Morse Theory

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0. Introduction

A first version of this essay was originally written in 1999 as one of my six Part III exams for the degree *Certificate of Advanced Study in Mathematics*, at the University of Cambridge, UK. Now this is a slightly corrected and improved version, containing more details both on some introductory and on some advanced topics. I would like to thank the help of *Prof. Graeme Segal*, who was my supervisor for this Part III essay topic, and *Prof. Nándor Simányi*, who patiently answers all my stupid questions concerning almost all parts of geometry. The summer school *Algebraic geometry in theoretical physics*, organized by the Association of Hungarian Physicist Students (MAΦHE) in 2000, was a very nice and useful experience, too.

Morse theory is a method to determine the topology of a finite or infinite dimensional manifold (such as the space of paths or loops on a compact manifold) from the critical points of only one suitable function on the manifold. The theory has many far-reaching applications: Gauss-Bonnet theorem, Poincaré-Hopf index theorem, the determination of the geodesic structure of a manifold, Lefschetz singularity theory of hypersurfaces, Milnor's exotic spheres, Bott's periodicity theorem about the homotopy groups of the unitary and orthogonal groups, Yang-Mills theory on vector bundles, the planar n-body problem, geometry of Hamiltonian dynamical systems, Floer homology, and has strong connections with the Atiyah-Bott-Lefschetz fixed point and Atiyah-Singer index theorems.

The purpose of our essay is two-fold. The starting idea was to describe and explain *E. Witten*'s supersymmetric approach to Morse theory. During writing this essay we found that this explanation should consist of not only a guide into tools used directly in this approach, but also of an overview of connected geometrical notions and results. So we try to give an account of a significant part of geometry from the point of view of Morse theory. We hope that the common roots and goals of the various mathematical theories described in this essay will help the Reader to gain a deeper insight into modern geometry.

The classical approach is beautifully written in *J. Milnor*'s book [M2] from 1963, containing some of the older applications. In the first part of our Introduction we describe this original proof of the basic result of the theory very briefly, just to give the Reader a general feeling about the topic. The second part of the Introduction contains some well-known and/or conceptually easier applications of Morse theory, such as Smale's refinement of the theory, the Poincaré-Hopf index theorem, the Gauss-Bonnet theorem and Poincaré duality.

In his 1982 paper [W1], Witten developed the whole theory with the help of some perturbated generalized Laplacian operators acting on the exterior bundle of differential forms on the manifold, using various ideas and techniques coming from supersymmetric quantum mechanics. Analyzing the asymptotic behaviour of these perturbated Laplacians he was able to prove not only the strong Morse inequalities and the Poincaré-Hopf index theorem, which are the main results of Morse theory, but also to develope a degenerate Morse and Poincaré-Hopf theory, and to describe (conjecturally) the whole cohomology of the manifold through Morse theory, which is a kind of finite-dimensional analogue of Floer homology. Since then this approach has been applied to serious problems in supersymmetry (SUSY) and also led to an asymptotic proof of the Atiyah-Singer index theorem, which has played a central role in the geometry of the past few decades.

First we overview the necessary background for Witten's proof, including: connexions and curvature in fibre bundles over differentiable and complex manifolds; the Dirac and Laplacian operators of Clifford bundles; some PDE and functional analytic methods to prove the Hodge theorem on the representation of cohomology by harmonic forms; the spectrum of the harmonic oscillator; the basics of quantum mechanics and supersymmetric theories. We also sketch the Atiyah-Bott-Lefschetz fixed point theorem together with some corollaries. As the key to the whole theory is the intrinsic geometric meaning of the Laplacian and other elliptic differential operators on the manifold, we give a few different approaches to grap the geometry of a manifold through the Laplacian, including some discrete probability theory, the Selberg trace formula, inverse spectral theory.

Next we give the proofs of Witten's results mentioned above. As the original paper of him concentrates on the mere ideas and not on the technicalities at all, there is a significant work in constructing the complete proofs in a clear way.

Among the many applications of Morse theory we focus on the description of the variational calculus of the geodesic flow and other Hamiltonian dynamical systems, and of the Yang-Mills functional defined on the space of connexions in a vector bundle. For instance, we prove the Cartan-Hadamard theorem, the Gauss-Bonnet-Chern theorem, and give some account of the Narasimhan-Seshadri theory of curvature in vector bundles. We also overview certain other topological consequencies of curvature, including some ergodic theory and geometric group theory. We end our essay with a short survey of discrete Morse theory, which is the analogue of the whole theory for simplicial complexes.

In understanding the background material for Witten's approach my main help was the book [R]. For a more elaborate treatise on differential geometry the Reader can always consult with [DFN] or [KN]. For the Atiyah-Singer index theorem I can recommend [BGV] and [G]. In particular, [BGV] follows a description having deep similarities with the asymptotic methods of [W1]. In writing this essay, the lecture notes [Se] and [Φ] were extremely useful. I also have to recommend the beautiful short survey [B], which is a personal overview of the development of Morse theory. Finally, there is a very nice book [N], which I have encountered only very recently. This seems to be an excellent survey with almost the same purposes as I had in mind.

0.1. The classical approach

In this paper M usually denotes a compact n-dimensional smooth manifold. The 'suitable function' we mentioned at the beginning is meant to be a so-called *Morse function.* If we have the manifold M, and a smooth function $f : M \longrightarrow \mathbb{R}$, we call $p \in M$ a *critical point* of f if in local coordinates around p we have $\partial f / \partial x^1 = \cdots = \partial f / \partial x^n = 0$, i.e. if the differential $f_* = Df$ has rank 0 (instead of 1). A critical point is *non-degenerate* if the *Hessian matrix* $f_{**} = (\partial^2 f \,/\, \partial x^i \partial x^j)_{i,j=1}^n$ is non-singular in p. In this case the number of its negative eigenvalues is called the *Morse index* of the critical point. If the critical point is degenerate, then the number of 0 eigenvalues is called the *nullity* of the critical point. A Morse function is a smooth function f with only non-degenerate critical points (which are always isolated, as can be seen from the next lemma). The following basic lemma gives a nice local description of a Morse function.

Lemma 0.1. (Morse lemma) *If $p \in M$ is a non-degenerate critical point of f then there exist a neigbourhood $U \ni p$ and local coordinates $y^1, \dots, y^n$ such that $y^i(p) = 0$ and $f(q) = f(p) - y^1(q)^2 - \cdots - y^\lambda(q)^2 + y^{\lambda+1}(q)^2 + \cdots + y^n(q)^2$ for $q \in U$, where λ is the index of f at p.* $\square$

Certainly, we can use our theory only if Morse functions do exist. One can find Morse functions among some kind of projection functions. The existence of a suitable point or direction for such a projection can be proved by *Sard's theorem* [DFN II, §10], [L], [MT], claiming that the set of critical values of a smooth function $f : M \longrightarrow N$ between smooth manifolds is always of measure 0 in N. Moreover, the set of Morse functions is dense in the space of all smooth functions on M in the C^2-topology, which result is not significantly more difficult to prove.

The basic idea of classical Morse theory is that the homotopy (or, in better situations, even the homeomorphism or diffeomorphism) type of the submanifold $M^a = \{p \in M \,|\, f(p) \leq a\}$ changes only at the critical points of f. If there is no critical value in the interval $[a, b]$, then the gradient flow of f provides a diffeomorphism between M^a and M^b. At a critical value a, we can suppose (after a small perturbation of f) that there is only one critical point p with $f(p) = a$. Now the key result is that we can get $M^{a+\epsilon}$ from $M^{a-\epsilon}$ by attaching a handle B^λ, a cell of dimension of the index λ of f at the critical point $f(p) = a$. This handle B^λ can be constructed via the description given by the Morse Lemma 0.1, and the attaching map is between ∂B^λ and $\partial M^{a-\epsilon}$. Then, using some technical results of *Whitehead* in homotopy theory, one can prove the following theorem of crucial importance:

Theorem 0.2. *If f is a Morse function on M such that M^a is compact for each $a \in \mathbb{R}$ then M has the homotopy type of a cell complex with one λ-dimensional cell for each critical point of index λ.* $\square$

Those results of Whitehead are needed to get not only a cell space, but a cell complex, which means that each cell is attached to cells of lower dimension. Instead of these homotopy theoretical methods there is a reformulation of the construction of the manifold by attaching handles in the language of algebraic topology: one may use a *Mayer-Vietoris* type argument, e.g. the *Excision theorem* [DFN III, Thm. 5.9], [BT]:

If b_k denotes the kth Betti number of M, i.e. the dimension of the cohomology group $H^k(M;\mathbb{R})$, and m_k is the number of critical points of a Morse function f with index k, then

Theorem 0.3. (Morse inequalities) *$b_k \leq m_k$, moreover, we have*

$$\begin{gathered} b_0 \leq m_0 \\ b_1 - b_0 \leq m_1 - m_0 \\ b_2 - b_1 + b_0 \leq m_2 - m_1 + m_0 \\ \cdots \\ \chi(M) = \sum_{k=0}^{n} (-1)^k b_k = \sum_{k=0}^{n} (-1)^k m_k. \quad \square \end{gathered}$$

The 'official example' of Morse theory is the height function on the torus embedded in $\mathbb{R}^3$ as a doughnut or bagel standing up vertically on a point of the great circle of the torus. This height function has four critical points: one minimum (with index 0), two saddle points (with index 1) and one maximum (with index 2). And, in fact, the torus can be built up starting with a point, then attaching a segment to it, which form a circle together, then attaching another segment, resulting in a bouquet of two circles, then attaching a disk to this bouquet, resulting in the whole torus. Another usual example is the complex projective space $\mathbb{CP}^n$, with the Morse function $f(z_0 : \cdots : z_n) = \sum_i c_i |z_i|^2$, using the standard homogeneous coordinates, where the c_i's are different real numbers. Now f has exactly $n+1$ critical points $(1 : 0 : \cdots : 0), \ldots, (0 : \cdots : 0 : 1)$, with indices $\lambda_i = 2|\{j : c_j < c_i\}|$. Thus $\mathbb{CP}^n \simeq B^0 \cup B^2 \cdots \cup B^{2n}$, or, inductively, the cohomology of $\mathbb{CP}^n$ differs from that of $\mathbb{CP}^{n-1}$ by a single free generator in dimension $2n$. A basically similar method of computing the Betti numbers is to use the *Weil conjectures* (proved by Deligne, see [Si]), which involves counting the number of points of an algebraic variety over finite fields. In particular, the inductive formula $|\mathbb{F}_q\mathbb{P}^n| = |\mathbb{F}_q\mathbb{P}^{n-1}| + q^n$ corresponds to the stratification of projective spaces we had earlier.

A closer look at the process of attaching the handles B^λ gives a decomposition of the manifold into handles in the smooth category:

Theorem 0.4. *Every connected closed smooth manifold M^n is diffeomorphic to a union of finitely many handles $H^n_\lambda = B^\lambda \times B^{n-\lambda}$, λ variable, where the handles H^n_λ are in one-to-one correspondence with the critical points of index λ. Conversely, given a decomposition of the manifold as a sum of handles, there exists a Morse function which gives rise to this given handle decomposition.* $\square$

Clearly, even if two manifolds have the same set of cells in their Morse decompositions, they do not have to be diffeomorphic, homeomorphic, or homotopic to each other — this still depends on the attaching maps between the handles!

A first interesting illustration of this problem is the following result, attributed to *Reeb* in [M2]: if f is an arbitrary smooth function having exactly 2 critical points on a compact n-manifold M, then M is homeomorphic to the n-sphere. For the case of non-degenarate critical points this is a trivial consequence of Morse theory. If we allow for degenarate ones, as well, then one can use the geodesic flow of f and the smooth Jordan curve (or sphere, in general) theorem for the level sets of f, which are diffeomorphic to an $(n-1)$-sphere. Then the manifold can be built up from two n-balls glued along an $(n-1)$-sphere.

This result identifies M^n with S^n up to homeomorphism, but the diffeomorphism type of M^n depends on the smooth attaching map between the two standard $(n-1)$-spheres at the boundaries. So the set of exotic differentiable n-spheres coincides with the set of essentially different orientation-preserving self-diffeomorphisms of the standard sphere S^{n-1}, i.e. with the mapping class group $\mathrm{Diff}^+(S^{n-1})$. This group is usually non-trivial: the first exotic spheres in $n = 7$ dimension were found by Milnor [M1], and the number of them grows exponentially with n. *Donaldson* showed exotic differentiable structures e.g. on $\mathbb{R}^4$, see [DK]. On exotic spheres see also [M4] and [DFN III, Chp. 3].

Now we can already see how important it would be to describe the attaching maps via Morse theory. A nice possibility could be to use the following transparent explanation on the decomposition of the manifold into the handles B_λ. Consider the gradient flow $\phi_f^t : M \longrightarrow M$ of the Morse function f, corresponding to some Riemannian metric on M. The rest points of this flow are exactly the critical points of f, and the *stable* and *unstable manifolds* of a critical point p,

$$W_p^s = \{x \in M : \lim_{t\to\infty} \phi^t(x) = p\}, \qquad W_p^u = \{x \in M : \lim_{t\to-\infty} \phi^t(x) = p\}$$

are $\lambda(p)$ and $(n-\lambda(p))$-dimensional balls, where $\lambda(p)$ is the Morse index at p. Therefore the stable manifolds give a decomposition of M into $\lambda(p)$-dimensional submanifolds, and we are roughly done. The problem is that this cell decomposition does not always represent a cell complex, and there could be problems with the attaching maps. *S. Smale* defined the gradient-like vector fields in [S1], which have two main properties: locally around every critical point there is a Riemannian structure in which the vector field is the gradient of a smooth function, and that the stable and unstable manifolds W_p^u and W_q^s intersect transversally in each of their common points. He proved that for such a vector field we can find a so-called nice function (nowdays it is called a *Morse-Smale function*), whose gradient vector field coincides with the given vector field plus a constant around each critical point. These nice functions are those that have only non-degenerate critical points, and have the self-indexing property: if $f(p) = \lambda(p)$ for all critical points p. Secondly, he proved that every Morse function can be C^1-approximated by a nice function. Note that C^2-approximation is impossible, since it would preserve Morse indices. As a corollary we can see that Morse-Smale functions always exist. An account of Morse-Smale functions can be found in [DFN III, §17], too.

A careful inspection of the 2-dimensional case of Theorem 0.4, now already using a Morse-Smale function, easily gives the following classical result:

Theorem 0.5. (Classification of compact 2-manifolds) *Every smooth, connected, compact 2-manifold without boundary is homeomorphic (moreover, diffeomorphic) to a sphere with a finite number of handles and Möbius bands (crosscaps), attached in the usual way. In the presence of at least one Möbius band, the sum of two Möbius bands is equivalent to one handle, and the manifold is orientable iff there are no Möbius bands at all. In such cases the number of handles is called the genus g. The Euler characteristic is given by $\chi(M) = 2 - 2g$.* $\square$

We have already mentioned that a manifold carrying a smooth function with only 2 critical points is necessarily homeomorphic to a sphere. A natural problem is to minimize the number of critical points on a given manifold M. First of all, when can the Morse inequalities $m_k \geq b_k$ be sharp? Because of the second part of Theorem 0.4, if there is a cell decomposition of M with c_k cells of dimension k, then we also have a Morse function with $m_k = c_k$, so the two minimalization problems are the same. I do not know about a manifold that forces a sharp Morse inequality, but Smale proved in [Sm3] that for $n \geq 5$ dimensions the equalities $m_k = b_k$ can always be achieved. In fact, the problem 'if an n-manifold M has the same Betti numbers as the n-sphere, does it follow that it can have a Morse function with exactly two critical points, and so it is homeomorphic to S^n?' is known as the *Poincaré conjecture*, which is solved for

$n \geq 5$ by Smale [Sm2], for $n = 4$ by *Freedman* [DK], and is unsolved for $n = 3$. In [M3] we can find a well-written Morse-theoretical proof of *Smale's h-cobordism theorem* [Sm3], which implies the two theorems for $n \geq 5$.

Theorem 0.6. (Smale's h-cobordism theorem) *If W^{n+1} is a compact manifold-with-boundary with a disjoint union $\partial W = V_1^n \cup V_2^n$ such that each V_i is 1-connected and is a deformation retract of W, then W is diffeomorphic to $V_i \times [0,1]$, and so V_1 is diffeomorphic to V_2.* $\square$

Here the basic idea is to pick a Morse-Smale function f on W, and, using the non-degeneracy conditions of Smale, to modify f such that the neighbouring critical points collapse (1st and 2nd Cancellation Theorems), and we result in a Morse function f' without critical points at all. On cobordisms and smooth structures, besides [Sm2-3], see the above mentioned reference [DFN III, Chp. 3].

For the case of the tori $\mathbb{T}^n$, now we desribe the solution to our problem. As $\mathbb{T}^n$ is a compact symmetric space of $\mathbb{R}^n$ and also of itself as Lie-groups, its cohomology ring coincides with the ring of its invariant differential forms [DFN III, Thm. 1.14], which is the free exterior algebra over the differentials dx^i in our case. So $b_k(\mathbb{T}^n) = \binom{n}{k}$. On the other hand, if we construct the torus by gluing the opposite faces of an n-cube, we get a cell decomposition with $c_k = \binom{n}{k}$, so the optimum can be reached. An optimal Morse function can also be easily found: consider the $(\pi, \dots, \pi)$- periodic function $f(x_1, \dots, x_n) = \cos^2 x_1 + \cdots + \cos^2 x_n$ on $\mathbb{R}^n$. This is in fact a smooth function on $\mathbb{T}^n_\pi$, having $\binom{n}{k}$ critical points on the same level set of f for $k = 0, 1, \dots, n$.

The next question is what happens if we do not insist on Morse-functions, but allow for isolated degenerate critical points as well. The answer is given by the *Lyusternik-Shnirelman theory*, see [DFN III, §19]. For a closed subset $A \subseteq X$ of a Hausdorff space let us define its *category* $\mathrm{cat}_X(A)$ to be the least number k for which A can be written as a union of k closed sets A_i that are contractible in X. The two main results of the theory are that the number of critical points of a smooth function on M is always at least $\mathrm{cat}(M)$, and that $\mathrm{cat}(M) \geq l + 1$, where $l = l(M)$ is the *cohomological length* of M. This length is the largest integer k such that there exist non-zero products $\alpha_1 \cdots \alpha_k$ of k elements of positive degree in the comology ring $H^*(M, \mathbb{Z})$. On cohomological multiplication see [DFN III, §7]. Calculating the exact value of $\mathrm{cat}(M)$ is usually very difficult, but e.g. $l(\mathbb{T}^n) \geq n$ is clear from our previous description of the ring $H^*(\mathbb{T}^n, \mathbb{R})$. On the other hand, there is a smooth function on $\mathbb{T}^n$ having exactly $n + 1$ critical points. Indeed, the Morse function f considered above can be perturbated so that the $\binom{n}{k}$ critical points on each level set collapse, and we result in a function with $n + 1$ critical points. Thus $\mathrm{cat}(\mathbb{T}^n) = n + 1$, and this is the minimal number of isolated critical points.

One can continue the search for the topological consequences of few critical points. For example, for exactly three critical points (then the indices are 0, $n/2$, n, by Poincaré duality, see later), we have manifolds which are like the real projective plane, see [M2, §4] and [EK]. There can be 4 non-degenerate critical points on each of the 4-manifolds S^4, $S^1 \times S^3$, $S^2 \times S^2$, and for manifolds with $\chi(M) \neq 0$ these are the only homotopically different possibilities, see [Hu].

For critical points from a variational point of view (see Chapter 4) the Reader should try [MW].

0.2. First applications

Critical points and their indices are central objects in the theory of vector fields. So, first of all, we briefly survey the corresponding degree theory. For more see [DFN II], [KH] and [MT].

The *degree* of a smooth map $f : M \longrightarrow N$ between to n-dimensional oriented manifolds is defined at a regular value $y \in N$ as $\deg f(y) := \sum_{f(x)=y} \operatorname{sign} \det Df(x)$. This is in fact independent of the choice of y, (moreover, invariant under homotopies on f), thus we have defined $\deg f$. The same notion can be expressed in two other languages: $\deg f := \int_M f^*\omega$, where ω is a normalized volume form on N, that is a smooth differential n-form which is positive on positively oriented frames, $\int_N \omega = 1$, and $f^*\omega$ is its pullback. Also, $\deg f := f_{*n}^{\mathbb{Z}}(1)$, where $f_{*n}^{\mathbb{Z}} : H_n(M,\mathbb{Z}) \longrightarrow H_n(N,\mathbb{Z})$ is the induced homomorphism between homology groups.

Now we can define the *index* of a smooth map $f : M \longrightarrow M$ at an isolated fixed point p. Locally we can think as $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and if V is an n-ball around p without more fixed points, then the smooth map

$$\nu_{f,V} : \partial V \longrightarrow S^{n-1}, \qquad x \mapsto \frac{x - f(x)}{\|x - f(x)\|}$$

is called the *Gauss map.* Now $\mathrm{ind}_p f := \deg \nu_{f,V}$, which is independent of V. For a so-called transverse fixed point this index is easy to compute: if the spectrum of $A = Df(p)$ does not contain the value 1, then $\mathrm{ind}_p f = \mathrm{ind}_p A = \mathrm{sign}\det(\mathrm{Id} - A)$ — these two equalities are easy exercises.

The critical or singular points of a vector field $X : M \longrightarrow TM$ are the rest points of its flow $\phi_X^t : M \longrightarrow M$, and the index of X at an isolated critical point p is the degree of the corresponding Gauss map $\nu_{X,V}(x) = X(x)/\|X(x)\|$, or equivalently, $\mathrm{ind}_p X := \mathrm{ind}_p \phi_X^t$, which is of course independent of $t \in (0, \epsilon)$.

If we consider a Morse function $f : M \longrightarrow \mathbb{R}$, and its gradient flow ϕ_f^t corresponding to some Riemannian metric, then the critical points of f will be the rest points of ϕ_f^t. Now $\mathrm{ind}_p \phi_f^t$ is independent of the choice of the Riemannian metric and in locally Euclidean coordinates around a critical point p we have $D\phi_f^t(p) = \exp(tH_p)$, where H_p is the Hessian f_{**} at p. The non-degeneracy of p means $0 \notin \mathrm{spec} H_p$, i.e. $1 \notin \mathrm{spec}\exp(tH_p)$, so p is a transverse fixed point of ϕ_f^t. Thus $\mathrm{ind}_p \phi_f^t = \mathrm{sign}\det(\mathrm{Id} - \exp H_p) = (-1)^{\lambda_p}$, where λ_p is the Morse index.

