size $k \times k$. Each of them is pure white in the initial configuration with a *positive* probability $(1-p)^{k^2}$. So we can find a pure white one in the whole plane with probability 1. If the exceptional event (namely, there is no initial pure white $k \times k$ subboard) is denoted by C_k, then $\mathrm{Prob}[C_k] = 0$, and

$$\begin{aligned}\mathrm{Prob}[t(p) = \infty] &\ge \mathrm{Prob}[\forall k\ \exists \text{ an initial white } k \times k \text{ square in the plane}] \\ &= 1 - \mathrm{Prob}[\cup_{k=1}^{\infty} C_k] \ge 1 - \sum_{k=1}^{\infty} \mathrm{Prob}[C_k] = 1,\end{aligned}$$

which proves our second statement. ∎

2.4. The general $P_{k,l}(n)$ case

1. For a successful disease process the exact thing we need is to have at least one initial black cube, so $(1 - P_{k,1}(n))^{n^k} = \frac{1}{2}$, that is

$$P_{k,1}(n) \sim \frac{\ln 2}{n^k}. \tag{16}$$

2. *Theorem 2.3* shows that

$$P_{k,2}(n) = \Theta\left((\ln n)^{-k+1}\right). \tag{17}$$

3. In the problem of $P_{k,k+1}(n)$ the complete painting cannot be carried out if there exists a $2 \times \ldots \times 2$ white cube in the initial configuration. Thus dividing the board into $(n/2)^k$ subboards containing 2^k cubes each, one can easily see that

$$P_{k,k+1}(n) \geq 1 - \frac{O(1)}{n^{k/2^k}}. \tag{18}$$

4. The complete painting is equivalent to the lack of two initial white cubes side-by-side and the total lack of white cubes on the border, so an easy computation gives

$$P_{k,2k}(n) \geq 1 - \frac{O(1)}{n^{k-1}}. \tag{19}$$

5. The simple methods above can easily be converted into the whole space case, so (17) and (18) give exact results for $l \leq 2$ and $l \geq k+1$. Nevertheless, the middle cases have been also solved, and the answer is what we expected in [BP]: the main theorem in [S 92] by *R. H. Schonmann* says that

$$P_{k,l}(\infty) = \begin{cases} 0, & \text{if } l \leq k \\ 1, & \text{if } l \geq k+1, \end{cases} \tag{20}$$

i.e. the $p_{k,l}$-random initial configuration of the infinite grid is a.s. contagious for $l \leq k$ with any $p_{k,l} > 0$ fixed, and it is a.s. contagious for $l \geq k+1$ only with $p_{k,l} = 1$. A problem related to this result is desribed in and after *Question 2* in the next section.

2.5. Some open problems

As it was shown in the previous section, $P_{k,1}(n)$ is very small and $P_{k,2k}(n)$ is very large. Thus a crucial question is the following: what is the maximal $f(k)$ and minimal $g(k)$ for which $P_{k,f(k)}(n) \to 0$ and $P_{k,g(k)}(n) \to 1$? In the deterministic version (see (1) in *Chapter 1*) we have

$$\lim_{n\to\infty} \frac{G_{k,k}(n)}{n^k} = 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{G_{k,k+1}(n)}{n^k} \geq \frac{1}{k+1},$$

so we can assume that the case $l = k$ has a special role. Actually, we think that the method of the lower bound in *Theorem 2.1* can be generalized, and gives a sharp result, which, together with the result $P_{k,k+1}(n) \to 1$ $(n \to \infty)$ in *Section 2.4*, would imply that $k = f(k) < g(k) = k+1$.

Conjecture 2.1. *For $l \leq k$*

$$P_{k,l}(n) = \Theta\left((\ln n)^{-k+l-1}\right). \tag{21}$$

This conjecture is settled by *Theorem 2.3* for $l = 2$. In particular, we can ask the following:

Question 1. *What is the exact order of $P_{k,l}(n)$? Is it true that $P_{k,l}(n) \le P_{k+1,l}(n)$?*

The appearance of the threshold $P(n)$ in the whole plane case is very natural, but is *Theorem 2.2 (a)* really sharp?

Question 2. *Is there a function $\widehat{P}_{k,l}(n)$ with $\widehat{P}_{k,l}(n)/P_{k,l}(n) \to 0$ for some k, l for which the $\widehat{P}_{k,l}(\|\underline{x}\|)$-random initial configuration of the infinite k-dimensional square grid is almost surely contagious?*

The similar question about the existence of an a.s. contagious $P_{k,l}(\infty)$-random configuration with a fixed $P_{k,l}(\infty) < \lim_{n\to\infty} P_{k,l}(n)$ would be answered negatively if *Conjecture 2.1* held, because of (20).

And, finally, we introduce a new model in order to understand the critical phase in all details. Here we are dealing only with the case of $P_{2,2}(n)$, but one can find the natural extensions of it.

Let us consider the uniform probability measure on the $n^2!$ orderings of the n^2 squares of the square or the torus board. Paint these squares one by one according to a fixed ordering, so we get a random sequence $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots \subset \mathcal{C}_{n^2}$ of initial configurations, where $\mathcal{C}_0$ is the empty, $\mathcal{C}_{n^2}$ is the all-black configuration. "Being contagious" is a monotone increasing property, so we have a random unique m such that $\mathcal{C}_m$ is a *maximal uncontagious initial configuration*, and $\mathcal{C}_{m+1}$ is already contagious. The final configuration $\mathcal{C}_c$ determined by $\mathcal{C}_m$ is called the *critical phase* of our model. For the torus board we denote this random configuration by $\mathcal{C}_c^T$.

It is equivalent to our previous results that

$$m = m(n) = \Theta\left(\frac{n^2}{\ln n}\right),$$

with high probability, and the sharp threshold of $P(n)$ is closely related to the large concentration of $m(n)$.

Question 3. *How large is the perimeter and the area of the black part in the critical phases $\mathcal{C}_c$ and $\mathcal{C}_c^T$? What are the distributions of the sizes of the maximal internally spanned rectangular regions in these critical phases?*

Question 4. *How concentrated is $m(n)$? How is the closeness of $\mathcal{C}_c$ and $\mathcal{C}_c^T$ related to the closeness of $P(n)$ and $P^T(n)$?*

Our last question is about the stopping time of our process in the finite board problems.

Question 5. *Is it true that the stopping time is large near the threshold (and near the critical phase) and small if we are far from it? How more time does a successful painting process take than an unsuccessful one?*

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