Now the most important application of Morse theory in this direction is the following:

Theorem 0.7. (Poincaré-Hopf index theorem) *For an arbitrary vector field $X : M \longrightarrow TM$, and its smooth flow $\phi_X^t : M \longrightarrow M$, we have*

$$\sum_{\phi_X^t(p) = p} \mathrm{ind}_p \phi_X^t = \chi(M).$$

Proof. The sum of indices of the critical points of a vector field on a manifold is a topological invariant, that is, it depends only on M, and not on X, see [DFN II, §15] or [MT]. (The main reason is the homotopy invariance of the degree.) So it is enough to compute this sum for the geodesic flow of a Morse function, which we have already done: it is $\sum_{Df(p)=0} (-1)^{\lambda_p}$. Now the last equality in Theorem 0.3 gives the result. $\square$

We will see a complete proof of the much more general degenerate Poincaré-Hopf theorem in Chapter 3.

V. Arnold in [A, Appendix 9] writes that in 2 dimension the index of a critical point of a gradient vector field can only be $1, 0, -1, -2, -3, \ldots$, while can be arbitrary in higher dimensions. Is it only by chance that these sets of possible indices coincide with the possible values of $\chi(M^n)/2$, for orientable manifolds of the corresponding dimension?

It would be a natural wish to develop a Morse theory for fixed points of self-diffeomorphisms of a compact manifold, moreover, for smooth maps (e.g. embeddings) between different manifolds, which would be a direct generalization of the standard Morse theory $M \longrightarrow \mathbb{R}$. However, this is a far too hard problem in general. We saw that the self-diffeomorphisms of the standard sphere S^{n-1} correspond to the exotic smooth structures on the n-sphere. The index theory of vector fields can be directly applied to a self-diffeomorphism f only if the self-diffeomorphism is isotopic to the identity, i.e. we have a flow ϕ^t with $\phi^0 = \mathrm{id}$ and $\phi^1 = f$, since in this case we have the vector field $\dot{\phi}^0$. For example, a self-diffeomorphism f of S^2 is of degree 1 or -1, and in the first case it is homotopic to id, in the second case this is true for $-f$. In the first case the Poincaré-Hopf theorem and an inspection of the indices at the critical points (compare the result mentioned in the previous paragraph) give that we have at least two fixed points. It the second case we have at least two fixed points of $-f$. For a torus $\mathbb{T}^2$ we had at least 4 non-degenerate critical points, but translation, for instance, has no fixed points at all. There is a deep theorem of *Conley* and *Zehnder* [CZ], resolving a special case of *Arnold's conjecture*: a certain natural class of the so-called symplectomorphisms of $\mathbb{T}^{2n}$ have at least $2n+1$ fixed points, and this number is at least 2^{2n} if all of them is non-degenerate. We will see more on this topic in Section 4.4.

From our degree and Morse theories the Gauss-Bonnet theorem and its natural generalizations can also be deduced [MT]:

Theorem 0.8. (Gauss-Bonnet theorem) *For a compact oriented 2-manifold M we have*

$$\frac{1}{2\pi}\int_M K(p)\,\mathrm{Vol}(p) = \chi(M),$$

where $K(p)$ is the Gaussian product curvature, which is the Jacobian of the Gauss map $n : M \longrightarrow S^2 \subset \mathbb{R}^3$ arising at some isometric embedding $M \hookrightarrow \mathbb{R}^3$.

Proof. First of all, note that the curvature $K(p)$ is in fact independent of the embedding, this is *Gauss's Theorema Egregium*, see [DFN I, §30].

Now we can choose (by Sard's theorem) a pair of regular antipodal points $p_\pm$ on S^2: if $n(p) = p_\pm$ then $K(p) \neq 0$. Now consider the projection $f : M :\longrightarrow \mathbb{R} \subset \mathbb{R}^3$ onto the line determined by the points $p_\pm$. This f has its critical points exactly at the points $p \in M$ for which $n(p) = p_\pm$. At any such point p, a neighbourhood of p can be parametrized by $(u, v, f(u, v))$, and in these local coordinates $K(p)$ can be expressed via the Hessian f_{**}. In fact, the determinant of f_{**} has the same sign as $K(p)$, so the Morse index of f at p is even iff $K(p) > 0$. Thus by the usual Morse equality we have

$$\chi(M) = |\{p \in M \,:\, n(p) = p_\pm,\ K(p) > 0\}| - |\{p \in M \,:\, n(p) = p_\pm,\ K(p) < 0\}|.$$

On the other hand, we have the same kind of expressions in the definition of $\deg n$ at the regular points $p_\pm$, so we get $\chi(M) = \deg n(p_+) + \deg n(p_-) = 2\deg(n)$. Now we can consider the pullback of the volume form on S^2 by n: $(n^*\mathrm{Vol}_{S^2})(p) = K(p)\mathrm{Vol}_M(p)$. Hence

$$\int_M K\,\mathrm{Vol}_M = \int_M n^*(\mathrm{Vol}_{S^2}) = (\deg n)\int_{S^2}\mathrm{Vol}_{S^2} = 4\pi\deg n,$$

which, together with the previous result on $\deg n$, gives the theorem. □

Instead of the Gaussian curvature in the tangent bundle we can consider arbitrary curvature forms in real or complex vector bundles, as well. We will see this generalization in Section 5.2.

One of the most important computational tricks of any (co-)homology theory is the *Poincaré duality theorem*, which has many different versions. In the smooth category [MT] it states $H^k(M) \simeq H_c^{n-k}(M)^*$ for the deRham cohomology groups, where the * stands for the dual vectorspace, $_c$ stands for the cohomology of forms with compact support, and M is not necessarily compact. Here the isomorphism is given by $(\omega_1, \omega_2) \mapsto \int_M \omega_1 \wedge \omega_2$ for $\omega_1 \in H^k(M)$, $\omega_2 \in H_c^{n-k}(M)$. In the simplicial category [Ma], [DFN III], we have the Alexander-Poincaré theorem: if K is an orientable homology-n-manifold with a pair of subcomplexes $L \geq M$, and G is an Abelian group, then $H^k(L, M; G) \simeq H_{n-k}(K - M, K - L; G)$; and the Lefschetz theorem for homology-n-manifolds-with-boundary: $H^k(M, \partial M; G) \simeq H_{n-k}(M; G)$. The cell-homological Poincaré theorem for compact manifolds follows easily from Morse theory [DFN III, §18]. The basic idea is that changing a Morse function f on M to $-f$ interchanges the stable and unstable manifolds, and thus we get a natural isomorphism between cells of complementary dimensions. The proof in the simplical category uses dual polyhedra constructed via the barycentic subdivision, and/or cap product, that is the simplicial counterpart of the smooth wedge product. We will see a proof for the smooth category in Section 1.6. For non-orientable manifolds everything remains true over $\mathbb{Z}_2$.

Here are some immediate corollaries to Poincaré duality. If $H_1(K;\mathbb{Z}_2) = 0$, then K is orientable. No non-orientable homology-n-manifold can be embedded into S^{n+1}. If $K \geq L$ are two homology-n-manifolds, then $K = L$. The cohomology ring $H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1}$, where $\deg \alpha = 2$. The Borsuk-Ulam theorem. If M is a homology-n-manifold-with-boundary, then $\chi(\partial M)$ is even. So, for instance, S^{2k} and $\mathbb{RP}^{2k}$ are not cobordant.

1. Preliminaries

In the following ten sections we give an account of the background material that we think to be necessary or at least useful in understanding the forthcoming more advanced topics in Morse theory. The most fortunate case would be if the Reader already knew almost everything to be described here, and this chapter served only as a warm-up. Nevertheless, we shall try to describe all the results in such details that seem to be enough to believe the validity of these theories. We will use the basics of differentiable manifolds, Riemannian geometry, algebraic topology, Lie groups and their Lie algebras without further warning — as we did it already in the Introduction. For more information the Reader may turn to [DFN], [KN], [H1-2], [Se], [R], [BGV], [G].

1.1. Bundles on surfaces

A *locally trivial fibre bundle* with fibre F over a base space X is a continuous map $\pi : Y \longrightarrow X$ such that every point $x \in X$ has a neighbourhood $U \subseteq X$ such that $Y_U = \pi^{-1}(U)$ is homeomorphic to $U \times F$ by a map $\phi_U : Y_U \longrightarrow U \times F$ taking each $Y_x = \pi^{-1}(x)$ to $\{x\} \times F$. Now Y is called the *total space*, X is the *base space*, π is the *projection*, and the Y_x's are the *fibres*. Sometimes the notation $F \longrightarrow Y \longrightarrow X$ is used. A map $\xi : X \longrightarrow Y$ with $\pi \circ \xi = \mathrm{id}_X$ is called a *section* or *cross-section*.

Analogously, we can define smooth bundles for smooth manifolds, and holomorphic bundles for complex manifolds.

Bundles usually have an additional structure, i.e. they are equipped with a *structural group* G. This G is a group of transformations of F in the appropriate category, so it is a topological group acting on F from the left. All the *transition maps* $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times F \longrightarrow U_\alpha \cap U_\beta) \times F$ are required to take the from $(u, f) \mapsto (u, g_{\alpha\beta}(u)f)$, with $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$.

These transition maps $g_{\alpha\beta}$ are clearly continuous, $g_{\alpha\alpha} = \mathrm{id}_F$, $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$, and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \mathrm{id}_F$. Conversely, if we are given such a family, we can construct a fibre bundle: $Y := \dot{\cup}(U_\alpha \times F)/\sim$, where the equivalence relation is $(u_\alpha, f) \sim (u_\alpha, g_{\alpha\beta}(u_\alpha)f)$. Two families $\{g_{\alpha\beta}\}$, $\{\tilde{g}_{\alpha\beta}\}$ describe the same bundle if there are maps $f_\alpha : U_\alpha \longrightarrow G$ such that $\tilde{g}_{\alpha\beta} = f_\beta g_{\alpha\beta} f_\alpha^{-1}$.

The two most important examples are vector bundles and homogeneous spaces.

A *rank k real vector bundle* $\pi : E \longrightarrow X$ is a bundle with fibre $\mathbb{R}^k$ and structural group $GL(k, \mathbb{R})$. For example, the *tangent bundle* TX of a smooth n-manifold is a rank n bundle over X. If the manifold has an atlas $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$, then this atlas gives a natural local trivialization $\phi_\alpha : TU_\alpha \longrightarrow U_\alpha \times \mathbb{R}^n$, and we have the transition maps $g_{\alpha\beta} = D(\varphi_\beta \circ \varphi_\alpha^{-1})$. Differential k-forms are the smooth sections of the kth exterior bundle $\Lambda^k(T^*X)$, which also has a structural group $GL(n, \mathbb{R})$. This space of sections is usually denoted by $\Omega^k(X)$. A bundle is *oriented* if its structural group is $G = GL^+(n, \mathbb{R})$, the group of matrices with positive determinant. For example, the Möbius band of unbounded width is a non-orientable rank 1 real vector bundle over the circle S^1. We can also speak of *complex vector bundles* over real manifolds, with structural group $GL(k, \mathbb{C})$. On a real vector bundle over a paracompact base space X, using partitions of unity, one can always construct an *inner product* $\langle \cdot, \cdot \rangle$; this is a collection of inner products on each fibre E_x so that the map $E \longrightarrow \mathbb{R}$, $v \mapsto \|v\|^2$ is continuous. Similarly, there is always a Hermitian inner product on complex vector bundles. Given already an inner product on E, we can find local trivializations respecting this inner product, and then this bundle can be regarded as a bundle with structural group $O(k)$ or $U(k)$ — these are the *Euclidean and Hermitian vector bundles.*

Besides proving the results of the previous paragraph, here are some usual exercises for the Reader. Show that a vector bundle is globally trivial, i.e. splits as a direct product, iff there is a nowhere-zero section of it. Show that the tangent bundle TS^2 is not globally trivial. Show that for any n-dimensional Lie group G, we have $TG \simeq G \times \mathbb{R}^n$. It is a very deep result that the only spheres with globally trivial tangent bundles are S^0, S^1, S^3 and S^7. The reason for them to be globally trivial is that they are the unit spheres of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the quaternions), and $\mathbb{O}$ (the Cayley octaves), respectively, and so they inherit something like a group structure.

With vector bundles we can do everything that is usual for single vector spaces: taking duals *, direct sums $\oplus$, tensor products $\otimes$, or form the bundle $\mathrm{End}(E)$ of fibrewise linear endomorphisms. It must also be clear for the Reader how to define the pulled-back bundle f^*Y over X' if we have a bundle Y over X, and a continuous map $f : X' \longrightarrow X$.

A complex vector bundle over a complex manifold is holomorphic iff the transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n, \mathbb{C})$ are holomorphic. If X is an n-dimensional complex manifold, then its standard holomorphic tangent bundle TX is the space of $\mathbb{C}$-linear maps from the germs of holomorphic functions to $\mathbb{C}$. This is a holomorphic bundle $TX = T^{10}X$ of complex rank n. But we can also consider the bundle $T^{01}X$ of $\mathbb{C}$-antilinear maps, which is a complex (but not holomorphic) vector bundle of complex rank n. If we consider X as a $2n$-dimensional real manifold, then its complexified real tangent bundle $T_{\mathbb{C}}X$ is a $2n$-dimensional complex bundle, which splits as $T_{\mathbb{C}}X = T^{10}X \oplus T^{01}X$. For the cotangent bundles we have $T^*_{\mathbb{C}}X = \Lambda^{10}(X) \oplus \Lambda^{01}(X)$, for the smooth sections we have $\Omega^1_{\mathbb{C}}(X) = \Omega^{10}(X) \oplus \Omega^{01}(X)$, and the complexified exterior derivative $d_{\mathbb{C}} : \Omega^0_{\mathbb{C}}(X) \longrightarrow \Omega^1_{\mathbb{C}}(X)$ breaks up as $d_{\mathbb{C}} = \partial + \overline{\partial}$. A complex valued smooth function $f \in \Omega^0_{\mathbb{C}}(X)$ is holomorphic iff $\overline{\partial} f = 0$ — in local coordinates this condition is the well-known Cauchy-Riemann equality.

Sometimes we do not have an honest complex manifold, but only an *almost complex manifold*, which is an even-dimensional smooth manifold X with a smooth family of linear maps $J_x \in \mathrm{End}(T_xX)$ satisfying $J_x^2 = -1$. In this case a locally defined map $f : U \longrightarrow \mathbb{C}$ is said to be holomorphic if $Df(x)(J_x v) = iDf(x)v$ for all $v \in T_xX$. In this way we can also define the splitting $\Omega^1(X) = \Omega^{10}(X) \oplus \Omega^{01}$ as above. For two-dimensional surfaces it is true that every almost complex structure comes from a complex structure.

On $\mathbb{R}^2$, every linear map $J^2 = -1$ is conjugated to the standard skew-symmetric matrix

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

by an element $g \in SL(2, \mathbb{R})$, and two such g_1, g_2 give the same J iff $g_1^{-1} g_2 \in SO(2)$, the centralizer of J_0. Thus the space of complex structures on $\mathbb{R}^2$ is isomorphic to $SL(2, \mathbb{R})/SO(2)$, which is actually the upper half plane model of the hyperbolic plane. Therefore, a complex structure on a 2-dimensional surface is given by a smooth map between two 2-dimensional real manifolds. Since a diffeomorphism of a surface is specified by the same amount of data, it is unsurprising that the space of isomorphism classes of Riemann surfaces of a given genus is finite dimensional. In fact, if the genus is g, then this space is $6g - 6$-dimensional.

The rank 1 complex vector bundles are called *complex line bundles*. The trivial line bundle $\mathcal{O}_M := M \times \mathbb{C}$, in the case of a compact complex M, has only constant holomorphic sections, by the maximum principle. In general, the complex vector space $H^0(M, E)$ of holomorphic sections of an arbitrary holomorphic vector bundle E over M is always finite dimensional — this is a special case of a finiteness theorem for sheaf cohomology. The *canonical line bundle* $\mathcal{K}_M := \Lambda^n(T^*M) = \Lambda^{n0}(M)$ is in fact a holomorphic line bundle if M is n-dimensional, and the manifolds with $\mathcal{K}_M = \mathcal{O}_M$ are of particular interest: they are the so-called *Calabi-Yau manifolds*.

For the projective spaces $\mathbb{CP}^n$ there is a holomorphic line bundle $\mathcal{O}_n(k)$ for every $k \in \mathbb{Z}$: the total space is $\big((\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}\big)/\mathbb{C}^*$, where the action of $\lambda \in \mathbb{C}^*$ is $(z_0, \ldots, z_n) \times w \mapsto (\lambda z_0, \ldots, \lambda z_n) \times \lambda^k w$. The bundle projection is the natural $(z_0, \ldots, z_n) \times w \mapsto [z_0 : \cdots : z_n]$. The sections of $\mathcal{O}_n(k)$ are clearly given by the holomorphic functions $f : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}$ satisfying $f(\lambda z_0, \ldots, \lambda z_n) = \lambda^k f(z_0, \ldots, z_n)$. For $k \geq 0$ these are the homogeneous polynomials of degree k, and there is none for $k < 0$. Hence some simple combinatorics shows that $\dim_{\mathbb{C}} H^0(\mathbb{CP}^n, \mathcal{O}_n(k)) =?$ (an exercise for the Reader). The line bundle $\mathcal{O}_n(-1)$ has another name: the *tautological line bundle* H_n. The reason for this name is that it can be desribed as $H_n = \{([z], w) \,|\, [z] \in \mathbb{CP}^n, w \in \mathbb{C}^{n+1}, w = \lambda z, \lambda \in \mathbb{C}\}$. The reason for the equality is that if one picks local trivializations over the neighbourhoods $U_i := \{[z_0 : \cdots : z_n] \,|\, z_i \neq 0\} \subseteq \mathbb{CP}^n$, then the transition maps are readily seen to be $g_{ij} = z_j/z_i$ in both cases. It is also easy to see that $H_n^* = \mathcal{O}_n(1)$, $\mathcal{O}_{\mathbb{CP}^n} = \mathcal{O}_n(0)$, and $\mathcal{O}_n(k) \otimes \mathcal{O}_n(l) = \mathcal{O}_n(k+l)$, so we have a nice group of line bundles here.

To any fibre bundle with structural group G we can associate its *principal bundle* P with the same structural group G, the same transition maps $g_{\alpha\beta}$, but fibre G, with the natural left action of the structural group G by left translations. Now, in any local trivialization $P_U \simeq U \times G$ we can consider the right action of G by right translations in each fibre. Since left and right translations of G commute, the result of this local right action is actually independent of the local trivialization chosen, and so defines a global right action on P. In general, a *principal G-bundle* P is a fibre bundle with structural group and fibre G. Clearly, $P/G \simeq X$.

A *homogeneous G-space* is a smooth manifold X with a smooth transitive right action by a Lie group G. Because of transitivity, the stabilizers G_x of points $x \in X$ are all isomorphic to the same Lie subgroup H, and the right quotient Lie-group G/H is naturally diffeomorphic to X. Here we have a fibre bundle with total space G, base space X, fibre H, and structural group H, so this is a principal H-bundle. More generally, if H is a Lie group acting from the right without fixed points on a manifold Y, then (under some mild conditions) the orbit space $X = Y/H$ inherits a smooth structure, and $H \longrightarrow Y \longrightarrow X$ is a principal H-bundle.

Some important examples of homogeneous spaces: $\mathbb{Z}^n \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{T}^n$, $SO(n-1) \longrightarrow SO(n) \longrightarrow S^{n-1}$, $SU(n-1) \longrightarrow SU(n) \longrightarrow S^{2n-1}$, $U(n-1) \longrightarrow SU(n) \longrightarrow \mathbb{CP}^{n-1}$. A famous special case of the last example is the *Hopf fibration* $S^1 \longrightarrow S^3 \xrightarrow{\gamma} S^2$, where $\gamma : S^3 \longrightarrow S^2$ is the following: $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \simeq SU(2)$, $S^2 \simeq \mathbb{CP}^1$, $\gamma : (z,w) \mapsto -z/\overline{w}$. Check that the fibres over any two distinct points are two circles which are linked. We will see an interesting consequence of this fibration after Theorem 1.2.

The basic homotopy-theoretic fact about bundles is that if $f_0, f_1 : X' \longrightarrow X$ are two homotopic maps, and Y is a bundle over X, then $f_0^*Y \simeq f_1^*Y$. A simple corollary is that any bundle on a contractible base is trivial. A more attractive application is the following:

Theorem 1.1. *Isomorphism classes of bundles on a closed oriented surface (i.e. on a compact 2-dimensional manifold) Σ with a connected structural group G correspond precisely to elements of $\pi_1(G)$.*

Main idea of proof: The surface Σ can be covered by two contractible neighbourhoods, U_1 and U_2, intersecting in an annulus U. Now any bundle Y is locally trivial over U_0 and U_1, so Y is given by a single transition map $g_{01} : U \longrightarrow G$. Using the homotopy-theoretical results above, it is easy to see that the bundle Y depends exactly on the homotopy class of g_{01}, i.e. on a class in $\pi_1(G)$. $\square$

An important example is $\pi_1(GL(n,\mathbb{C})) \simeq \mathbb{Z}$, where the isomorphism is given by assigning to a loop γ the winding number of $\det(\gamma) : S^1 \longrightarrow \mathbb{C}^*$. For a complex vector bundle E on Σ the corresponding class $C_1(E) \in \mathbb{Z}$ is called the *first Chern class* of E. Now it is already less surprising that the complex line bundles $\mathcal{O}_n(k)$ form a group: a good exercise is to prove $C_1(\mathcal{O}_1(k)) = k$.

The following generalization can be found in [DFN II, §24]: the isomorphism classes of G-bundles over the sphere S^n are given by the homotopy group $\pi_{n-1}(G)$.

A second result about the nice behaviour of fibre bundles under homotopies is the *(relative) homotopy lifting property*, which has a corollary of crucial role in computing higher homotopy groups $\pi_k(X)$:

Theorem 1.2. (Homotopy exact sequence) *Let $\pi : Y \longrightarrow X$ be a fibre bundle with some arbitrary base points $x \in X$, $y \in Y$, $\pi(y) = x$, and with fibre $F = \pi^{-1}(x)$, with its inclusion map $\iota : F \longrightarrow Y$. Now there exists a homomorphism $\partial : \pi_k(X,x) \longrightarrow \pi_{k-1}(F,y)$ such that* *the following sequence is exact:*

$$\cdots \longrightarrow \pi_{k+1}(B) \xrightarrow{\partial} \pi_k(F) \xrightarrow{\iota_*} \pi_k(E) \xrightarrow{p_*} \pi_k(B) \xrightarrow{\partial} \pi_{k-1}(F) \longrightarrow \cdots \qquad \square$$

Some homotopy groups of the spheres are easy to determine. By Sard's theorem, a differentiable map $S^k \longrightarrow S^n$ cannot be surjective for $k < n$, and from this $\pi_k(S^n) = 0$ follows. Using the notion of degree defined in the Introduction, $\pi_n(S^n) = \mathbb{Z}$. However, the other cases are very far from trivial. Using Theorem 1.2, the striking corollary to the Hopf fibration we promised is $\pi_3(S^2) = \mathbb{Z}$. The higher homotopy groups of S^2 are all known to be non-trivial, but they are not completely described even conjecturally.

Using the examples of homogeneous spaces above, one can immediately see that $\pi_k(O(n)) = \pi_k(O(n+1))$ for $k \leq n-2$, and $\pi_k(U(n)) = \pi_k(U(n+1))$ for $k \leq 2n-1$. Thus, for example, $\pi_1(U(n)) = \pi_1(U(1)) = \mathbb{Z}$ and $\pi_1(O(n)) = \pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$. The second equality is because of the group isomorphism $SO(3) \simeq SU(2)/\{\pm 1\}$, which yields $SO(3) \simeq \mathbb{RP}^3$. For the group isomorphism, consider $SU(2)$ as the unit sphere of the quaternionic algebra $\mathbb{H}$, and the action of it by conjugation on the three dimensional real subspace of pure quaternions.

On the homotopy groups of the orthogonal and unitary groups a famous result is *Bott's periodicity theorem*, a Morse theoretical proof of which can be found in [M]. This says that $\pi_k(U(n)) = \pi_{k+2}(U(n))$ and $\pi_k(O(n)) = \pi_{k+8}(O(n))$ for all $k \geq 0$ and all n (≥ 3 in the orthogonal case).

Note that $GL(n,\mathbb{R})$ is homotopic to $SO(n)$ and $GL(n,\mathbb{C})$ is to $U(n)$, so we have another proof of $\pi_1(GL(n,\mathbb{C})) \simeq \mathbb{Z}$.

1.2. Connexions and curvature

On $\mathbb{R}^n$ there is a natural identification between any two tangent spaces $T_x\mathbb{R}^n$ and $T_y\mathbb{R}^n$. However, on a general manifold we have to choose a connexion for transportation between different tangent spaces. In this section we overview some different definitions of parallel transport, connexion, and the curvature of a connexion in fibre bundles $F \longrightarrow Y \xrightarrow{\pi} X$, with structural group G.

Definition 1. *A connexion is a rule assigning to each smooth path $\gamma : [a,b] \longrightarrow X$ an isomorphism $T_\gamma : Y_{\gamma(a)} \longrightarrow Y_{\gamma(b)}$ called the parallel transport along γ, with three properties:*

(i) *Transitivity: $T_{\gamma_2} \circ T_{\gamma_1} = T_{\gamma_2 * \gamma_1}$, if the composite path $\gamma_2 * \gamma_1$ is well-defined and smooth.*

(ii) *Strong parametrization independence: if $\theta : [a,b] \longrightarrow [c,d]$ is smooth with $\theta(a) = c$, $\theta(b) = d$, but it is not necessarily bijective, then $T_\gamma = T_{\gamma\circ\theta}$.*

(iii) *T_γ depends smoothly on γ, where the space of paths $[a,b] \longrightarrow X$ is equipped with the topology of uniform convergence of γ and all its derivatives.*

Note that we have parallel transport for piecewise smooth paths, as well, because every path can be reparametrized to be stationary on some $[a, a+\epsilon]$ and on $[b-\epsilon, b]$, and so any composition of two smooth paths will be smooth.

Connexion is a local object in the sense that if we have a connexion in Y_{U_α} for each member U_α of an open covering of X, and they agree in each $U_\alpha \cap U_\beta$, then we have a connexion in Y. Also, a connexion in Y gives a connexion in the principal bundle P, and vice versa. For a principal G-bundle we can associate a vector bundle with fibre V and structural group G via any linear representation $\rho : G \longrightarrow GL(V)$. These facts mean that it is enough to understand connexions in principal bundles or in trivial vector bundles $U \times \mathbb{R}^k$, $U \subseteq \mathbb{R}^n$.

So now we continue with a definition for principal bundles $P/G = X$.

Definition 2. *Let $V = \ker(D\pi) \leq TP$ be the sub-bundle of 'vertical tangent vectors' over P. A connexion is a G-equivariant splitting $T_pP = H_p \oplus V_p$, i.e. a choice of a complementary sub-bundle of 'horizontal tangent vectors'. This splitting defines a 'horizontal lift' $h(t,p) \in H_p$ for any tangent vector $t \in T_xX$ with $\pi(p) = x$.*

Given such a connexion, and a path γ from x to y in X, we can consider the tangent vector field of γ, and for any $p \in P_x$ we have a horizontal lift of this vector field, and so also a horizontal lift of the path, which lifted path $\tilde{\gamma}$ starts at p and ends at a point $q \in P_y$. This map $P_x \ni p \mapsto q \in P_y$ is a parallel transport described in Definition 1, so we can really speak of a connexion here.

As the vertical tangent space V_p is naturally identified with the tangent space of the fibre G, i.e. with the Lie algebra $\mathfrak{g}$ of G, we can consider the splitting above as a projection $\alpha_p : T_pP \longrightarrow V_p \simeq \mathfrak{g}$, where $H_p = \ker(\alpha_p)$. As G acts on P, it also acts on TP, and each $u \in \mathfrak{g}$ induces a *Killing vector field* $K_u : P \longrightarrow TP$. The fact that G acts inside a fibre P_x is equivalent to $\alpha(K_u) \equiv u$. So a neat reformulation of the notion of a connexion is the following:

Definition 3. *A $\mathfrak{g}$-valued 1-form $\alpha \in \Omega^1(P;\mathfrak{g})$ on P is a connexion 1-form if it is G-equivariant: $\alpha(v.g) = \mathrm{Ad}(g^{-1})\alpha(v)$ for all $v \in TP$ and $g \in G$; and it represents a projection: $\alpha(K_u) \equiv u$ for all $u \in \mathfrak{g}$.*

The three conditions for parallel transport in Definition 1 may appear to an inventive Reader as results of an infinitesimal description by a differential equation. So our next definition will be such an infinitesimal one. It resembles the previous definition, but I do not know whether there is a nice formula describing the forthcoming 1-forms A_τ in terms of the global 1-form α.

Let $\tau : Y_U \longrightarrow U \times F$ be a local trivialization, and $A_\tau \in \Omega^1(U,\mathfrak{g})$ a Lie-algebra valued 1-form on U. Then we can locally define parallel transport along a curve $\gamma : [a,b] \longrightarrow U$ by integrating the first order ordinary differential equation

$$\frac{d}{dt}\xi(t) + A_\tau\big(\gamma(t);\dot{\gamma}(t)\big)\xi(t) = 0 \quad \text{in } T_{\xi(t)}Y;$$

if $\xi(a) \in Y_{\gamma(a)}$, then $T_\gamma\xi(a) = \xi(b) \in Y_{\gamma(b)}$. The three conditions of Definition 1 are fulfilled by standard ODE theory.

Of course, we want our notion to be independent of the trivialization chosen. If we change trivialization from τ to $\tilde{\tau}$ by a map $g : U \longrightarrow G$ (now acting from the left), then for a section $\xi(t)$ parallel along $\gamma(t)$ we have $\dot{\xi} + A_\tau(\gamma;\dot{\gamma})\xi = 0$ in the old trivialization, and $(g\xi)^{\cdot} + A_{\tilde{\tau}}(\gamma;\dot{\gamma})(g\xi) = 0$ in the new one. After substituting $\xi = g^{-1}\eta$ we find $A_{\tilde{\tau}}(\gamma;\dot{\gamma})\eta = gA_\tau(\gamma;\dot{\gamma})g^{-1}\eta - \dot{g}g^{-1}\eta$. Since γ and ξ were arbitrary, this means

$$A_{\tilde{\tau}} = \mathrm{Ad}(g^{-1})A_\tau - dg \cdot g^{-1},$$

which is the basic transition equality in gauge theory.

Definition 4. *A connexion is a collection $\{A_\tau\}$ of 1-forms in $\Omega^1(U_\tau;\mathfrak{g})$ defined for all local trivializations τ, and related by the transitions described in the equality above.*

Finally, we describe connexions as a global way of differentiating sections in vector bundles $V \longrightarrow E \longrightarrow X$. Suppose we have local trivializations over an open covering U_τ, and a global section $\xi : X \longrightarrow E$ given locally by $\xi_\tau : U_\tau \longrightarrow V$. Let $A_\tau \in \Omega^1(U_\alpha;\mathrm{End}(V))$ be a connexion, and define

$$d_{A_\tau}\xi_\tau := d\xi_\tau + A_\tau\xi_\tau \in \Omega^1(U_\tau;V).$$

An easy calculation with transition maps shows that these local 1-forms $d_{A_\tau}\xi_\tau$ fit together to define a global 1-form $d_A\xi \in \Omega^1(X;E)$. So our last definition is the following:

Definition 5. *A connexion A is given by a covariant derivative $d_A = \nabla : \Omega^0(X;E) \longrightarrow \Omega^1(X;E)$ which is linear and satisfies the Leibniz rule $d_A(f\xi) = fd_A\xi + df\xi$ for every section $\xi \in \Omega^0(X;E)$ and function $f \in \Omega^0(X;\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, depending on the base field of the vector space V.*

Note that in a holomorphic vector bundle E, d_A acts on all smooth sections, and not only on the holomorphic ones.

For a tangent vector field $v : X \longrightarrow TX$ the following notation is used sometimes: $\nabla\xi(v) = \nabla_v\xi$, which suggests that we differentiate sections in the direction of a vector field. This also makes it useful to define $\nabla_v f = v(f)$, i.e. $d_A f = df$ for $f \in \Omega^0(M, \mathbb{K})$, for every connexion A. Moreover, we can define $d_A : \Omega^p(X; E) \longrightarrow \Omega^{p+1}(X; E)$ for all p by means of the formula $d_A(\xi\omega) = d_A\xi \cdot \omega + \xi d\omega$ for $\xi \in \Omega^0(X; E)$ and $\omega \in \Omega^p(X)$.

In general, a purely algebraic definition of connexion in an arbitrary module is the following. Let M be a module over a ring R; e.g. $R = \Omega^0(X)$ has been our case so far. Then let $\mathcal{D}$ be the module of derivations of R, and $\Omega^1_R = \mathrm{Hom}_R(\mathcal{D}; R)$. Now a connexion in M is a map $M \longrightarrow M \otimes_R \Omega^1_R$ of R-modules satisfying the Leibniz rule. At a cursory glance, the Lie-derivative is also such a connexion, but it is of course not. For a tangent vector field $w \in \Omega^0(X; TX)$ we actually have $[w, \cdot] \notin \Omega^1(X; TX)$, since $[w, v]_x$ depends not only on v_x, but also on the whole v in a neighbourhood of x. One might speculate that the prize for the connexion-freeness of the Lie-derivative is this non-tensorial property.

We also have a natural extension of covariant derivative for sections of $\mathrm{End}(E)$. If $\Theta \in \Omega^0(X; \mathrm{End}(E))$, and $\xi \in \Omega^0(X; E)$, then we would like to have $d_A(\Theta\xi) = \Theta d_A\xi + (d_A\Theta)\xi$, which means $d_A\Theta = d\Theta + [A, \Theta]$.

We can also see that any two connexions A and $\tilde{A}$ in the same vector bundle E satisfy $d_A\xi - d_{\tilde{A}}\xi = a\xi$ for some $a \in \Omega^1(X; \mathrm{End}(E))$. In other words, connexions in E form an affine space for $\Omega^1(X; \mathrm{End}(E))$.

If our vector bundle E is equipped with some additional structure, e.g. with an Euclidean or Hermitian inner product, that one usually would like to use connexions compatible with these structures. This means that the parallel transports T_γ are required to take values in the structural group G, that is to be e.g. Euclidean or Hermitian isometries. It is easy to verify that for the covariant derivative this amounts to the compatibility condition $d\langle\xi, \eta\rangle = \langle d_A\xi, \eta\rangle + \langle\xi, d_A\eta\rangle$. Using partitions of unity one can always construct such *orthogonal or unitary connexions* in a Euclidean or Hermitian vector bundle, respectively.

In the tangent bundle of a smooth manifold one is sometimes interested in *torsion-free (or symmetric) connexions*. Clearly, ∇ itself is not tensorial, but its torsion tensor $\mathcal{T}_x(v, w) = \nabla_{w_x} v - \nabla_{v_x} w - [v, w]_x$ depends only on the single tangent vectors v_x and w_x. Now ∇ is said to be torsion-free if $\mathcal{T}(v, w) = 0$ for all vector fields v, w. If $\varepsilon \in \Omega^1(X; TX)$ is the tautological 1-form defined by $\varepsilon(x; v) = v$, then torsion-freeness is the property $\nabla_A\varepsilon = 0$. (Why?) On a Riemannian manifold there is exactly one symmetric connexion compatible with the Riemannian metric; it is called the *Levi-Cività connexion*.

In a complex vector bundle $E \longrightarrow X$ any connexion splits as $d_A = \partial_A + \overline{\partial}_A$, according to the splitting $\Omega^1(X; E) = \Omega^{10}(X; E) \oplus \Omega^{01}(X; E)$ of $\mathbb{C}$- linear and $\mathbb{C}$-antilinear maps. On the other hand, in a holomorphic bundle there is a canonical map $\overline{\partial} : \Omega^0(X; E) \longrightarrow \Omega^{01}(X; E)$, defined without choosing a connexion. This map $\xi \mapsto \overline{\partial}\xi$ can be given locally in a holomorphic trivialization τ as $\xi_\tau \mapsto \overline{\partial}\xi_\tau$, using the standard $\overline{\partial}$ map of vector spaces, and these local sections fit together to define the global $\overline{\partial}\xi$, because $\overline{\partial}$ commutes with the action of transition maps, which is multiplication by holomorphic matrix-valued functions in this case. Thus we can make the following definition: a connexion d_A is *compatible with the holomorphic structure* of E if $\overline{\partial}_A = \overline{\partial}$.

Conversely, if X has complex dimension 1, then any connexion in a complex vector bundle E defines a holomorphic structure, by claiming a locally defined section ξ holomorphic if $\overline{\partial}_A\xi = 0$, and then finding independent solutions $\xi_1, \ldots, \xi_k$ to have holomorphic basis section for E. The local solubility of $\overline{\partial}_A\xi = 0$ what holds only for $\dim_{\mathbb{C}} X = 1$, and this is the same phenomenon that we already encountered at the integrability of an almost-$\mathbb{C}$-structure.

Now, if E is a holomorphic bundle with a Hermitian inner product, then there is a unique connexion d_A which is both unitary and compatible with the holomorphic structure. For if the connexion 1-form A is given by $A = P_1(z_1, \ldots, z_n)dz_1 + \cdots + P_n(z_1, \ldots, z_n)dz_n + Q_1(z_1, \ldots, z_n)d\overline{z}_1 + \cdots + Q_n(z_1, \ldots, z_n)d\overline{z}_n$ with respect to an orthonormal trivialization, then the Q_i's are determined by the holomorphic structure, and then $P = -Q^*$ ensures unitarity. Alternatively, if we use a holomorphic trivialization, the inner product is given by a Hermitian-matrix-valued function h, and the unitary condition $dh = A^*h + hA$ together with the holomorphic compatibility implies $A = h^{-1}\partial h$.

Summarizing these observations:

Theorem 1.3. *For a complex Hermitian vector bundle E over a Riemann surface Σ (a complex 1-dimensional manifold), the space $\mathcal{A}(E)$ of unitary connexions, which is an affine space for $\Omega^1(\Sigma; \mathrm{End}(E))$, is isomorphic to the space $\mathcal{C}(E)$ of holomorphic structures.* □

We also have a refinement of Theorem 1.1, which we do not prove here; a careful Reader might be able to do it alone.

Theorem 1.4. *On a connected smooth manifold X, the isomorphism classes of principal G-bundles with connexion are in bijection with the conjugacy classes of homomorphisms $\pi_1(X) \longrightarrow G$. This correspodence is given by parallel transport along closed curves in X.* □

Using the extension $d_A : \Omega^p(X; E) \longrightarrow \Omega^{p+1}(X; E)$ as defined above, we can consider $d_A^2 : \Omega^p(X; E) \longrightarrow \Omega^{p+2}(X; E)$, which, in contrast with $d^2 = 0$, is an important object. It is called the *curvature* of the connexion, and in fact it is a tensor: $F_A = d_A^2 \in \Omega^2(X; \mathrm{End}(E))$. If we write $d_{A_\tau} = d + A_\tau$ in a local trivialization, then

$$F_{A_\tau} = dA_\tau + A_\tau \wedge A_\tau.$$

The second (usually non-zero) term may look strange, but some contemplation should result in the right definition. If we write $A_\tau = \sum_i A_i dx^i$ in local smooth coordinates, then $dA_\tau = \sum_{i,j} A_{i,j} dx^i dx^j$ and $A_\tau \wedge A_\tau = \sum_{i,j} A_i A_j dx^i dx^j$, where $A_{i,j} = \partial A_i / \partial x^j$.

We can define curvature in a principal G-bundle, too. For a family $\{A_\tau\}$ of local $\mathfrak{g}$-valued connexion 1-forms the appropriate formula is $F_{A_\tau} = dA_\tau + 1/2[A, A]$. If we change local trivialization from τ to $\tilde{\tau}$ by a transition map g, then $F_{A_{\tilde{\tau}}} = \mathrm{Ad}(g^{-1}) F_{A_\tau}$, which ensures the existence of a global $\mathfrak{g}$-valued 2-form F_A.

A fundamental result is that $F_A = 0$ iff in every local trivialization there exists a basis $\xi_1, \ldots, \xi_n$ of smooth sections with $d_A \xi_i = 0$. For example, if X is simply connected, then $F_A = 0$ implies that E is globally trivial. In general, a connexion is called *flat*, if $F_A = 0$. After the previous result it is not surprising that a connexion is flat iff the result of parallel transport along γ depends only on the homotopy class of the curve. This can also be seen from the following description:

$$\frac{\partial}{\partial u} T_u(p, q) = \int_p^q T_u(t, q) F_A \left(\gamma_u(t); \dot{\gamma}_u(t), \frac{\partial \gamma_u(t)}{\partial u} \right) T_u(p, t)\, dt,$$

where $\gamma_u(t)$ is a smooth family of paths, and $T_u(p, q)$ is parallel transport from $\gamma_u(p)$ to $\gamma_u(q)$ along γ_u.

Another definition of curvature which helps to visualize its meaning:

$$F_A(x; v, w) = \lim_{s,t \to 0} \frac{1}{st} \left(T_{\gamma_{s,t}} - \mathrm{Id} \right),$$

where $\gamma_{s,t}$ is the closed 'rectangular' path from x to x with side lenghts s, t, and side directions v, w.

Let us mention the Bianchi identity: $d_A F_A = 0$.

Finally, let us have a look at the special case of a real oriented two-manifold Σ embedded into $\mathbb{R}^3$, with the Riemannian metric inherited from $\mathbb{R}^3$. In a coordinate patch of Σ we have an orthonormal tangent frame e_1, e_2, and the unit normal vector field is $n = e_1 \times e_2$. Regarding e_1, e_2, n as maps $\Sigma \longrightarrow \mathbb{R}^3$, using the orthonormality of this system, we can write

$$\begin{pmatrix} de_1 \\ de_2 \\ dn \end{pmatrix} = \begin{pmatrix} 0 & -\alpha & \lambda \\ \alpha & 0 & \mu \\ -\lambda & \mu & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ n \end{pmatrix},$$

for some $\alpha, \lambda, \mu \in \Omega^1(\Sigma)$. Because of $d^2 e_1 = 0$ we have $\lambda\mu = d\alpha$.

The area form Vol_{S^2} on the unit sphere in $\mathbb{R}^3$ is $1/2\langle x, dx \times dx\rangle$, where $x : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the identity map. So let us define the pulled back 2-form $F = n^*(\mathrm{Vol}_{S^2}) = 1/2\langle n, dn \times dn\rangle = \lambda\mu = d\alpha = K\mathrm{Vol}_\Sigma$. This K, known as the *Gaussian product curvature*, see also Introduction 0.2, measures the rate at which n sweeps out area on S^2, so it seems to be a right generalization of the curvature of curves. But what is the underlying connexion then?

In an orthonormal local trivialization $\{e_1, e_2\}$ of $T\Sigma$, a connexion compatible with the inner product is a skew-symmetric matrix-valued 1-form, i.e. an element $A \in \Omega^1(\Sigma) \otimes \mathfrak{so}(2, \mathbb{R})$. If we write out the torsion-freeness condition $d_A\varepsilon = 0$ in terms of the basis $\{e_1, e_2\}$, we find that the unique Levi-Cività connexion on Σ is

$$A = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix},$$

with the $\alpha \in \Omega^1(\Sigma)$ we had earlier! And what is the associated curvature $F_A = dA + A \wedge A$? Since $\mathfrak{so}(2, \mathbb{R}) \simeq \mathbb{R}$ is a commutative Lie-algebra, the second term vanishes, and we find $F_A = dA$, i.e.

$$F_A = \begin{pmatrix} 0 & -d\alpha \\ d\alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & -K\mathrm{Vol}_\Sigma \\ K\mathrm{Vol}_\Sigma & 0 \end{pmatrix}.$$

Thus we have expressed the Gaussian curvature as the curvature of the Levi-Cività connexion in $T\Sigma$. Also note the naturality of the Levi-Cività connexion: $\nabla_{\dot\gamma(t)}\xi(t)$ is the 3-dimensional derivative of $\xi(t)$ along the curve $\gamma(t)$, orthogonally projected to the tangent space $T_{\gamma(t)}\Sigma$. So the parallel transport of $\xi(t)$ along $\gamma(t)$ means that the 3-dimensional derivative of $\xi(t)$ has no tangential component at any time — the transportation is as economic as possible.

On Σ we can easily relate curvature to volume growth. In geodesic polar coordinates (r, θ) centered at some point of Σ, the metric is $ds^2 = dr^2 + r^2 f^2 d\theta^2$, where f is a smooth function with $f(0, \theta) = 1$ and $df(0, \theta) = 0$. Taking $\varepsilon_1 = dr$ and $\varepsilon_2 = rf d\theta$ as an orthonormal basis for 1-forms, we find

$$\alpha = -\frac{\partial}{\partial r}(rf)d\theta \quad \text{and} \quad F = -\frac{1}{rf}\frac{\partial^2}{\partial r^2}(rf)\mathrm{Vol}_\Sigma = K\mathrm{Vol}_\Sigma,$$

where K is the Gaussian curvature at $r = 0$. This gives $f(r, \theta) = 1 - 1/3Kr^2 + O(r^3)$. Knowing that the metric on the unit sphere is given by $dr^2 + \sin^2 r d\theta^2$, we can calculate now the area of the disc of radius r on Σ:

$$A(r) = \pi r^2 \left(1 - \frac{1}{12}Kr^2 + O(r^4)\right).$$

1.3. Clifford bundles and Dirac complexes

In this section we make a rather abstract setup in order to define the 'square root of the Laplacian', the significance of which will become clear later. These Dirac operators on Clifford bundles were intruduced the first time in [GrL].

If V is an n-dimensional real vectorspace with a symmetric bilinear form $(\cdot, \cdot)$, we can define its *Clifford algebra* $\mathrm{Cliff}(V)$ as a unital algebra A together with a structure map $c : V \longrightarrow A$ satisfying $c(v)^2 = -(v, v) \cdot 1 \in A$ for every $v \in V$ such that A is universal with respect to this property; i.e. if A' is another unital algebra with the same properties, with structure map c', then there is a unique algebra homomorphism $\phi : A \longrightarrow A'$ with $c' = \phi \circ c$.

The uniqueness (up to a unique isomorphism) of such a universal algebra follows from the usual and trivial category theory argument. The existence can also be shown by the standard method, factorizing the tensor algebra $T(V)$ by a suitable ideal. More explicitly, if V

has a basis $e_1, \ldots, e_n$, then A can be taken to be the linear span of the 2^n possible products $c(e_1)^{\epsilon_1} \cdots c(e_n)^{\epsilon_n}$, $\epsilon_i \in \{0,1\}$, with multiplication determined by $c(v)c(w) + c(w)c(v) = -2(v,w)$. It is not difficult to see, that the dimension of $\mathrm{Cliff}(V)$ is really 2^n, and the structure map is injective. So we will consider V as a subspace of $\mathrm{Cliff}(V)$.

(Note that sometimes the structure map is determined by $c(v)^2 = (v,v) \cdot 1$ in the definition. Certainly, the only effect of this alteration is that multiplication by -1 or $i = \sqrt{-1}$ occurs in different places in the theory.)

The simplest example is when the bilinear form is identically zero. Then the exterior algebra $\Lambda(V)$ is the Clifford algebra, with multiplication $\wedge$. Another important example is the Clifford algebra of $\mathbb{R}^3$ with the standard positive definite inner product, for which the notion was originally introduced: it is isomorphic to $\mathbb{H} \oplus \mathbb{H}$, where $\mathbb{H}$ is the quaternionic algebra. Clearly, Clifford algebras are Lie algebras with the usual commutator Lie bracket.

Clifford algebras naturaly arise in representation theory. Every representation of the Lie groups $SL(n,\mathbb{C})$ and $Sp(2n,\mathbb{C})$ can be found inside tensor powers of some standard representations, but only half of the representations of $SO(n,\mathbb{C})$ arise this way. In some sense the reason can be that the first two groups are simply connected, while $\pi_1(SO(n,\mathbb{C})) = \pi_1(SO(n,\mathbb{R})) = \mathbb{Z}_2$ for $n \geq 3$. Clifford algebras help us to construct the *spin groups*, which are the double coverings of these groups. For example,

$$S^3 \simeq \mathrm{Spin}(3,\mathbb{R}) \simeq SU(2) \longrightarrow SO(3,\mathbb{R}) \simeq \mathbb{RP}^3.$$

In this essay we will not use the spin groups, but they are very important for the general theory. For more details on Clifford algebras see [R] and [HF].

Let us generalize our flat-space construction to the Riemannian manifold M. For any $p \in M$ we have the complexified Clifford algebra $\mathrm{Cliff}(T_pM \otimes \mathbb{C}) \simeq \mathrm{Cliff}(T_pM) \otimes \mathbb{C}$. Now let S be a vector bundle of Clifford modules, i.e. each fibre S_p is a module over (that is, a representation of) $\mathrm{Cliff}(T_pM \otimes \mathbb{C})$. To differentiate the smooth sections we need a connexion in S.

Let S be a complex vector bundle of Clifford modules over the Riemannian manifold M. S is called a *Clifford bundle* if it is equipped with a Hermitian metric and a connexion compatible with it such that

(i) the hermitian metric is invariant under the Clifford action, that is, the action of a vector $v \in T_pM$ by $c(v) \equiv v$ on S_p is skew-adjoint: $(vs_1, s_2) + (s_1, vs_2) = 0$;

(ii) the connexion in S is compatible with the Levi-Cività connexion on M in the sense that for any two vector fields X, Y and section $s \in C^\infty(S)$, we have $\nabla_X(Ys) = (\nabla_X Y)s + Y\nabla_X s$.

The *Dirac operator* D of a Clifford bundle S is the first order differential operator on $C^\infty(S)$ defined by the composition

$$C^\infty(S) \longrightarrow C^\infty(T^*M \otimes S) \longrightarrow C^\infty(TM \otimes S) \longrightarrow C^\infty(S),$$

where the first arrow is given by the connexion $s \mapsto \nabla_{\cdot} s$, the second by the duality coming from the metric, and the third by the Clifford action. In terms of a local orthonormal basis e_i of sections of TM, one can write

$$Ds = \sum_i e_i \nabla_i s.$$

In the simplest case, when M is the flat vector space V, we have $Ds = \sum_i e_i(\partial_i s)$, and a simple computation gives $D^2 s = -\sum_i \partial_i^2 s$, which is the Euclidean *Laplacian*. In the general case the generalized Riemannian *curvature* of the connexion, the $\mathrm{End}(S)$-valued 2-form $R = R^\nabla$

$$R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

also appears. We can choose the local orthonormal tangent base e_i s.t. $[e_i, e_j] = 0$; in this base the coefficients of R will be $R_{ij} = \nabla_i \nabla_j - \nabla_j \nabla_i$, and we have $D^2 s = -\sum_i \nabla_i^2 s + \sum_{i<j} e_i e_j R_{ij} s$.

Clearly, the Laplace operator on functions on a Riemannian manifold can be written locally as the sum of second partial derivations along geodesic coordinates, that is, as the image of the ordinary Laplace on the tangent space under the exponential map.

If $\nabla^* : C^\infty(T^*M \otimes S) \longrightarrow C^\infty(S)$ is the adjoint of ∇, then some computation shows that the previous equation can be written as the *Weitzenböck formula*, see [R, (2.6)],

$$D^2 s = \nabla^* \nabla s + Ks,$$

where $K \in \mathrm{End}(C^\infty(S))$ is the *curvature operator* of S, defined by $\sum_{i<j} e_i e_j R_{ij}$. A similar computation shows the important result that D is self-adjoint. The operator $\nabla^*\nabla$ is positive, K is self-adjoint, so the Weitzenbock formula easily gives the following

Theorem 1.5. (Bochner) *If the least eigenvalue of K at each point of the compact M is strictly positive, then there are no nonzero solutions of the equation $D^2 s = 0$.* $\square$

In the next section we will see that in the case of the exterior bundle our curvature operator is a generalization of the *Ricci curvature operator*, which is the $(1,1)$-tensor field given by raising one index of the *Ricci curvature tensor* $K(X,Y) = \mathrm{Tr}(Z \mapsto R(X,Z)Y)$, using the Riemannian metric g_{ij}. We will see an application to Bochner's theorem in Section 4.3.

We say that a Clifford bundle S on a compact oriented manifold is a *Dirac complex* if it is a direct sum $S = \oplus_{j=0}^k S_j$ of vector bundles with Hermitian metrics and compatible connexions, together with a sequence of differential operators $d_j : C^\infty(S_j) \longrightarrow C^\infty(S_{j+1})$ satisfying $d_{j+1} d_j = 0$, and the Dirac operator of the Clifford bundle is $D = d + d^*$, where $d = \oplus_{j=0}^k d_j$. Certainly, we also can define the $\mathbb{Z}_2$-grading, as above.

The main example of a Dirac complex will be the deRham complex on the exterior bundle. A further very important example is the *Dolbeault complex* on an (almost) complex manifold, see [R, (2.20)], [KN], and first of all, [GH]. Using the spin groups mentioned in the previous section one can introduce the spin manifolds and spin complexes. These are also Clifford bundles, and the Atiyah-Singer index theorem for spin manifolds gives strong results for each of the usual Dirac complexes. In our paper some kind of perturbated deRham complexes will play a central role, to be discussed in Chapters 2 and 3.

1.4. The exterior bundle as a Dirac complex

The most important example of a Clifford bundle is the exterior bundle of M. As a vector bundle, $S := \Lambda T^*M \otimes \mathbb{C}$ is naturally isomorphic to $\mathrm{Cliff}(TM \otimes \mathbb{C})$; the basis element $e_1 \wedge \cdots \wedge e_k$ (the e_i's are orthonormal) of the exterior algebra corresponds to the basis element $e_1 \cdots e_k$ of the Clifford algebra. (This is not an isomorphism of algebras, since $\mathrm{Cliff}(V) \simeq \Lambda(V)$ only in the case of the zero bilinear form.) So S is a natural module over $\mathrm{Cliff}(TM \otimes \mathbb{C})$, thus is a bundle of Clifford modules. Moreover, equipped with a natural metric and connexion it is a Clifford bundle indeed. We describe it now in more details.

On an n-dimensional orientable compact Riemannian manifold M there is exactly one n-form Vol such that $\mathrm{Vol}_x(e_1, \ldots, e_n) = 1$ for every orthonormal basis T_xM. This is called the *volume form*, and equals $\sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$, where $g = g_{ij}$ is the Riemannian metric. Now we can define the *Hodge $*$ operation* on the exterior bundle by

$$* : \Omega^r(M;TM) \longrightarrow \Omega^{n-r}(M;TM), \qquad (*\omega)(v_1, \ldots, v_{n-r})\mathrm{Vol} = \omega \wedge \check{v}_1 \wedge \cdots \wedge \check{v}_{n-r},$$

where $\check{v}$ is the dual 1-form to the vector field v, w.r.t. the metric g, i.e. $\check{v}(w) = (v,w)_g$ for every vector field w. We can extend g to a metric on arbitrary r-forms α, β locally at any point: if g^{ij} is the inverse matrix of g_{ij}, then define $(dx^i, dx^j) = g^{ij}$, and $(dx^{i_1} \wedge \cdots \wedge dx^{i_k}, dx^{j_1} \wedge \cdots \wedge dx^{j_k}) = \det\big((dx^{i_r}, dx^{j_s})\big)_{r,s=1}^k$. Then we have a global inner product defined by $\int_M (\alpha, \beta)\mathrm{Vol}$. Now

$$(\alpha, \beta)\mathrm{Vol} = \beta \wedge *\alpha = \alpha \wedge *\beta,$$

and $** = (-1)^{r(n-1)}$ on $\Lambda^r T^*M$, see [R, Chp. 2] and [L, Chp. X and XIII].

A vector field v defines two important operations on differential forms, independently of the Riemannian metric. The *interior multiplication* or *contraction* $i(v) : \Omega^r \longrightarrow \Omega^{r-1}$ is defined by $(i(v)\omega)_x(v_2, \dots, v_r) = \omega(v(x), v_2, \dots, v_r)$ for any r-form ω. Then we have $*i(v)*\omega = (-1)^{nr+n+1}\check{v}\wedge\omega$, and the operator $i(v)$ is the adjoint of the exterior multiplication $\check{v}\wedge$, w.r.t. the global metric of forms. Sometimes the following notation is used: $\check{v}\lrcorner\omega = (-1)^{n-1}i(v)\omega$. Clearly, contraction is the unique antiderivation such that $i(v)\omega = \omega(v)$ for 1-forms.

Secondly, we have the Lie-derivative $\mathcal{L}_v : \Omega^r \longrightarrow \Omega^r$, the unique derivation which commutes with $d : \Omega^r \longrightarrow \Omega^{r+1}$, the usual differential operator of forms, and extends the derivation of functions alnog vector fields. Moreover, the Lie-derivative can be defined for any tensor field: it has to be the extension of the Lie-bracket of vector fields, as well: $\mathcal{L}_v w = [v, w] = vw - wv$. A very improtant local description can be given by the flow α_t generated by v:

$$\mathcal{L}_v\omega = \frac{d}{dt}\alpha_t^*\omega\bigg|_{t=0}.$$

Interior multiplication and Lie derivative are related by *Cartan's formula*

$$di(v) + i(v)d = \mathcal{L}_v,$$

which also expresses the homotopy invariance of deRham cohmology. For the computations see [L, Chp. V] or [DFN I].

Using Stoke's theorem it turns out that the operator $d^* : \Omega^r \longrightarrow \Omega^{r-1}$ defined by $d^* = (-1)^{rn+n+1} * d *$ is the adjoint of d. For example, in $\mathbb{R}^n$, $d\omega = \sum_i dx^i \wedge (\partial\omega/\partial x^i)$ and $d^*\omega = \sum_i i(\partial/\partial x^i)(\partial\omega/\partial x^i)$. With some additional formal calculations [R, Chp. 2] we can summarize our description in the following result:

Theorem 1.6. *The bundle $S = \Lambda T^*M \otimes \mathbb{C}$ is a Clifford bundle with Clifford action $c(v)\omega = \check{v}\wedge\omega + \check{v}\lrcorner\omega$, and Dirac operator $D = d + d^*$. The Laplacian is $D^2 = dd^* + d^*d$. The restriction of the curvature operator K to 1-forms is equal to the Ricci curvature operator.* $\square$

The exterior bundle splits as a direct sum of the even dimensional forms Λ^+T^*M and odd dimensional forms Λ^-T^*M. We can introduce a $\mathbb{Z}_2$-grading: there is an involution ι with the $+1$ and -1-eigenspaces Λ^+T^*M and Λ^-T^*M, and

$$\left(\Lambda^{(-1)^\nu}T^*M\right) \wedge \left(\Lambda^{(-1)^\mu}T^*M\right) \subseteq \Lambda^{(-1)^{\nu+\mu}}T^*M.$$

Moreover, we have $\iota D + D\iota = 0$. Then the graded D splits as $D_+ \oplus D_-$, and D_+ and D_- are the adjoints of each other.

1.5. Analytic properties of the Dirac operator

Now we will explain the main analytic properties of the Dirac operator on a Dirac complex. Our standard reference will be [R].

Let us define the *Sobolev space* $W^k(M)$ obtained by completing $C^\infty(M)$ in the Sobolev k-norm

$$\|f\|_k = \sum_{|\alpha|\le k} \|D^\alpha f\|_{L^2},$$

where α stands for multiindices $(\alpha_1, \dots, \alpha_n)$, and D^α is the partial derivative $(\partial^{\alpha_i} / \partial x_i^{\alpha_i})_i$. If V is a vector bundle over M, we can define similarly the Sobolev space $W^k(V)$ of W^k sections of V. The most important facts about these spaces are the *Sobolev embedding theorem* and *Rellich's theorem*, see [R, Chp. 3]. The first one is that for $p > n/2$, the space W^{k+p} is continuously included in C^k. The second one states that if $k < l$, then the inclusion operator $W^l \longrightarrow W^k$

is compact. Clearly, multiplication by a C^∞ function acts as a bounded operator on each W^k, and linear differential operators of order l act boundedly from W^k to W^{k-l}.

Since the Dirac operator D is a first order opearator, we have $\|Ds\|_0 \le C\|s\|_1$ with some constant C, for any $s \in C^\infty(S)$ section. Some kind of converse is also true, which is the main analytic property of D; see [R, (3.15)]:

Lemma 1.7. (Elliptic estimate) *For any $k > 0$ there is a constant C_k s.t., for any $s \in C^\infty(S)$,*

$$\|s\|_{k+1} \le C_k(\|s\|_k + \|Ds\|_k). \quad \square$$

D is an unbounded operator on the Hilbert space $H = L^2(S)$, defined on $C^\infty(S)$. Its graph is the subspace of $H \oplus H$ defined as $G = \{(x; Dx) \,|\, x \in \mathrm{dom}(D)\}$. It is easy to see that the closure $\overline{G}$ is also a graph, so defines an extended unbounded operator $\overline{D}$. By the elliptic estimate its domain is exactly the Sobolev space $W^1(S)$. Moreover, if for $x, y \in L^2(S)$ we have $Dx = y$ weakly, i.e. $(x, Ds) = (y, s)$ for all $s \in C^\infty(S)$, then $x \in W^1(S)$ and $\overline{D}x = y$. More generally, if $x, y \in W^k(S)$, $(k \ge 1)$, and $\overline{D}x = y$, then $x \in W^{k+1}(S)$. From now on we can consider D as a bounded operator from $W^1(S)$ to $L^2(S)$.

Theorem 1.8. *The Hilbert-space $H = L^2(S)$ has a direct sum decomposition into countably many orthogonal subspaces H_λ, where each H_λ is a finite dimensional space of smooth sections, an eigenspace for D with eigenvalue λ. The eigenvalues λ form a discrete subset of $\mathbb{R}$.*

Main idea of the proof. Using standard Hilbert space arguments we can construct a compact bounded operator $Q : H \longrightarrow W^2(S)$ satisfying $1 = (1 + \overline{D}^2)Q$ on H. The boundedness and compactness of Q are due to Rellich's theorem and the elliptic estimate on D. Now we can use the standard knowledge of the spectrum of the compact Q, and translate this for the spectrum of D. To show that the eigenvectors are smooth, one can use the results of the previous paragraph and the Sobolev embedding theorem. For details see [R, (3.25)]. $\square$

If f is a bounded function on the spectrum $\sigma(D)$, we can define a bounded operator $f(D)$ on $L^2(S)$: the operator which acts as multiplication by $f(\lambda)$ on each H_λ. One can easily prove the following observation:

Lemma 1.9. *The map $f \mapsto f(D)$ is a unital $*$-ring homomorphism from $B(\sigma(D))$ to $B(H)$ (the spaces of bounded functions and operators, respectively). If D commutes with an operator A, so does every $f(D)$. Moreover, every $f(D)$ maps $C^\infty(S)$ to $C^\infty(S)$.* $\square$

The spectrum of the ordinary Laplacian on a compact Riemannian manifold contains a large amount of important geometrical data. First of all, it determines the dimension, the volume and the total scalar curvature. For the number of eigenvalues not larger than λ the following estimate holds, see [R, (7.3)]:

$$N(\lambda) \sim \frac{1}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)} \mathrm{Vol}(M)\lambda^{n/2}.$$

There was a long-standing question whether there exist isospectral manifolds which are not isometric. The answer is affirmative; for a nice introductory survey see [Br]. For other signs of the geometric significance of the Laplacian see Section 1.10.

As we will see in the next section, the finite-dimensional null-space of the Dirac operator D, which coincides with the null-space of D^2 (the subspace $\mathcal{H}$ of the so-called harmonic sections), also has a crucial geometrical meaning, that is, it completely represents the cohomology of the Dirac complex on the manifold. This already explains the importance of the *heat operator* e^{-tD^2}, $t > 0$, acting on $L^2(S)$ as defined by Lemma 1.9, which connects the identity operator (for $t = 0$) with the projection $L^2(S) \longrightarrow \mathcal{H}$ (for $t = \infty$). This operator naturally appears in the heat equation

$$\frac{\partial s_t}{\partial t} + D^2 s_t = 0;$$

in fact, for an initial state $s_0 \in C^\infty(S)$ we have the unique $C^\infty(S)$ solution $s_t = e^{-tD^2}s_0$. The analytical properties we have discussed so far assert that this heat operator is a smoothing operator, see [R (5.7)], that is, there exists a smooth *heat kernel* $k_t(m_1, m_2)$ on $M \times M$, $t > 0$, with values $k_t(m_1, m_2) \in \mathrm{Hom}(S_{m_2}, S_{m_1})$, such that

$$e^{-tD^2}s(m_1) = \int_M k_t(m_1, m_2)s(m_2)\,\mathrm{Vol}(m_2).$$

A smoothing operator on M is of trace-class, which we do not define here, but it is worth noting that if the bounded operator A has the smoothing kernel $k(m_1, m_2)$, then

$$\mathrm{Tr}(A) = \int_M k(m, m)\,\mathrm{Vol}(m),$$

and this trace coincides with the sum of the eigenvalues for self-adjoint operators.

The asymptotic properties of this heat kernel play a key role in the theory of the Atiyah-Singer index theorem. We will see some applications (without real proofs) in Section 1.7.

1.6. Hodge theory and the index of the Dirac operator

A bounded operator $A : B_1 \longrightarrow B_2$ between Banach spaces is called *Fredholm* if $\ker(A)$ is finite dimensional in B_1 and $\mathrm{im}(A)$ is closed in B_2 and of finite codimension. For such an operator we can define its *index* by

$$\mathrm{ind}(A) = \dim\,\ker(A) - \mathrm{codim}\,\mathrm{im}(A).$$

Clearly, if both B_i are finite dimensional, then $\mathrm{ind}(A) = \dim(B_1) - \dim(B_2)$, which is rather stable under perturbations of A. In general Banach spaces we can allow for perturbations by compact operators, for instance.

Lemma 1.10. *The Dirac operator D is a Fredholm operator from $W^1(S)$ to $L^2(S)$. More exactly,* $\mathrm{im}(D)$ *is the orthogonal complement of* $\ker(D)$ *in* $L^2(S)$.

Proof. By Theorem 1.8, $\ker(D)$ is finite dimensional. From the self-adjointness of D it is obvious that $\mathrm{im}(D) \subseteq \ker(D)^\perp$. On the other hand, let f be a function on the spectrum $\sigma(D)$ with $f(0) = 0$ and $f(\lambda) = \lambda^{-1}$. Then for $x \in \ker(D)^\perp$ we have $Df(D)x = x$, see Theorem 1.8 and Lemma 1.9. As usually, $Df(D)x \in L^2(S)$ implies $f(D)x \in W^1(S)$, so $x \in \mathrm{im}(D)$. $\square$

We can define the *cohomology* $H^j(S, d)$ of smooth sections of a Dirac complex as for any chain complex. For the deRham complex on the exterior bundle it is of course the usual cohomology of the manifold, this is *deRham's theorem*, see [BT] or [DFN III]. This cohomology is independent on the metric, but we have the *Laplacian* $\Delta = dd^* + d^*d = D^2$, and the spaces of smooth *harmonic* sections satisfying $\Delta s = Ds = 0$. The central idea of *Hodge theory* is that we can choose a canonical harmonic representative for each cohomology class.

In the basic case we are given a (real or complex) graded Hilbert space $A = \oplus_{p=0}^n A^p$ with a linear map $d = d^p : A^p \longrightarrow A^{p+1}$ satisfying $d^2 = 0$. Let $H^p(A) = \ker d^p/\mathrm{im} d^{p-1}$. Define the Laplacian $\Delta = dd^* + d^*d$, where d^* is the adjoint of d with respect to the Hilbert space inner product, and define the harmonic subspace $\mathcal{H} = \ker\Delta$. If the following two Hodge conditions

(i) $\ker\Delta$ is finite dimensional

(ii) $\mathcal{H}^\perp = \Delta A$

hold, then elementary Hilbert space manipulations give the Hodge decomposition [L, Chp. X.5]:

$$A = \mathcal{H} \oplus dA \oplus d^*A$$

and

$$\mathcal{H} = \mathop{\oplus}_{p=0}^{n} \mathcal{H}^p \text{ with } \mathcal{H}^p \simeq H^p(A),$$

where $\mathcal{H}^p = \mathcal{H} \cap A^p$.

In the case of a Dirac complex we already know that the Hodge conditions are satisfied, by Lemma 1.10. So, being a little bit careful about the difference between $C^\infty(S)$ and $L^2(S)$, we get immediately the following theorems of crucial importance:

Theorem 1.11. (Hodge theorem) *Each cohomology class has a unique harmonic representative, i.e. $\mathcal{H}^j \simeq H^j(S,d)$ as vector spaces.* $\square$

The trivial example is the harmonic functions on M, which are constant on each connected component [KN II, Note 14], and the group $H^0(M)$ with dimension being equal to the number of connected components. In [R, (4.2)] there is a more direct proof of the Hodge Theorem, constructing a chain equivalence between the Dirac complex and its subcomplex of harmonic sections with 0 differential. The chain homotopy operator (see [BT] or [MT]) is $H = d^*G$, where $G = g(D)$ is the *Green's operator* defined by the function $g(0) = 0$, $g(\lambda) = \lambda^{-2}$ on the spectrum $\sigma(D)$.

One can think of these harmonic representatives from a geometric point of view. If $C \subseteq C^\infty(S_j)$ is a cohomology class, then it is an affine subspace of $L^2(S)$ with $\alpha - \beta \in dC^\infty(S_{j-1})$ for all $\alpha, \beta \in C$. Therefore we expect a norm-minimizing element α to be perpendicular to $dC^\infty(S_{j-1})$, which translates to say that $d^*\alpha = 0$. Since $d\alpha = 0$, this is equivalent to $D^2\alpha = 0$. Unfortunately, the existence of this norm-minimizing element in a general Hilbert space is not necessary, so this was not a proof for the Hodge theorem.

Corollary 1.12. *The cohomology of a Dirac complex is finite dimensional.* $\square$

Corollary 1.13. (Poincaré duality) *If M is a compact connected oriented n-manifold, then $H^k(M) \simeq H^{n-k}(M)$.* $\square$

From Lemma 1.10 we know that $\mathrm{ind}(D) = 0$, but the index of D_+, which is also a Fredholm operator, already contains some important geometrical data.

Corollary 1.14. *The index of D_+ equals to the Euler characteristic of the Dirac complex (S,d).* $\square$

1.7. The Atiyah-Bott-Lefschetz fixed-point theorem

The *Lefschetz number* $L(\phi)$ of a smooth self-map of M^n, with isolated non-degenerate fixed point only, can be defined in three different ways:

$$L(\phi) = \sum_{\phi(m)=m} \operatorname{sign}\det(I - D_m\phi) = \sum_{k=0}^{n} (-1)^k \mathrm{Tr}(\phi_k^*) = \Delta(\phi) \circ \Delta,$$

where $\Delta \subseteq M \times M$ is the diagonal, $\Delta(\phi) = (x, \phi(x))_{x\in M}$, and $\circ$ is the intersection index of submanifolds in $M \times M$, see our Introduction 0.2 and [DFN II]. The usual *Lefshetz theorem* is the equivalence of these definitions. Now we are going to consider a generalization.

If $\phi : M \longrightarrow M$ is a smooth map, then it induces an endomorphism $\phi^* : C^\infty(S) \longrightarrow C^\infty(\phi^*S)$. Assuming we have a bundle map $\zeta : \phi^*S \longrightarrow S$, for example, the natural $\zeta = \Lambda^*D^*\phi$ in the case of the deRham complex, where $D^*\phi$ is the dual to the derivative $D\phi$, then we have the composite map $F = \zeta\phi^* : C^\infty(S) \longrightarrow C^\infty(S)$. (I hope there will be no confusion from the two meanings of D, one for the derivative, one for the Dirac operator.) Now, if F satisfies $Fd = dF$, then we call the pair (ζ, ϕ) a geometric endomorphism of the Dirac complex, and we can define the *Lefschetz number*

$$L(\zeta, \phi) = \sum_{q=0}^{n} (-1)^q \mathrm{Tr}(F^* \text{ on } H^q(S,d)).$$

The Lefschetz number of the identity map is clearly the Euler characterictic of the Dirac complex.

Now it is not difficult to prove that if P_q denotes the orthogonal projection $L^2(S_q) \longrightarrow \mathcal{H}^q$, then $\mathrm{Tr}(F^*$ on $H^q(S)) = \mathrm{Tr}(FP_q)$, and the smoothing kernel of $e^{-t\Delta_q}$ tends to the smoothing kernel of P_q in the C^∞ topology, where Δ_q is the restriction of D^2 to $C^\infty(S)$. Thus we have $\mathrm{Tr}(FP_q) = \lim_{t\to\infty} \mathrm{Tr}(Fe^{-t\Delta_q})$, and a corresponding formula for the Lefschetz number. But a 'supersymmetric cancellation of eigenspaces' yields the same result for every $t > 0$; we will meet a similar phenomenon in Chapter 3:

Lemma 1.15. $\sum_{q=0}^n (-1)^q \mathrm{Tr}(Fe^{-t\Delta_q}) = L(\zeta, \phi)$, *independently of* t.

Sketch of the proof. After differentiating our equation by t, we seee that we have to prove $\sum_{q=0}^n (-1)^q \mathrm{Tr}(F(dd^* + d^*d)e^{-t\Delta_q}) = 0$. Using $dF = Fd$ and $\Delta_q d = d\Delta_{q+1}$, the terms in the sum will cancel in pairs, resulting in 0; the only problem is that d is an unbounded operator, so we can have problems with computing trace, but this is only some kind of technicality; for details see [R, (8.6)]. $\square$

This formula immediately gives $\mathrm{ind}(D_+) = \mathrm{Tr}_s(e^{-tD^2})$, cf. Corollary 1.14, where $\mathrm{Tr}_s(A) = \mathrm{Tr}(\iota A)$ is the *super-trace* of the graded operator A, ι is the grading operator.

Looking at the diagonal of the heat kernel, Lemma 1.15 gives a generalized *Lefschetz fixed-point theorem*: if ϕ has no fixed points, then $L(\zeta, \phi) = 0$. A detailed analysis of the heat kernel gives the following strong result, see [AB1] and [R, (8.11)]:

Theorem 1.16. (Atiyah – Bott) *If (ζ, ϕ) is a geometric endomorphism having only simple fixed points, that is,* $\det(1 - T_m\phi) \neq 0$ *(implying that m is an isolated fixed point), then*

$$L(\zeta, \phi) = \sum_{\phi(m)=m} \sum_{q=0}^{n} \frac{(-1)^q \mathrm{Tr}(\zeta_q(m))}{|\det(1 - D_m\phi)|}. \quad \square$$

For the ordinary deRham complex we cannot expect more than the original *Lefschetz index theorem.* Using the fact that $\mathrm{Tr}(\Lambda^q T)$ is nothing else but the qth elementary symmetric polynomial of the eigenvalues of T, one can see that $\sum_q (-1)^q \mathrm{Tr}(\Lambda^q T) = \det(1 - T)$. On the other hand, for an isolated simple fixed point we have $\mathrm{ind}_m\phi = \mathrm{ind}_m D_m\phi = \mathrm{sign}\det(1 - D_m\phi)$, as we saw in the Introduction 0.2. Thus we have

$$L(\phi) = \sum_{\phi(m)=m} \mathrm{sign}\det(1 - D_m\phi) = \sum_{\phi(m)=m} \mathrm{ind}_m\phi.$$

Usual applications are the *Brouwer fixed point theorem* and the *Poincaré-Hopf index theorem.* For a flow $\phi^t : M \longrightarrow M$ each ϕ^t is homotopy equivalent to the identity ϕ^0, thus we have

$$\sum_{\phi^t(m)=m} \mathrm{ind}_x\phi^t = L(\mathrm{Id}) = \chi(M).$$

If we have a Morse function f on M, then we can apply this for the gradient flow of f, and have $\sum_k M_k = \chi(M)$, as in the strong Morse inequalities Theorem 0.3.

We will see a degenerate Poincaré-Hopf theorem in Chapter 3.

More interesting corollaries to the Atiyah-Bott Theorem 1.16 can be achieved if we consider Dirac complexes other than the ordinary deRham complex, such as the Dolbeault complex, spin manifolds, and representations of a Lie groups. Lots of beautiful applications can be found in [AB1], and we will also give some of them at the end of Chapter 3. A typical example for the complex analytic case is the following:

Let X be a connected compact complex manifold with $H^{0,q}(X) = 0$ for $q > 0$. Then any holomorphic map $f : X \longrightarrow X$ has a fixed point. In particular, if X is a complex projective space, then any holomorphic automorphism has a fixed point. Sometimes we can guarantee lots of fixed points, see the end of Chapter 3, again.

1.8. The harmonic oscillator and quantum mechanincs

The *harmonic oscillator* is the unbounded operator

$$H = -\frac{d^2}{dx^2} + a^2x^2 \qquad (a > 0)$$

on $L^2(\mathbb{R})$. The aim of this section is to determine its spectrum and give some introduction to the very important role of this operator in physics.

There are the corresponding *annihilation (or lowering) operator* $A = ax + \frac{d}{dx}$, the *creation (or raising) operator* $A^* = ax - \frac{d}{dx}$. Elementary computations give

$$AA^* = H + a, \qquad A^*A = H - a$$

$$[A, A^*] = 2a, \qquad [H, A] = -2aA, \qquad [H, A^*] = 2aA^*.$$

The *ground state* of H is the function $\psi_0 \in L^2(\mathbb{R})$ satisfying the differential equation $A\psi_0 = 0$ and $\|\psi_0\| = 1$. Then it is clearly an eigenfunction $H\psi_0 = a\psi_0$, and it is easy to calculate it explicitly:

$$\psi_0(x) = a^{1/2}\pi^{-1/4}\, e^{-ax^2/2}.$$

For $k \geq 1$ we can define the *excited states* of H inductively by

$$\psi_k = (2ka)^{-1/2} A^* \psi_{k-1}.$$

They are also eigenfunctions $H\psi_k = (2k+1)a\psi_k$, with $\|\psi_k\| = 1$. The names 'raising' and 'lowering' operators come from the fact that if $H\psi = \lambda\psi$, then $H(A^*\psi) = (\lambda + 2a)A^*\psi$ and $H(A\psi) = (\lambda - 2a)A\psi$. (Note that this does not mean that we could produce an eigenfunction from ψ_0 with smaller eigenvalue than a, since $A\psi_0 = 0$.)

Now it is clear by induction that $\psi_k(x) = h_k x e^{-ax^2/2}$, where h_k is a polynomial of degree k. So these ψ_k's span the space $\mathcal{P} = \{p(x)e^{-ax^2/2} \,|\, p \text{ is a polynomial}\}$, and it is not too difficult to see that this is a dense subspace of $L^2(\mathbb{R})$. Thus we have proved the following result; for the details see [R Chp. 10]:

Theorem 1.17. *The space $L^2(\mathbb{R})$ has a complete orthogonal decomposition into 1-dimensional eigenspaces of the harmonic oscillator with discrete spectrum $\{(2k+1)a \,|\, k = 0, 1, \dots\}$.* $\square$

From the respect of the the general theory the heat equation of the harmonic oscillator,

$$\frac{\partial s_t}{\partial t} + Hs_t = 0,$$

is also very important.

The harmonic oscillator is an extremely important example for quantum mechanics and the supersymmetry theories of mathematical physics. The 1-dimensional classical harmonic oscillator is the dynamical system governed by the Hamiltonian

$$H(q, p) = \frac{1}{2m}(p^2 + m^2\omega^2q^2),$$

where $q(t)$ is the coordinate for the space, $p(t)$ is the momentum, m is a constant for the mass and ω is a constant for the frequency. Time evolution is given by the Hamiltonian equations

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q},$$

or, in terms of the *Poisson bracket* for functions $F(q,p)$ and $G(q,p)$,

$$\{F,G\} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q},$$

they can be written as

$$\dot{q} = \{q,H\}, \qquad \dot{p} = \{p,H\}.$$

The functions of the canonical variables p and q form an infinite- dimensional Lie algebra with this Poisson-bracket, and time evolution is always given by $\dot{F} = \{f,H\}$. Poisson bracket can be more generally defined in case of *symplectic manifolds*, which arise naturally in the theory of Hamiltonian dynamical systems; see also Chapter 4 and [KH].

In quantum mechanics the states of a system are not single coordinates (q,p), but complex wave functions $\psi(q)$ with real density functions $|\psi(q)|^2$. The momentum wave function $\hat{\psi}(p)$ is just the Fourier transform of $\psi(q)$, and $|\hat{\psi}(q)|^2$ is again a probability density. For any classical function $f(q,p)$, we would like to introduce a Hermitian operator F acting on the complex Hilbert space of the states, such that to the classical function $i\hbar\{f,g\}$ always the operator $[F,G]$ has to correspond, where $\hbar = h/2\pi$, h is Planck's constant. In particular, the classical canonical variables p and q satisfy the commutation relations $\{q,q\} = \{p,p\} = 0$, $\{p,q\} = 1$, so we would like to produce quantum mechanical operators P and Q with the commutation relations $[Q,Q] = [P,P] = 0$, $[Q,P] = i\hbar I$, this is called *quantization.* One realization of such an operator Lie algebra is given by the operators

$$Q\psi(q) = q\psi(q), \qquad P\psi(q) = -i\hbar\frac{\partial\psi(q)}{\partial q}.$$

Setting $A = 1/\sqrt{2}(P - iQ)$ and $A^* = 1/\sqrt{2}(P + iQ)$, we have the so-called bosonic commutation relations $[A,A] = [A^*,A^*] = 0$ and $[A,A^*] = I$, which (apart from constants) is the Lie algebra generated by A, A^*, I, we had before; this is the three-dimensional *Heisenberg algebra*, with Lie group consisting of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \; a,b,c \in \mathbb{R},$$

called the Heisenberg group.

For a dynamical system with Hamiltonian H, which is expected to be Hermitian, time translation is given by the operator e^{iHt}. In particular, if $\psi(q)$ is an eigenfunction of H with eigenvalue λ, then $\psi_t = e^{iHt}\psi$ will differ from ψ only in phase, $\psi_t(q) = \psi(q)e^{i\lambda t}$. Thus $|\psi_t(q)|^2 \equiv |\psi(q)|^2$, which means that $\psi(q)$ is an equilibrium state for the system.

From the relation $[Q,P] = i\hbar I$ one can easily deduce the Heisenberg uncertainity relation, stating that $(\Delta P)(\Delta Q) \geq \hbar/2$ on every state ψ, where $(\Delta P)^2$ is the variation according to the probability density $|\psi|^2$. Clearly, the physical relevance of the above construction by quantization is known by experimental results. For an introduction to quantum mechanics see [Li], and [SW] for the Heisenberg algebra.

Raising and lowering operators also appears in the representations of every Lie algebra. Consider, for example, the most important semisimple Lie algebra, $\mathfrak{sl}_2 = \mathfrak{sl}(2,\mathbb{C})$, having a natural complex basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with commutation relations

$$[E,F] = H, \qquad [H,E] = 2E, \qquad [H,F] = -2F.$$

If V is a finite dimensional representation, i.e. a left $\mathfrak{sl}_2$-module, and $v \in V$ is a $Hv = \lambda v$ eigenvector (a so-called *weight vector* with weight λ), then $HEv = (\lambda + 2)Ev$ and $HFv = (\lambda - 2)Fv$, so E increases, F decreases the weight by 2. The main result is the following

Theorem 1.18. *The finite dimensional irreducible representations of $\mathfrak{sl}_2$ are classified by their 'highest weight', a positive integer d. The irreducible representation V_d has dimension $d + 1$ and has a unique 'highest weight vector' (up to scalar multiple) $v \in V_d$ satisfying $Hv = dv$ and $Ev = 0$. If $v_0 = v$ and $v_j = j^{-1}Fv_{j-1}$ for $j = 1, \dots, d$, then these v_j's form a basis of V_d with $Hv_j = (d - 2j)v_j$.* $\square$

For the structure and representations of Lie algebras see [FH] and [H2].

In general, we are speaking about a system of *bosonic annihilation and creation operators* a_i and a_i^*, if they satisfy the commutation relations

$$[b_i, b_j] = [b_i^*, b_j^*] = 0, \qquad [b_i, b_j^*] = \delta_{ij} I,$$

and *fermionic annihilation and creation operators* f_i, f_i^*, if they satisfy the anticommutation relations

$$\{f_i, f_j\} = \{f_i^*, f_j^*\} = 0, \qquad \{f_i, f_j^*\} = \delta_{ij} I,$$

with the brackets $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$.

For example, the Lie algebra $\mathfrak{sp}(2k, \mathbb{C})$ can be represented by bosonic operators, the Lie algebra $\mathfrak{so}(2k, \mathbb{C})$ is by fermionic operators. In our paper the crucial example is the exterior bundle $\Lambda(T^*M)$ with the fermionic creation operators $a_i^* = dx^i\wedge$ (exterior multiplication) and annihilation operators $a_i = i(\partial/\partial x^i)$ (interior multiplication), as introduced in Section 1.4. One can see immediately that they really satisfy the anticommutation relations prescribed above.

1.9. Supersymmetry theories

In fact, the $\mathbb{Z}_2$-grading introduced at the end of Section 1.4 is the key connection to supersymmetry [W1]. By definition, a *supersymmetry theory* is a Hilbert space $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ decomposed into the spaces of *bosonic* and *fermionic* states, together with Hermitian 'symmetry operators' Q_i, $i = 1, \dots, N$, mapping $\mathcal{K}^+$ into $\mathcal{K}^-$ and vica versa. If $(-1)^F$ is the involution which distinguishes the bosonic and fermionic subspaces, then this supersymmetry condition reads as

$$(-1)^F Q_i + Q_i (-1)^F = 0.$$

Secondly, we have a Hamiltonian operator H which generates time translations, e.g. a differential operator governing a smooth dynamical system. The supersymmetry operators should commute with it,

$$Q_i H - H Q_i = 0.$$

In the simplest form of supersymmetric quantum mechanics we have the further condition specifying the algebraic structure:

$$Q_i^2 = H \qquad \text{and} \qquad Q_i Q_j + Q_j Q_i = 0 \text{ (for } i \neq j).$$

If one wants the structure fit to the relativistic quantum field theory (now only with one space and one time dimension), then we also need a momentum operator P, with the modified conditions

$$Q_1^2 = H + P, \qquad Q_2^2 = H - P, \qquad Q_i Q_j + Q_j Q_i = 0.$$

Now it follows that

$$[Q_i, H] = [Q_i, P] = 0, \qquad [H, (-1)^F] = [P, (-1)^F] = 0$$

and

$$H = \frac{1}{2}(Q_1^2 + Q_2^2).$$

The central question in a supersymmetry theory is the existence of a state $|\Omega\rangle \in \mathcal{K}$ annihilated by all the supersymetry operators, $Q_i|\Omega\rangle = 0$. For such a state we have $P|\Omega\rangle = H|\Omega\rangle = 0$, so this state has zero energy, it is a *vacuum state*. (Clearly, for any state in $\mathcal{K}$ we have $H|\Omega\rangle \geq 0$.) Supersymmetry basically means the existence of a vacuum state; in this case fermions and bosons have equal mass. If there is a minimal enery state, but it has strictly positive energy, then supersymmetry is 'spontaneously broken', fermions and bosons are not of equal mass, despite the underlying supersymmetry structure. In physics it is not clear at the moment whether supersymmetry does play a role in nature or not.

For a vacuum state it is enough to look in the subspace $\mathcal{K}_0$ consisting of states annihilated by P, and within this subspace our system looks like the simplest supersymmetry theory. That is, $Q_i^2 = H$ for any i, and a state annihilated by one Q_i is annihilated by all of them. So we are looking for a zero eigenstate of one supersymmetry operator Q in $\mathcal{K}_0$. We have a decomposition $Q = Q_+ \oplus Q_-$ according to the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, where Q_+ and Q_- are the adjoints of each other. Now it is worth considering the *index* $\mathrm{ind}Q_+ = \dim\ker Q_+ - \dim\ker Q_- = \mathrm{Tr}(-1)^F$, since a nonzero index implies a zero eigenvalue of Q.

The reason for calling the subspaces $\mathcal{K}^\pm$ as bosonic and fermionic subspaces is that in a supersymmetry theory they are usually the subspaces generated by excited states created from the vacuum state by some bosonic and fermionic creation operators, respectively. In our geometrical example one can think of the symmetric and exterior algebras $S(V)$ and $\Lambda(V)$ as the subspaces of bosonic and fermionic states of the tensor algebra $T(V)$, satisfying canonical commutation and anticommutation relations, respectively, see the end of Section 1.8; $S(V)$ is a commutative algebra with multiplication $\alpha \cdot \beta = S(\alpha \otimes \beta)$, while $\Lambda(V) = \oplus_{i=0}^n \Lambda^n V$ is a graded commutative algebra with $\alpha \wedge \beta = \Lambda(\alpha \otimes \beta)$. Graded commutativity means $\alpha \wedge \beta = (-1)^{\deg\alpha \deg\beta} \beta \wedge \alpha$. A general Clifford algebra can also be decomposed into bosonic and fermionic subalgebras, by the canonical involution $\iota(x_1 \cdots x_k) = (-1)^k (x_1 \cdots x_k)$ for $x_l \in c(V)$.

1.10. Around the Laplace operator

This section is still to be written. At the moment we can give only a brief indication of the desired contents.

In Sections 1.6 and 1.7 we have already seen how important can be the Laplacian. Another kind of geometric significance is revealed by the fact that the only differential operators invariant under the isometries of a Riemannian manifold are more or less always the polynomials of the Laplacian, see [H1, Chp. X] and [Ha].

Let X be a Riemannian manifold with a given (usually discrete) group Γ of isometries. Now Γ also acts on function spaces over X, e.g. on $L^2(X, \mathbb{R})$, and we can speak of Γ-invariant functions an operators on X. The *Selberg trace formula* [Ha] relates the Γ-invariant eigenvalues and eigenfunctions of invariant differential and integral operators, such as the Laplacian, to the geometry of the qutient space X/Γ. This is a generalization of the *Poisson summation formula.* It can be viewed as the statement that two isospectral tori are isometric. For arbitrary manifolds it is not true, though we have seen in Section 1.5 that the spectrum contains a lot of geometric data. The *Sunada trace formula* gives a nice group-theoretic characterization of isospectral but non-isometric manifolds. See [Br], which is a very nice survey on the subject of inverse spectral geometry.

There is a *discrete Laplacian* for finite and infinite graphs and groups, which is basically the same as the *Markov operator* of simple random walks on these graphs. This operator acts on the L^2-space of the graph, and the spectrum of it is closely related to the geometry of the graph, such as expanding properties, *amenability*, and geometric properties of random processes on the graphs. If our graph is a lattice in a Riemannian manifold, then the geometric properties of the continuous and discrete Laplacians are the same. For example, the representation theory of Lie groups has led to the construction of *expanders*. There is also a *discrete potential theory* on infinite graphs, and the probabilistic boundary of the random walks, according to which we would like to solve Dirichlet etc. problems, often has a geometric meaning, such as the *Gromov boundary of hyperbolic groups*. On these topics see [Lu] and [Wo].

2. Supersymmetry and Morse theory

In this chapter we will prove the weak and strong Morse inequalities of Theorem 0.3, using asymptotic methods based on supersymmetry theories. We also outline the generalization of our arguments to degenerate Morse theory, and formulate a sharpening of the Morse inequalities, which is in close connection with the so-called Floer cohomology, see later.

As we have already indicated before, we have a supersymmetry theory on the exterior bundle: we can define

$$Q_1 = d + d^* = D, \qquad Q_2 = i(d - d^*), \qquad H = dd^* + d^*d = D^2,$$

because of $d^2 = d^{*2} = 0$ we have the supersymmetry relations

$$Q_1^2 = Q_2^2 = H, \qquad Q_1Q_2 + Q_2Q_1 = 0,$$

and we have the bosonic and fermionic subspaces of forms, Λ^+T^*M and Λ^-T^*M.

Now we define a Dirac complex (a perturbated deRham complex) on the exterior bundle, using a given Morse function $h : M \longrightarrow \mathbb{R}$:

$$d_t = e^{-ht}de^{ht}, \qquad d_t^* = e^{ht}de^{-ht}$$

for any $t \geq 0$. We have $d_t^2 = d_t^{*2} = 0$ again, and so if we define the corresponding operators

$$Q_{1t} = d_t + d_t^* = D_t, \qquad Q_{2t} = i(d_t - d_t^*), \qquad H_t = D_t^2,$$

then the supersymmetry relations are still satisfied for any t.

In this Dirac complex we have all the properties of the Dirac operator D_t we have obtained in the Preliminaires. We can define the Betti numbers $B_k(t)$ as the dimension of the cohomology group $H^k(M, d_t)$. By the Hodge Theorem 1.11, this $B_k(t)$ is equal to the number of zero eigenvalues of H_t in the space Λ^kT^*M of k-forms. However, d_t differs from d by only a conjugation by the invertible operator e^{ht}, so the mapping $\omega \mapsto e^{th}\omega$ is an invertible mapping from $H^k(M, d)$ to $H^k(M, d_t)$, so $B_k(t)$ is independent of t, and thus equals the original Betti number B_k. So H_t contains the same geometric data as H did, while the for large t the spectrum of H_t simplifies dramatically, and gives information about the critical points of h — this will be the way of establishing the Morse inequalities Theorem 0.3.

Similarly to the computations determining the Clifford structure of the exterior bundle, Section 1.4, one can easily show that

$$d_t\omega = d\omega + tdh \wedge \omega, \quad d_t^*\omega = d^*\omega - tdh \lrcorner \omega,$$

and if $x^i,\ i = 1, \ldots, n$ are local coordinates with orthonormal tangent basis $\partial/\partial x^i$, then

$$H_t = dd^* + d^*d + t^2\|dh\|^2 + \sum_{i,j} t \frac{\partial^2 h}{\partial x^i \partial x^j}[a_i^*, a_j],$$

where $\|dh\|^2 = \sum_{i,j} g^{ij}(\partial h/\partial x^i)(\partial h/\partial x^j)$, by the Riemannian metric extended to differential forms (and now this is the square of the gradient of h). The operator a_i is the interior multiplication by the tangent basis vector $\partial/\partial x^i$, and a_i^* is the exterior product by dx^i, the fermionic annihilation and creation operators at the end of Section 1.8. As we have seen in Section 1.4, a_i^* is the adjoint of a_i.

For large t, the 'potential energy' $V(x) = t^2\|dh\|^2$ becomes very large, except at the critical points of h, where $dh = 0$. Around a critical point p we can introduce locally Euclidean coordinates, i.e. $x(p) = 0$ and $\partial g_{ij}/\partial x^k = 0\ \forall\, i, j, k$, such that $h(x) = h(0) + 1/2 \sum \lambda_i (x^i)^2 + O(x^3)$, for some real numbers λ_i; this is a modified version of the Morse Lemma 0.1.

Near this critical point, H_t can be approximated as

$$\overline{H}_t = \sum_{i=1}^{n} \left(-\frac{\partial^2}{(\partial x^i)^2} + t^2 \lambda_i^2 (x^i)^2 + t\lambda_i [a_i^*, a_i] \right),$$

which would be equal to H_t in case of a flat manifold. Now we can easily calculate the spectrum of

$$\overline{H}_t = \sum_{i=1}^{n} (H_i + t\lambda_i K_i),$$

where

$$H_i = -\frac{\partial^2}{(\partial x^i)^2} + t^2 \lambda_i^2 (x^i)^2, \qquad K_j = [a_j^*, a_j].$$

The operators H_i and K_j mutually commute so can be simultaneously diagonalized. Now H_i is the Hamiltonian of a simple harmonic oscillator in the ith direction, and so, by a simple extension of Theorem 1.17, the operator $\sum_i H_i$ has the eigenvalues $t \sum_i |\lambda_i|(1+2p_i)$, $p_i = 0, 1, 2, \ldots$, each of which appears with multiplicity 2^n (the fiber dimension of the exterior bundle). The eigenfunctions of H_i vanish rapidly if $|\lambda_i x^i| \gg 1/\sqrt{t}$. We also outline the generalization of our arguments to degenerate Morse theory.

The operators K_j act on each of the eigenspaces as involutions, splitting them into $+1$- and -1-eigenspaces for each K_j; actually,

$$K_j dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \begin{cases} -dx^{i_1} \wedge \cdots \wedge dx^{i_k}, & \text{if } j \notin \{i_1, \ldots, i_k\} \\ +dx^{i_1} \wedge \cdots \wedge dx^{i_k}, & \text{if } j \in \{i_1, \ldots, i_k\}. \end{cases}$$

So the $t \sum_i |\lambda_i|(1 + 2p_i)$-eigenspace for $\sum_i H_i$ splits into $t \sum_i (|\lambda_i|(1 + 2p_i) + \lambda_i q_i)$-eigenspaces for $\overline{H}_t$ with $q_i \in \{1, -1\}$. So, looking at the eigenspaces of the K_j's, we have proved the following

Lemma 2.1. *The spectrum of $\overline{H}_t$ acting on k-forms is*

$$t \sum_{i=1}^{n} (|\lambda_i|(1 + 2p_i) + \lambda_i q_i),$$

where $p_i = 0, 1, \ldots$ and $q_i \in \{1, -1\}$, $|\{q_i = 1\}| = k$. The multiplicity of an eigenvalue with given p_i's and q_i's is one. □

No we are looking for the zero eigenvalues of $\overline{H}_t$. To get a zero eigenvalue, all p_i's have to be zero, and q_i has to be $+1$ if and only if λ_i is negative. Thus the approximation $\overline{H}_t$ at the critical point p has exactly one zero eigenvalue, which is a k-form if the critical point p has Morse index k. All the other eigenvalues tend to infinity, proportionally to t.

Using the fact that the eigenforms of $\overline{H}_t$ for large t are very much concentrated near the critical point at which we localized H_t, one can show that our approximation is close enough to H_t to imply that for each critical point, H_t has exactly one eigenform (concentrated at the critical point) with eigenvalue not tending to infinity, and this is a k-form if the index of the critical point is k. The exact verification of this approximation needs very technical and careful analytic arguments, so we would rather believe it for now; a detailed proof can be found in [R, Chp. 12].

Certainly, not all the low energy eigenstates of H_t must be annihilated indeed, but we have just proved the weak Morse inequalities: the number of k-forms with zero eigenvalue is at most the number of critical points of index k: $B_k \leq M_k$. Actually, we have almost proved the strong Morse inequalities, the complete Theorem 0.3, as well. The only thing we have to notice is that we have in fact constructed a chain complex of the vector spaces $X_k,\ k = 0, \dots, n$, spanned by the low energy eigen-k-forms of H_t localized at the critical points of Morse index k, and coboundary operator d_t restricted to X_k. This restriction works because of $d_t H_t = H_t d_t$, and so d_t preserves the eigenvalues of H_t. The kth cohomology group of this complex has dimension B_k, and the dimension of X_k is M_k; a standard argument about chain complexes (see [M2, §5]) then shows the strong Morse inequalities for the *Poincaré polynomial* $P(t) = \sum_k B_k t^k$:

$$\sum_k M_k t^k - \sum_k B_k t^k = (1+t) \sum_k Q_k t^k,$$

for some nonnegative integers Q_k. $\square$

It is also possible to treat the degenerate case in this framework, that is, when the critical points of h form an arbitrary submanifold N. The only thing we have to assume is that at any point on any connected component N_0 of N, the Hessian matrix h_{**} restricted to the directions orthogonal to N_0 is nonsingular. Then the number of negative eigenvalues is constant along N_0, this is the Morse index k of N_0. The negative eigenstates form a k-dimensional bundle over N_0, this is called the negative bundle $\Gamma(N_0)$. As before, the potential energy $V(x) = t^2 \|dh\|^2$ vanish on N_0, but for large t, is very large elsewhere. This will imply that the low-lying spectrum of H_t, acting on states localized near N_0, converges to the spectrum of the Laplacian on N_0.

An open neighborhood $M(N_0)$ of N_0 in M can be regarded as a fiber bundle over N_0 by projecting each point in $M(N_0)$ onto the nearest point of N_0. Using the Riemannian metric of M it makes sense to think of the exterior derivative $\tilde{d}$ of N_0 as acting on the deRham complex of the whole neighbourhood $M(N_0)$. Then for H_t on $M(N_0)$ we have

$$H_t = (\tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d}) + H',$$

where the first term is just the Laplacian of N_0 acting on the deRham complex of $M(N_0)$, and H' contains all terms which act in the directions transverse to N_0. This is true basically because of the fact that the Laplacian at a point can be calculated using the partial derivatives along any orthonormal coordinate system, which follows from the isometry invariance of the Laplacian.

For any point p in N_0, one can think of H' as a differential operator acting on forms of the fiber over p in $M(N_0)$. For large t, this restriction of H' can be approximated similarly to what we did before, and we have a single zero-energy k-form, denoted by $\alpha(q,p)$, where q stands for the points in the fiber over p. For each p we have such a k-form, and now we should glue them together to get a low-lying state ψ of H_t on the whole of $M(N_0)$. This can be done in the form

$$\psi(q,p) = \chi(p) \otimes \alpha(q,p),$$

where χ will be a low-lying eigenstate of $\tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d}$. Now this correspondence between the low-lying eigenstates of H_t on $M(N_0)$ and those of the standard Laplacian $\tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d}$ on N_0 leads to the degenerate Morse inequalities.

There is only one subtlety we have to be careful with. The form $\alpha(q,p)$ gives an orientation of the fiber over p in the negative bundle $\Gamma(N_0)$, and so the state χ can be an ordinary differential form only if $\Gamma(N_0)$ is orientable. If the negative bundle is not orientable, then χ will be a section of the twisted deRham complex of N (see [BT, §7]), and then, consequently, we have to use the Poincaré polynomial of the twisted complex for the strong Morse inequalities.

Actually, we have done significantly more than the original Morse inequalities, since we have got not only local attachments, but also the operator d_t acting on the chain complex determined by the critical points. We could try to refine the Morse inequalities by computing the action of d_t on the spaces X_k, but, in fact, we can not expect any improvement, since these asysptotic computations involve only local data, and the strong Morse inequalities are already the best possible in this sense. So the only way of refining them could be a somewhat global connection, some kind of 'tunneling' between the different critical points — the aim would be an exact description of the cohomology of the manifold in terms of Morse theory. We give only a very rough extract of the ideas appearing in [W1], where the exact realizations and computations use deep methods from the theory of the *supersymmetric nonlinear sigma model*, see [DMS], such as the so-called instanton calculation.

We consider paths of steepest descent between pairs of critical points, determined by the gradient flow of h. If we use a Morse-Smale function for h, as defined in the Introduction, then all these paths are generated by paths connecting critical points of index differing by one. Consider two critical points, p and q, with indices k and $k+1$. The low-lying eigenforms α and β localized at these critical points give oritentations to the k and $k+1$-dimensional vectorspaces V_p and V_q consisting of negative eigenvectors of the Hessian of h at p and q. Now if we consider the tunneling path γ from q to p then the tangent vector v at q to γ, $v \in V_q$, induces an orientation of the k-dimensional subspace $\tilde{V}_q$ orthogonal to v, just using the interior multiplication of v with the $k+1$ form we have. Now we can parallelly transport this vectorspace $\tilde{V}_q$ along γ onto the space V_p. We define n_γ to be $+1$ or -1 depending on whether the two orientations agree or disagree. Finally, let us define $n(\alpha,\beta) = \sum_\gamma n_\gamma$, where the sum runs over all paths of steepest descent from q to p.

Now we are ready to define a coboundary operator

$$\delta : \, X_k \longrightarrow X_{k+1}, \quad \delta\alpha = \sum_\beta n(\alpha,\beta)\beta,$$

where the sum runs over all basis elements β of X_{k+1}. It is not obvious at all that $\delta^2 = 0$, but it can be computed from the large t limit of d_t.

Now the statement is that the number Y_k of the zero eigenvalues of $\delta\delta^* + \delta^*\delta$ acting on X_k furnish the usual upper bounds for the ordinary Betti numbers B_k of the manifold. On the other hand, Witten's conjecture was that $Y_k = B_k$ is always true.

There is another context where the integer $n(\alpha,\beta)$ appears: it is the intersection number of the ascending sphere from p and the descending sphere from q, determined by the gradient flow of the Morse-Smale function h.

The construction in [W1] is in fact the finite dimensional version of *Floer's instanton cohomology theory*, see [F], and it is proved in [S] that this cohomology is independent of the given Morse-Smale function we started with, and in fact is isomorphic to the usual cohomology of the manifold. Floer's homology theory was developed in order to solve Arnold's conjecture about certain Hamiltonian dynamical systems: see [HoZ] and the end of Chapter 4.

3. Supersymmetry and Killing vector fields

A smooth vector field on the compact manifold M with Riemannian metric g is called a *Killing vector field* if the Lie derivative of the metric tensor vanishes: $\mathcal{L}_K g = 0$. Locally, and in the case of a compact manifold, even globally, this is equivalent to the condition that the flow $\alpha_t : M \longrightarrow M$ of K is an isometry for every $t \in \mathbb{R}$. Locally, if the Riemannian metric is Euclidean, $g_{ij} = \delta_{ij}$, then being a Killing field reads as $\partial K_i/\partial x^j + \partial K_j/\partial x^i = 0$, where $K = \sum_{i=1}^n K_i \partial/\partial x^i$. Moreover, this local description can be transported even for an arbitrary metric, as follows. Locally we can choose geodesic coordinates, which are the images of Euclidean coordinates on T_pM under the exponential map at p; they will be locally (approximately) Euclidean coordinates. The condition being a Killing field is preserved by the exponential map, and so in locally Euclidean coordinates the matrix $\big(\partial K_i/\partial x^j\big)_{i,j=1}^n$ will be skew-symmetric. In odd dimension this matrix is singular, therefore no zero of K can be isolated. In even dimension, around an isolated zero, the following expansion can be achieved in locally Euclidean coordinates:

$$K = \sum_{i=1}^{n/2} \lambda_i \left(x_{2i-1} \frac{\partial}{\partial x^{2i}} - x_{2i} \frac{\partial}{\partial x^{2i-1}} \right)$$

with some constants $\lambda_1, \dots, \lambda_{n/2}$. In this case K has eigenvalues $e^{\pm i\lambda_k}$; one can think of the λ_k's as rotation angles near the fixed point. For details on the computations above see [DFN I, §23].

Now we will consider a perturbated deRham complex again as a Dirac complex on the exterior bundle:

$$d_s = d + si(K),$$

where $i(K)$ is the interior multiplication by K, and s is an arbitrary real number. This operator d_s maps a k-form into a linear combination of a $(k+1)$- and a $(k-1)$-form, so we can use the decomposition $\Lambda^+T^*M \oplus \Lambda^-T^*M$ into even and odd dimensional forms. To calculate the Dirac operator of our complex (which, in fact, is not a Dirac complex, yet), we first mention a few simple properties of the operators $i(K)$ and $\kappa\wedge$, where κ is the 1-form dual to K, w.r.t. the metric. As for any vector field, $di(K) + i(K)d = \mathcal{L}_K$, and now, for a Killing field, also $d^*(\kappa\wedge) + (\kappa\wedge)d^* = -\mathcal{L}_K$. Furthermore, $i(K)d\kappa = -d(K,K)$ and $i(K)(\kappa \wedge \omega) + (\kappa\wedge)(i(K)\omega) = (K,K)\omega$ for every differential form ω. Using these computations we have

$$d_s^2 = -d_s^{*2} = s\mathcal{L}_K$$

and

$$H_s = d_s d_s^* + d_s^* d_s = dd^* + d^*d + s^2(K,K) + s\big((d\kappa) \wedge + i(d\kappa)\big),$$

where $i(d\kappa)$ is the adjoint of $(d\kappa)\wedge$.

We would like to count the number of zero eigenvalues of H_s. Clearly, any such eigenvalue ψ also has to obey $d_s\psi = d_s^*\psi = 0$, and thus it is annihilated by $s\mathcal{L}_K$. Therefore we lose nothing if we restrict ourselves to the subspace of states annihilated by $\mathcal{L}_K$, the states which are invariant under the one-parameter group of isometries generated by K. Within this subspace we have $d_s^2 = 0$, and thus we already have a Dirac complex. By the Hodge Theorem 1.11, the number n_+ of zero eigenvalues of H_s within the subspace of even dimensional forms is equal to the sum of the even dimensional Betti numbers of our Dirac complex on M, and similarly for the odd dimensional zero eigenvalues, n_-. Moreover, these numbers are independent of s, provided $s \neq 0$. By the usual $\mathbb{Z}_2$-grading, the Hermitian operator $D_s = d_s + d_s^*$ can be decomposed as $D_s = D_{s+} + D_{s-}$, and $\mathrm{ind}(D_{s+}) = n_+ - n_-$. By standard arguments, similarly to Lemma 1.15, this index is independent on s, so $n_+ - n_- = \mathrm{ind}(D_+) = \chi(M)$.

Now we are going to associate the numbers n_+ and n_- to the Betti numbers of the submanifold N, consisting of the zeros of the Killing vector field K.

Theorem 3.1. *For any Killing vector field on M, we have $n_+ = N_+$ and $n_- = N_-$, where N_+ and N_- are the sums of the even and odd dimensional standard Betti numbers of N. In particular, $\chi(M) = \chi(N)$.*

Proof. We show first that $n_+ \geq N_+$ for n even, and $n_- \geq N_-$ for n odd. Then, using the asymptotic expansion of H_s for large values of s, we will prove some kind of strong Morse inequalities between the k-dimensional Betti numbers n_k and N_k. In particular, $n_+ - n_- = N_+ - N_-$, which implies the desired result. This second step will be very similar to the method we proceeded with in Chapter 2, so we will do it very briefly.

Let N_0 be any connected component of N, and let ψ be any differential form on it, which is a representative of the standard deRham cohomology of N_0. We will construct a corresponding form $\overline{\psi}$ on M, which is closed but not exact in the sense of d_s.

An open neighborhood $M(N_0)$ of N_0 in M can be regarded as a fiber bundle over N_0, as we did in Chapter 2. Using the fiber bundle structure, we can extend ψ to a form $\tilde{\psi}$, defined on $M(N_0)$, for which $d\tilde{\psi} = 0$ and $i(K)\tilde{\psi} = 0$ on $M(N_0)$, since the projection commutes with the action of K. So $d_s\tilde{\psi} = 0$, and it is impossible to have $\tilde{\psi} = d_s\alpha$, because it would reduce to $\psi = d\alpha$ on N (being $K = 0$ on N), which does not have a solution, by hypothesis. The only problem is that on the boundary of $M(N_0)$ we do not have $d\tilde{\psi} = 0$, so we have to modify $\tilde{\psi}$ a little bit.

Consider the set of points M_ϵ satisfying $K^2 = (K, K) \leq \epsilon$, and choose $\epsilon > 0$ such that the component of M_ϵ containing N_0 is contained in $M(N_0)$. Let $\phi(x)$ be a smooth real function with $\phi(0) = 1$ and $\phi(x) = 0$ for $x \geq \epsilon$.

We now define

$$\sigma = \phi(K^2) + \frac{1}{s}\phi'(K^2)d\kappa + \frac{1}{2s^2}\phi''(K^2)d\kappa \wedge d\kappa + \frac{1}{3s^3}\phi'''(K^2)d\kappa \wedge d\kappa \wedge d\kappa + \cdots .$$

This series terminates since M is finite dimensional. Now, using the properties of $i(K)$ we have listed earlier, one can immediately see that $d_s\sigma = 0$ if n is even, while if n is odd, then $d_s\sigma$ has only n-dimensional non-zero terms.

Now consider the form $\overline{\psi} = \tilde{\psi} \wedge \sigma$, with a ψ being an even (respectively odd) dimensional representative of the cohomology of N, depending on whether n is even or odd. Being M compact, finitely many iterations of this procedure yields in a form on the whole of M, and this form shows our first desired inequality in both cases.

For the opposite (Morse-type) inequalities, assume first that K has only isolated zeros, and thus n is even. As in Chapter 2, we approximate H_s by its ‘flat version’

$$\overline{H}_s = -\sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2} + s^2 \sum_{k=1}^{n/2} \lambda_k^2\big((x^{2k-1})^2 + (x^{2k})^2\big) + 2s \sum_{k=1}^{n/2} \lambda_k(a^*_{2k-1}a^*_{2k} - a_{2k-1}a_{2k}),$$

where a_i and a^*_j are the ‘fermionic creation and annihilation operators’ introduced in Section 1.8. Now $\overline{H}_s$ can again be d diagonalized explicitly, and it turns out that there altogether N_+ states in Λ^+T^*M whose H_s-energy does not go to infinity with s, and none in Λ^-T^*M. This implies $n_+ \leq N_+$, $n_- = N_- = 0$. Together with our first inequality, we have $n_+ = N_+$, as well.

In the general case, similarly to our discussion of degenerate Morse theory, the low-energy eigenvalue problem for H_s reduces for large s to the eigenvalue problem of the ordinary Laplacian $H_N = dd^* + d^*d$ on N. In fact, H_s has N_+ even dimensional and N_- odd dimensional eigenvalues which vanish in the large s limit. This clearly implies $n_+ \leq N_+$ and $n_- \leq N_-$.

Moreover, just as for the strong Morse inequalities, we have constructed a suitable chain complex showing $n_+ - n_- = N_+ - N_-$. Hence, combining our equalities and inequalities, we have proved our theorem. $\square$

In the case when N consists of only isolated zeros, this theorem agrees with the Poincaré-Hopf index theorem, see Sections 0.2 and 1.7, as

$$\chi(M) = \sum_{x \in N} \mathrm{ind}_x K = |N| = N_+ = \chi(N).$$

Moreover, we can look at an arbitrary vector field, not necessarily a Killing field. Then we will not have a general formula for the number of eigenvalues of H_s, but, with some small additional efforts we can achive the full Poincaré-Hopf index theorem, even for the degenarate case.

One can also prove an index theorem for the *Hirzebruch signature* of M and N, first proved by Atiyah and Bott [AB1], as a consequence of Theorem 1.16. On a $2k$-dimensional manifold we can consider the bilinear form $\mu: H^k(M) \times H^k(M) \longrightarrow \mathbb{R}$ given by $\mu(\alpha, \beta) = \int_M \alpha \wedge \beta$. This is a symmetric form if k is even, and the signature of μ (the number of positive eigenvalues minus the number of negative eigenvalues) is the Hirzebruch signature of a $4l$-dimensional oriented manifold. For more details, for example, for Hirzebruch's formula which is of the kind of the Gauss-Bonnet-Chern theorem, Section 5.1, see [BT, §22] and [DFN III, §27]. The key step is to introduce the Hermitian operator $Q_s = i^{1/2} d_s + i^{-1/2} d_s^*$, and for any real θ, let $I(\theta) = \mathrm{Tr} * \exp(\theta \mathcal{L}_K)$, where the trace is to be evaluated among the states annihilated by Q_s. Then, near an isolated zero p of K, we have

$$I_p(\theta) = n_p \prod_k \frac{1 + e^{i\theta\lambda_{p,k}}}{1 - e^{i\theta\lambda_{p,k}}},$$

where the $\lambda_{p,k}$'s are the rotation angles at p, and $n_p = \pm 1$, according to some kind of orientation. The index formula is then

$$\operatorname{sign} M = \operatorname{sign} N = \sum_p I_p(\theta).$$

A similar formula can be achieved for the rotation angles of the derivative of any isometry on M having only isolated fixed points. A nice group of corollaries is the following.

Let M be a compact connected oriented manifold, and let f be a homeomorphism of M of prime power order $n = p^l$ with p odd. Then f can not have just one fixed point. If M is a homology sphere and f is a $\mathbb{Z}_p$-action with precisely two fixed points then the induced representations of $\mathbb{Z}_p$ on the tangent spaces of the two fixed points are isomorphic. An involution of an oriented $4k$-dimensional compact manifold of odd Euler characteristic (such as the complex projective spaces $\mathbb{CP}^{2k}$) must have a fixed point set of $\dim > 0$.

Learning these result the Reader may have the rather bold idea that the *Borsuk-Ulam theorem* could be deduced from the Atiyah-Bott-Lefschetz fixed point theorem, as the Brouwer fixed point theorem is usually proved from the ordinary Lefschetz theorem. However, we could not find any reference or proof to this possible application.

The method of examining the asymptotic behaviour of a perturbated deRham complex can also work in the case of path spaces, but because of the infinite dimensional features the correct mathematical footing is usually extremely hard. In [W] there is a sketch of the generalization of the Killing vector field methods, with the following starting idea. If we are given the loop space $\Omega(M; S^1)$ consisting of smooth maps from the circle S^1 into the complete Riemannian manifold M, than the rotation group $U(1)$ of the circle acts naturally on Ω by $\sigma(t) \mapsto \sigma(t+a)$, for $\sigma \in \Omega$, $a \in S^1$. Thus we have a continuous group of isometries on Ω, equipped with the natural metric. Then we can consider the corresponding Killing vector field K, and the perturbated differential operator $d_s = d + si(K)$. The zeros of the vector field are loops invariant under the action of $U(1)$, that is, the constant maps. The space of zeros thus can be identified with the manifold M, and the zero eigenvalues of H_s are encoded in the Betti numbers of M.

For example, if the manifold M is orientable, then the sum of its Betti numbers is non-zero, and thus we have a zero eigenvalue of H_s, which, as we mentioned in Section 1.9, is usually the main question in quantum physics. One can also define Lefschetz numbers and Hirzebruch signature, but it is very important to notice that the results are seriously lack of mathematical rigour. Nevertheless, the method reveals a deep interaction between the supersymmetric non-linear sigma model of quantum field theory and important mathematical problems, see [AG] and [W2]. This way has also lead to a supersymmetric proof of the Atiyah-Singer and Atiyah-Bott-Lefschetz index theorems in [AG], but this is of course heavily supported by physical intuition.

4. Hamiltonian dynamical systems and Morse theory

One of the most powerful applications of Morse theory is the determination of the geodesic structure of a manifold. In this section we give a brief overview of the classical approach, together with some beautiful applications of the theory, see [M2], [DFN I-II] and [A]. For the basics of variational calculus we also refer the reader to these books. For the dynamical system point of view (symplectic geometry, orbit growth, entropy, ergodicity, etc.) our standard reference is [KH] and [HoZ]. For more advanced bits of the theory see [Ho], [MW], [DFN II], and the Appendices of [A]. For Yang-Mills theory see the next Chapter.

4.1. Hamiltonian dynamical systems

Let M be a smooth n-dimensional manifold, $p, q \in M$ arbitrary points, and $\Omega = \Omega(M; p, q)$ the set of piecewise smooth paths $\omega : [0,1] \longrightarrow M$ connecting p and q. This set has a natural topology, and we can view it as an infinite dimensional manifold; this will be correctly defined later. For a given path $\omega \in \Omega$ we can consider a 'smooth curve' through it, that is a smooth *variation* $\alpha : (\epsilon, \epsilon) \times [0,1] \longrightarrow M$ with $\alpha(0,t) = \omega(t)$; the endpoints are usually fixed, $\alpha(u,0) = p$, $\alpha(u,1) = q$, so $\alpha(u,t) \in \Omega$ for all u. There is a variation vector field

$$W_t = \left.\frac{\partial \alpha(u,t)}{\partial u}\right|_{u=0}$$

associated to α; the tangent space $T_\omega\Omega$ is the set of these vector fields. The parameter neighbourhood $(-\epsilon, \epsilon)$ can be replaced by $(-\epsilon, \epsilon)^n$, in this case we speak about an n-parameter variation.

We are going to consider the *energy functional*

$$E(\omega) = \int_0^1 \|\dot{\omega}(t)\|^2 \, dt$$

on the path space of a Riemannian manifold M. Clearly, this functional has a close relation to the arc-length $S(\omega)$; for example, $S(\omega)^2 \leq E(\omega)$, by Schwarz's inequality. However, one should notice that this energy functional is more sensitive than length; for example, it is not independent on reparametrization. Using the *First variational formula* it is easy to show that the *critical points* of E, i.e. the paths ω for which

$$\left.\frac{dE(\alpha(u,t))}{du}\right|_{u=0} = 0$$

for every variation α of ω, are exactly the geodesics connecting p and q.

In fact, this is a special case of the *Euler-Lagrange principle* in the variational calculus of dynamical systems. If there is a Lagrangian on TM given by $L(x,v) = K(x,v) - V(x)$, $v \in T_xM$, $K(x,v)$ is the 'kinetic' and $V(x)$ is the 'potential energy', in our case $L(x,v) = (v,v)$, then the energy function E, given by integrating the Lagrangian $L(x(t), \dot{x}(t))$ along a path, has its critical points at the solutions of the *Euler-Lagrange equation*

$$\frac{d}{dt}\frac{\partial L(x,v)}{\partial v} = \frac{\partial L(x,v)}{\partial x},$$

where $v(t) = \dot{x}(t)$, that is, at geodesics in our special case. One can consider the *Legendre transform* $\mathcal{L} : TM \longrightarrow T^*M$, given by

$$\mathcal{L}(x,v) = \left(x, \frac{\partial K(x,v)}{\partial v}\right) = (q,p);$$

this transforms the Euler-Lagrange equation into the *Hamiltonian equations*

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p},$$

where $H(x, v) = K(x, v) + V(x)$ is the total energy.

A Lagrangian $L(x, v)$ or a Hamiltonian $H(x, v)$ governs a dynamical system on TM by postulating that a particle on M moves in such a way that the Euler-Lagrange equation, or equivalently, the Hamiltonian equations, hold all the time. Clearly, if we have an initial state $(x_0, v_0) \in TM$, then these equations determine time evolution uniquely, and the Hamiltonian $H(x, v)$ is constant along the orbits. The flow of the Hamiltonian dynamical system given in the special case above is called the *geodesic flow*: the $x(t)$-coordinates of the orbits are precisely the geodesics on M, with constant speed $v = \dot{x}(t)$.

Hamiltonian flows are the most important special cases of symplectic flows. A *symplectic manifold* (M, ω) is a smooth $2n$-dimensional manifold M equipped with a non-degenarate closed 2-form ω. This non-degeneracy can be defined in two equivalent ways: for every $x \in M$ it defines an isomorphism between T_xM and T_x^*M, or, the nth exterior power ω^n is a nonzero volume form on M. The basic example is $\mathbb{R}^{2n}$ with the antisymmetric bilinear form

$$\omega(u, v) = \langle u, Jv \rangle, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix};$$

this is also a complex structure on $\mathbb{R}^{2n}$ with $J^2 = -1$. Moreover, locally every symplectic manifold is of this form. First of all, in $(\mathbb{R}^{2n}, \omega)$ there exists a basis in which the 2-form ω is given by J, as above — these coordinates are called symplectic or canonical ones. Secondly, as *Darboux's theorem* claims, around any $x \in M$ in a symplectic manifold (M, ω) one can introduce local coordinates $q^1, \dots, q^n, p^1, \dots, p^n$, $q^i(x) = p^i(x) = 0$, such that

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp^i,$$

that is, ω is given by the constant matrix J in each point of the coordinate chart! This shows that symplectic manifolds are much more rigid than Riemannian manifolds — in fact, they can be regarded as the analogues of flat Riemannian manifolds, where the condition $d\omega = 0$ stands for flatness.

We say that a smooth map $\phi : M \longrightarrow N$ between symplectic manifolds (M, ω) and (N, η) is symplectic if $\phi^*\eta = \omega$ for the pull-back. As the most special case, the symplectic linear transformations of a symplectic vector space $(\mathbb{R}^{2n}, \omega)$ are the members of the Lie group

$$Sp(2n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) \, : \, A^T JA = J\}.$$

One can immediately see that symplectic maps are orientation and volume preserving.

If we are given a smooth function $H : M \longrightarrow \mathbb{R}$ on the symplectic manifold (M, ω), it defines a so-called *Hamiltonian vector field* X_H by the formula

$$i(X_H)\omega = dH,$$

and its *Hamiltonian flow* ϕ_H^t, by $\dot{\phi}^t = X_H$. With this we have arrived to the Hamiltonian dynamical systems. If there is a Hamiltonian $H(x, v)$ given on the $2n$-dimensional tangent bundle TM of an n-manifold M, then the coordinates (q, p) on the cotangent bundle T^*M defined by the Legendre transform $\mathcal{L} : TM \longrightarrow T^*M$ above are canonical coordinates for the symplectic manifold $(T^*M, \sum_{i=1}^n dq^i \wedge dp^i)$, and the flow determined by the Hamiltonian equations is exactly the Hamiltonian flow X_H.

It is worth noting that there exist even dimensional manifolds with no symplectic form; a good exercise is to show that the spheres S^{2n} are such manifolds. Another exercise could be to construct symplectic flows which are not Hamiltonian.

On a symplectic manifold the *Poisson bracket* of functions is defined as $\{f, g\} = \omega(X_f, X_g)$. In canonical coordinates (q, p) this can be written in the form we met in Section 1.8:

$$f, g = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

The linear space $C^\infty(M)$ is an infinite dimensional Lie algebra with Lie bracket $\{\cdot,\cdot\}$, and $X_{\{f,g\}} = -[X_f, X_g]$. A function f is invariant under the flow generated by a Hamiltonian H, i.e. constant on orbits, iff $\{f, H\} = 0$ (for example, $\{H, H\} = 0$). These functions are called the first integrals of H, and the Poisson bracket of two first integrals is also a first integral. *Noether's theorem* says that if H is invariant under a 1-parameter family of symplectic diffeomorphisms generated by a Hamiltonian f, then f is a first integral of H. This theorem is usually quoted as 'each symmetry of the system induces a quantity preserved during the motion'. There can be at most $2n - 1$ first integrals with pointwise linearly independent derivatives. *Liouville's theorem* states that Hamiltonian systems are *completely integrable*: if we can find n first integrals f_i with $\{f_i, f_j\} = 0$, which are pointwise linearly independent on a joint level set $M_z = \{x \in M : f_i(x) = z_i, i = 1, \dots n\}$, then $\omega|_{M_z} = 0$, M_z is invariant under each Hamiltonian flow $\phi^t_{f_i}$, and if M_z is compact and connected, then it is diffeomorphic to an n-torus, and $\phi^t_{f_i}|_{M_z}$ is conjugate to a linear flow via this diffeomorphism. This theorem also shows that understanding Hamiltonian dynamics on tori is of crucial importance.

4.2. The Morse theory of geodesics

Returning to the geodesic flow and the variational calculus of geodesics, for a geodesic $\gamma \in \Omega$, which is a critical point of E in the sense we described above, we can consider the Hessian of E:

$$E_{**}(W_1, W_2) = \left.\frac{\partial^2 E(\alpha(u_1, u_2, t))}{\partial u_1 \partial u_2}\right|_{u_i = 0},$$

where $\frac{\partial \alpha}{\partial u_i}(0, 0, t) = W_i(t)$. By the *Second variation formula* this is a well-defined symmetric bilinear form $T_\gamma\Omega \times T_\gamma\Omega \longrightarrow \mathbb{R}$. If γ is a minimal geodesic from p to q then E_{**} is positive semi-definite, i.e. the index of E is zero. A vector field J_t along a geodesic $\gamma(t)$, $t \in [0, 1]$, is called a *Jacobi field* if it satisfies the equation

$$\nabla^2_{\dot\gamma(t)} J_t + R(\dot\gamma(t), J_t)\dot\gamma(t) = 0,$$

where R is the Riemann curvature tensor on M. This is exactly the equation appearing in the Second variation formula. It is not too difficult to see that these Jacobi fields are exactly the variation vector fields arising from free geodesic variations α of γ: the endpoints are not fixed and $\alpha(u, t)$ is a geodesic for each u. The endpoints p and q are called *conjugate* along γ if there is a non-zero Jacobi field which vanish at $t = 0$ and $t = 1$. The multiplicity of this conjugacy is the number of linearly independent Jacobi fields of this kind; this is at most $n - 1$ (an exercise for the Reader). Note that the conjugacy of p and q does not mean that they can be connected by lots of different geodesics — the non-trivial free geodesic variation has to vanish only in first order. An immediate corollary of the Second variation formula is that a vector field $W_1 \in T_\gamma\Omega$ belongs to the null space of E_{**} if and only if it is a Jacobi field, so the nullity of E_{**} is equal to the multiplicity of p and q as conjugate points. A key result of the theory is the following:

Theorem 4.1. (Morse index theorem) *The index of E_{**} is equal to the number of points $\gamma(t)$, $0 < t < 1$, such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ; each conjugacy is counted with multiplicity. This index is always finite.* □

To visualize this theorem a good example can be a standard n-sphere with an antipodal pair of points, p and p^*, and a geodesic from p through p^* to a third point q. The conjugacy of p and p^* means an $(n-1)$-dimensional nullspace of E_{**} at the geodesic segment from p to p^*, and an $(n-1)$-dimensional index at the geodesic from p to q: any geodesic from from p to p^* different from our original one, continued with the original geodesic segment from p^* to q, results in a broken geodesic, which represents a kind of local maximum of E in the path space $\Omega(S^n; p, q)$.

Now we can define a natural metric on $\Omega(M; p, q)$, using the metric ρ coming from the Riemannan structure of M: given $\omega, \omega' \in \Omega$ with arc-lengths $s(t), s'(t)$, then their distance is

$$d(\omega, \omega') = \max_{0 \le t \le 1} \rho(\omega(t), \omega'(t)) + \left(\int_0^1 (\dot{s}(t) - \dot{s}'(t))^2 \, dt \right)^{1/2}.$$

The second term is added in order to make E continuous. Now we can consider finite dimensional approximations to $\Omega^a = \{\omega \in \Omega \mid E(\omega) \le a\}$, consisting of broken geodesics from p to q, and Theorem 0.2 gives our main result:

Theorem 4.2. (Fundamental theorem of Morse theory) *Let M be a complete Riemannian manifold and let $p, q \in M$ be two points which are not conjugate along any geodesic of length $\le \sqrt{a}$. Then Ω^a has the homotopy type of a finite cell complex, with one cell of dimension λ for each geodesic in Ω^a at which E_{**} has index λ. If p and q are not conjugate along any geodesic, then Ω has the homotopy type of a countable cell complex, with cells corresponding to the indices of geodesics from p to q.* $\square$

We can apply this theory both to spaces of negative and positive curvature. Negative curvature is measured by the sectional curvatures $(R(A, B)A, B)$, where $A, B \in T_pM$. A minute's thought about the defining equation of Jacobi vector fields shows that if $(R(A, B)A, B) \le 0$ for every pair of tangent vectors A, B, then no two points of M are conjugate along any geodesic. On the other hand, we can consider the *exponential map* $\exp_p : T_pM \longrightarrow M$, $v \mapsto \gamma_v(1)$, where $\gamma_v(t)$ is the geodesic starting at p in the direction $\dot{\gamma}_v(0) = v$, and it is not difficult to prove that the point $\exp_p(v)$ is conjugate to p along the geodesic γ_v if and only if the exponential mapping is critical at v. So we have the following corollary:

Theorem 4.3. (Cartan) *Suppose M is simply connected complete Riemannian manifold, with sectional curvature $(R(A, B)A, B) \le 0$ everywhere. Then M is diffeomorphic to the Euclidean space $\mathbb{R}^n$.*

Proof. There are no conjugate points, so by the Morse index theorem every geodesic segment has index 0. Thus Theorem 4.2 asserts that for each pair $p, q \in M$ the path space $\Omega(M; p, q)$ has the homotopy type of a 0-dimensional cell complex, with one point for each geodesic from p to q. But then the simply connectedness of M implies that $\Omega(M; p, q)$ is connected, so there is at most one geodesic from p to q. On the other hand, M is complete, therefore we have shown that the exponential map $\exp_p$ is a bijection. Moreover, by the observations of the previous paragraph we know that $\exp_p$ has no critical points, hence it is a local diffeomorphism. Combining these results we obtain that $\exp_p : T_pM \longrightarrow M$ is a global diffeomorphism, which completes the proof. $\square$

After showing that the exponential map is a covering map, a simpler solution to finish the proof could have been that the simply connectedness of M implies that it is diffeomorphic to its universal cover, T_pM. Nevertheless, the original argument give also the following more general result:

Corollary 4.4. *If M is complete with nonpositive sectional curvature, then the homotopy groups $\pi_i(M)$ are zero for $i > 1$, and $\pi_1(M)$ is torsion-free.* $\square$

Inspite of the fact that no complete surface of constant negative curvature can be embedded in $\mathbb{R}^3$, they do exist, and can be even compact: the compact factors of the hyperbolic plane with fundamental domain a regular hyperbolic $4k$-gon with $k \geq 2$, which are surfaces of genus k, see [BP].

Positive curvature can be measured by the Ricci curvature $K(U, V)$.

Theorem 4.5. (Myers) *If the Ricci curvature satisfies $K(U,U) \geq (n-1)/r^2$ for every unit vector U, where r is a positive constant, then every geodesic on M of length $> \pi r$ contains conjugate points, and hence is not minimal. In particular, if M is complete, then it is compact, with diameter $\leq \pi r$.* □

Proof. Consider a geosedic $\gamma : [0,1] \longrightarrow M$ of length L, and a system of n orthonormal vector fields $P_1, \ldots P_n$, parallel along γ. We can assume $LP_n(t) = \dot\gamma(t)$, and have $\nabla_{\dot\gamma(t)} P_i(t) = 0$. Now let $W_i(t) = (\sin \pi t) P_i(t)$. Then, by the Second variation formula,

$$\frac{1}{2} E_{**}(W_i, W_i) = \int_0^1 (\sin \pi t)^2 \left(\pi_2 - L^2(R(P_n, P_i)P_n), P_i)\right) dt,$$

and so for $K(P_n, P_n) \geq (n-1)/r^2$ and $L > \pi r$ we have $\frac{1}{2}\sum_{i=1}^{n-1} E_{**}(W_i, W_i) < 0$. This means that every geodesic of length $L > \pi r$ contains conjugate points, which proves the theorem. □

Theorem 4.6. *If M is compact with positive definite Ricci curvature, then Ω has the homotopy type of a cell complex with only finitely many cells in each dimension.* □

[M2, §19] mentions the problem of characterizing the manifolds which can carry a metric so that all sectional curvatures are positive. For example, the Ricci tensor on the standard product $S^m \times S^k$ is everywhere positive definite, but the sectional curvatures in directions corresponding to the flat tori $S^1 \times S^1 \subseteq S^m \times S^k$ are zero. It is not clear whether $S^m \times S^k$ admits a metric in question.

Here it is worth mentioning a corollary to Bochner's Theorem 1.5. If G is a compact semisimple Lie group, then its Killing metric coming from the Killing form of its Lie algebra $\mathfrak{g}$ is positive definite, with positive definite Ricci curvature operator (see [H2, Chp. II, §5, §6] and [DFN I, §24, §30]). Hence Theorems 1.5, 1.6 and 1.11 give that G has no harmonic 1-forms, and thus its first Betti number is zero.

For more on spaces of negative and positive curvature see [DFN] and [KN]. For harmonic forms and curvature see especially [YB] and [GL].

4.3. Volume growth and the complexity of the geodesic structure

One kind of moral of the above results could be that spaces of positive curvature (e.g. Euclidean spheres) are small, especially compared with those of negative curvature (e.g. hyperbolic spaces). Actually, the spaciousness of spaces of negative curvature can be formulated in a lot of ways. Let M be a compact connected Riemannian manifold with infinite fundamental group; then the universal cover $\tilde M$ is non-compact. SM is the unit tangent bundle of M, and $\mathcal{M} \subseteq SM$ is the set of unit tangent vectors for which the geodesic γ_v on M has a lift to $\tilde M$ of infinite length such that it is length-minimizing along each of its finite segments. It is easy to see that $\mathcal{M}$ is non-empty. SM has a natural finite measure, coming from the product of the Riemannian volume on M and the volume on the balls S_pM. Clearly, the *geodesic flow* g_t on SM preserves this measure (this is the so-called *Liouville measure* for the geodesic flow). Moreover, if M has negative sectional curvature, then g_t is ergodic and Anosov w.r.t. this Liouville measure, see [KH, Thm. 5.4.16 and 17.6.2].

The universal cover $\tilde M$ inherits the Riemannian metric from M. Let $x \in \tilde M$, and $B(x,r)$ the ball around x of radius r. Then we can define the *volume growth* of M as

$$v(M) = \lim_{r\to\infty} \frac{1}{r} \log \mathrm{vol}\,(B(x,r));$$

this limit exists and independent of x. This quantity represents the exponential rate of the volume growth, and it is clearly the same for all compact factors of $\tilde{M}$. If the sectional curvature on M is bounded from below by $-K^2$ and above by $-k^2$, then $k \leq v(M)/(n-1) \leq K$. Another description of the volume growth is that $v(M) = \lim_{r\to\infty} \frac{1}{r} \log |\{\gamma \in \pi_1(M) : \gamma(x) \in B(x,r)\}|$, where the curves γ in the fundamental group are considered as covering transformations of $\tilde{M}$. From this one can see that $v(M) > 0$ if and only if the group $\pi_1(M)$ has exponential growth. Here we can not stand citing the famous *Gromov theorem* stating that a group has polynomial growth if and only if it has a nilpotent subgroup of finite index [Gr]. This theorem has important geometrical connections, both in its proof and in applications, and a related geometrical notion, the *Gromov-norm* plays a key role for example in the proof of the *Mostow rigidity theorem*: if two connected hyperbolic n-manifolds with $n > 2$ have the same fundamental groups then they are isometric [BP].

The connection between volume growth and the richness of the geodesic structure can be described via the topological entropy of the geodesic flow. If f_t is a flow on a compact metric space (X, d), we can define a sequence of metrics by $d_T^f(x,y) = \max_{0\leq t\leq T} d(f_t(x), f_t(y))$. Let $S_d(f,\epsilon,T)$ be the minimal cardinality of an ϵ-covering set of points in X, w.r.t. the metric d_T^f. Then define the *topological entropy* of f_t as

$$h(f) = h_d(f) = \lim_{\epsilon\to 0} \overline{\lim_{T\to\infty}} \frac{1}{T} \log S_d(f,\epsilon,n).$$

Being X compact, this exponential rate of growth is independent of the choice of the original metric d; that is why this is a topological notion. For the geodesic flow g_t on the invariant set $\mathcal{M} \subseteq SM$ of any compact manifold with infinite fundamental group the following important result holds [KH, Thm. 9.6.7]:

$$h(g_t\big|_{\mathcal{M}}) \geq v(M).$$

Finally, we state a celebrated result of this type, see [KH, Thm. 20.6.10]:

Theorem 4.7. (Margulis) *Let M be a compact Riemannian manifold with negative sectional curvature, $G(t)$ is the number of different closed geodesics of length at most t, and h the topological entropy of the geodesic flow g_t on SM. Then* $\lim_{t\to\infty} G(t)2the^{-th} = 1$. $\square$

There is an important and impressive connection between the 'audible' and 'visible' spectra of a compact manifold, where the audible spectrum is the spectrum of the Laplacian on the manifold, and the visible spectrum is the set of lengths of closed geodesics, which is usually a discrete set, by Theorem 4.2. In fact, one can recover these spectra from each other; this connection was first discovered by *Colin de Verdiere*, and was completely solved by *Duistermaat*, *Guillemin*, and *Chazarain*, see [DG] and [Ch], using elliptic pseudo-differential operators.

Similarly to the problem in Section 1.5, one can ask some kind of converse to what we have done so far: what is known about the geometry of the underlying space, if we have, for example, only few critical points? The *Hopf conjecture* states that a Riemannian metric on an n-dimensional torus without conjugate points has to be flat. The $n = 2$ case was proved by Hopf in 1948, while the general case has been solved only in 1994, see [BI1]. The method of the proof is associating a Banach norm to the given Riemannian metric on the universal cover of our torus, and using an integral geometric condition it can be shown that for a metric without conjugate points this norm is in fact Euclidean. Another application of the method is the main result of [BI2]: if $V(r)$ is the volume of an r-ball in the universal cover, and $\epsilon_n r^n$ is the standard volume in $\mathbb{R}^n$, then $\lim_{r\to\infty} V(r)/\epsilon_n r^n \geq 1$ with equality if and only if our torus is flat. Summarizing, if a Riemannian torus does not have one of the two crucial properties of positive curvature we have discussed, then it is flat, indeed.

There should be a lot more about the connections between ergodic theory and geometry here, but we do not have time at the moment, unfortunately.

4.4. Morse theory on symplectic manifolds

This section consists of two parts. First of all, recall Noether's theorem from Section 4.1, showing the importance of the symmetries of a Hamiltonian system. So let us suppose that G is a Lie group which acts on the symplectic manifold M and preserves the symplectic form ω. Each element $a \in \mathfrak{g}$ in the Lie algebra of G determines a one-parameter symplectic flow in M, with Hamiltonian H_a. Now we have

$$H_{[a,b]} = \{H_a, H_b\} + C(a,b),$$

where $C(a,b)$ is a constant satisfying

$$C([a,b],c) + C([b,c],a) + C([c,a],b) = 0,$$

i.e. it is a 2-cocycle of $\mathfrak{g}$. The action of G is called a *Poisson action* if $H_{[a,b]} = \{H_a, H_b\}$ always. In this case the map $p_x(a) = H_a(x)$ is a Lie-algebra homomorphism, and we can define the so-called *moment map*

$$P: M \longrightarrow \mathfrak{g}^*, \quad P(x) = p_x.$$

This map is G-equivariant, and so if H is a Hamiltonian function that is invariant under G, then P is a first integral of H. Note that G acts on $P^{-1}(0)$, and under some natural conditions the quotient $N = P^{-1}(0)/G$ is a smooth symplectic manifold. The point of this construction, called *symplectic reduction*, is that this N represents exactly the so called *stable points* of M under the action of G, which means that the G-invariant functions on M are in a 1-to-1 correspondance with the functions on N. The significance of this theorem, proved by Morse theory, is that it is not always clear how one can 'see' the G-invariant functions on a manifold. The Reader is encouraged to construct some pathological examples. The name of this topic is geometric invariant theory; for more on symplectic reduction see [A, Appendix 5], [MΦ], [Se]. We will see an application to Yang-Mills theory in Section 5.3.

In the Introduction 0.2 we encountered the problem of relating the number of fixed points of certain diffeomorphisms to the number of critical points of a Morse function on the same manifold.

Let $\mathrm{Diff}_0(M,\omega)$ be the identity component of the group of all symplectic diffeomorphisms of a symplectic manifold (M,ω). It is proven by *A. Banyaga* that the commutator subgroup of $\mathrm{Diff}_0(M,\omega)$ consists precisely of the symplectomorphisms generated by a time-dependent Hamiltonian flow. In [A, Appendix 9] there is an outline of the result that if such a symplectomorphism is close enough to the identity, then the statement about the number of its critical points is true. As a special case we can consider the measure-preserving diffeomorphisms of the torus $\mathbb{T}^2$. Being in $\mathrm{Diff}_0(M,\omega)$, on the covering space $\mathbb{R}^2$ it must be of the form

$$\psi: x \mapsto x + f(x), \quad x \in \mathbb{R}^2,$$

with a periodic function f. As claimed in [A], and proved in [CZ], being in the commutator group is equivalent to the condition that $[f] = \int_{\mathbb{T}^2} f(x)\,dx = 0$. So the above mentioned theorem applies for such torus maps. Arnold conjectured that the condition of closeness to identity can be dropped. For the standard symplectic manifolds $\mathbb{T}^{2n}$ it is proved by *Conley* and *Zehnder* in [CZ], and for more general symplectic manifolds by *A. Floer*, see [F], [HoZ], [Ho].

5. The Yang-Mills theory of connexions

This chapter is to briefly overview Morse theoretical methods in relating connexions in vector bundles to the topology of the bundles, from characteristic classes to Yang-Mills theory.

5.1 Characteristic classes and the Gauss-Bonnet-Chern theorem

It is a well-known sentence that 'curvature has strong topological consequences'. Of this we have already seen some examples: the Gauss-Bonnet theorem in the Introduction 0.2, some geometric group theory in Section 1.10, and some dynamical system approaches in Chapter 4. Inspired by these results, and by the ideas around Theorems 1.1 and 1.4, one might think it helpful to use connexions and curvature to define topological invariants of bundles. The beautiful and useful theory of characteristic classes of vector bundles is invented by Stiefel, Whitney, Pontryagin and Chern to fulfill this ambition. To give the main idea, we begin with the special case of unitary complex line bundles over a compact manifold M.

The Lie algebra of the structure group $U(1)$ can be identified with the imaginary numbers $i\mathbb{R}$, thus the curvature of a connexion A can be written as $F_A = -2\pi i\phi$ for a real 2-form ϕ, that is closed because of the Bianchi identity $d_A F_A = 0$. Taking another connexion $A' = A + a$, $a \in \Omega^1(M; i\mathbb{R})$, we have $F_{A'} = F_A + da$, so $[\phi'] = [\phi] \in H^2(M)$ in deRham cohomology. Thus we have a cohomology class that is independent of the chosen connexion, i.e. it is a characteristic of the line bundle itself. That is why it is called a *characteristic class*.

Now let us consider *invariant polynomials* in n^2 complex variables, written as functions on complex $n \times n$ matrices; the invariance of a polynomial p means $p(gAg^{-1}) = p(A)$ for all $A \in M_n(\mathbb{C})$, $g \in GL_n(\mathbb{C})$. We will restrict ourselves to homogeneous polynomials. The main examples come from the characteristic polynomial $\sigma(t) = \det(I + tA) = \sum_{i=0}^n \sigma_i(A)t^i$, and $s(t) = -t\frac{d}{dt}\log(\det(I - tA)) = \sum_{k=0}^\infty s_k(A)t^k$: each of the functions $\sigma_i(A)$ and $s_k(A)$ is an invariant polynomial. Clearly, $\sigma_1(A) = s_1(A) = \mathrm{Tr}(A)$. It is not difficult to see that $s_k(A) = \mathrm{Tr}(A^k)$ and that there exist polynomials $Q_k, P_k \in \mathbb{Q}[x_1, \ldots, x_k]$ such that $s_k(A) = Q_k(\sigma_1(A), \ldots, \sigma_k(A))$ and $\sigma_k(A) = P_k(s_1(A), \ldots, s_k(A))$. The effects of matrix operations are given by $\sigma_k(A_1 \oplus A_2) = \sum_{i=0}^k \sigma_i(A_1)\sigma_{k-i}(A_2)$, $s_k(A_1 \oplus A_2) = s_k(A_1) + s_k(A_2)$, $s_k(A_1 \otimes A_2) = s_k(A_1) \cdot s_k(A_2)$. For diagonal matrices $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, the polynomial $\sigma_k(D)$ becomes the kth elementary symmetric polynomial in the variables $\lambda_1, \ldots, \lambda_n$. Hence, the fundamental theorem of symmetric polynomials, together with the fact that diagonalizable matrices are dense in $M_n(\mathbb{C})$, give that every (homogeneous) invariant polynomial can be written as a (homogeneous) polynomial of the σ_i's. Thus, our examples of invariant polynomials are in fact the most important ones.

Finally, we define an almost invariant polynomial, the *Pfaffian* $\mathrm{Pf} : \mathfrak{so}(2n, \mathbb{R}) \longrightarrow \mathbb{R}$, by $\omega(A) \wedge \cdots \wedge \omega(A) = n!\,\mathrm{Pf}(A)\,\mathrm{Vol}$, where $\omega(A) = \sum_{i<j} A_{ij} e_i \wedge e_j \in \Lambda^2(\mathbb{R}^{2n})$, and $\mathrm{Vol} = e_1 \wedge \cdots \wedge e_{2n}$. For instance, for the block diagonal matrix $D = \mathrm{diag}(a_1 J_0, \ldots, a_n J_0)$, where J_0 is the standard 2-dimensional symplectic matrix, we have $\mathrm{Pf}(A) = a_1 \cdots a_n$. Furthermore, for $A \in \mathfrak{so}(2n, \mathbb{R})$ and $B \in M_n(\mathbb{C})$, we have $\mathrm{Pf}(A)^2 = \det(A)$ and $\mathrm{Pf}(BAB^T) = \mathrm{Pf}(A)\det(B)$. Taking the realification map $\mathfrak{su}_n \longrightarrow \mathfrak{so}_{2n}$, $A \mapsto A_\mathbb{R}$, we get $\mathrm{Pf}(A_\mathbb{R}) = (-i)^n \det(A)$.

Now we are going to evaluate these polynomials on the $\mathrm{End}(E)$-valued curvature forms F_A of a rank-m complex vector bundle $E \longrightarrow M$, where M is a compact smooth manifold, see Section 1.2. First of all, $\wedge$ is commutative on even-dimensional differential forms, so if p is a homogeneous invariant polynomial of degree k, and $A_{ij} \in \Omega^2(M; E)$, $1 \le i, j \le m$, then we can speak of $p(A) \in \Omega^{2k}(M; E)$. If we have a connexion in E, given locally by the 2-forms A_τ, then the invariance of p gives a globally defined $2k$-form $p(F_A)$. If E' is an isomorphic vector bundle, then using that isomorphism, we get a connexion A' in E' corresponding to A in E, and we have $p(F_A) = p(F_{A'})$, up to the isomorphism. From this the following basic facts can be deduced: $p(F_A)$ is always a closed form, and its de Rham cohomology class is independent of the chosen connexion A. In particular, using the flat connexion in trivial vector bundles, we see that $[p(E)] = 0$ if E is trivial. And we can define the topological invariants, the *kth Chern class* of any bundle E by

$$c_k(E) = \left[\sigma_k\left(\frac{-1}{2\pi i}F_A\right)\right] \in H^{2k}(M;\mathbb{C}),$$

and the *kth Chern character class* by

$$\mathrm{ch}_k(E) = \frac{1}{k!}\left[s_k\left(\frac{-1}{2\pi i}F_A\right)\right] \in H^{2k}(M;\mathbb{C}).$$

We have $c_0(E) = 1$, $\mathrm{ch}_0(E) = \dim E$, $c_1(E) = \mathrm{ch}_1(E)$, $\mathrm{ch}_2(E) = \frac{1}{2}c_1(E)^2 - c_2(E)$, for instance. The normalization in the definition suggests that we should have some nice numbers for certain nice bundles. Indeed, some computation in the Fubini-Study metric of $\mathbb{CP}^1$, the isomorphism $H^2(\mathbb{CP}^n;\mathbb{C}) \simeq H^2(\mathbb{CP}^1;\mathbb{C})$, and the integration isomorphism $I : H^2(\mathbb{CP}^n;\mathbb{C}) \longrightarrow \mathbb{C}$ give $I(c_1(H_n)) = -1$ for the tautological line bundle $H_n = \mathcal{O}_n(-1)$ of Section 1.1.

Actually, Chern classes can also be defined as the unique family of cohomology classes $c_k(E) \in H^{2k}(M;\mathbb{C})$ satisfying the three conditions

(i) $I(c_1(H_1)) = -1$, $c_k(H_n) = 0$ when $k > 1$, and $c_0(H_n) = 1$

(ii) $f^*c_k(E) = c_k(f^*(E))$

(iii) $c_k(E_1 \oplus E_2) = \sum_{i=0}^k c_i(E_1)c_{k-i}(E_2)$.

Note that all Chern classes lie in $\mathbb{R}$-cohomology. The uniqueness part of the theorem follows from the extremely useful *splitting principle*: for any complex vector bundle E on M there exists a manifold T and a proper smooth map $f : T \longrightarrow M$ such that $f^* : H^k(M) \longrightarrow H^k(T)$ is injective and $f^*(E) \simeq \gamma_1 \oplus \cdots \oplus \gamma_n$ for certain complex line bundles γ_i. There is a similar result for real vector bundles, too.

If E is a real rank-$2k$ oriented vector bundle over M, with inner product $\langle\cdot,\cdot\rangle$, then in orthogonal trivializations the curvature 2-forms F_{A_τ} of a metric connexion $A = \{A_\tau\}$ are skew-symmetric, the transition maps of the bundle are in $SO(2k,\mathbb{R})$, so, with the help of the Pfaffian, we can define a global real $2k$-form

$$e(E) = \left[\mathrm{Pf}\left(\frac{F_A}{2\pi}\right)\right] \in H^{2k}(M;\mathbb{R}),$$

the so-called *Euler class* of the bundle E. Also, if E is a complex rank-k Hermitian vector bundle, then $e(E) := e(E_\mathbb{R})$, and $e(E) = c_k(E)$. We have $e(E_1 \oplus E_2) = e(E_1)e(E_2)$ and $e(E^*) = -e(E)$. Note that almost every textbook has its own sign conventions and sign mistakes in the theory of characteristic classes. I did my best to be correct and precise, but full success cannot be guaranteed.

A main example is the tangent bundle of a 2-dimensional compact Riemannian manifold Σ embedded isometrically into $\mathbb{R}^3$ with the Levi-Cività connexion A. By the computation of F_A at the end of Section 1.2, we have $e(T\Sigma) = K\mathrm{Vol}_\Sigma/(2\pi)$. So the Gauss-Bonnet theorem can be reformulated as $I(e(T\Sigma)) = \chi(\Sigma)$.

Recall that once we have already defined the first Chern classes $C_1(E) \in \mathbb{Z}$ for complex vector bundles over closed oriented surfaces Σ in Section 1.1. These numbers behave well under the usual vector bundle operations, and we have $I(c_1(\mathcal{O}_n(k))) = k = C_1(\mathcal{O}_n(k))$, so the uniqueness of the Chern class gives $I(c_1(E)) = C_1(E)$ for all complex bundles E on Σ. Let us see another check of the compatibility of the two definitions. An almost complex structure on the real bundle $T\Sigma$ can be integrated to give an honest complex line bundle, for which $C_1(T\Sigma) = \chi(\Sigma)$ can be computed via Morse-Poincaré-Hopf index theory and Theorem 1.1. And this is the same as the above-mentioned form of the Gauss-Bonnet theorem. Unfortunately, I do not know a transparent explanation on the equality of the 'homotopical Chern classes'

$C_1(E) \in \pi_1(GL(n,\mathbb{C}))$ and the 'homological Chern classes' $c_1(E) \in H^2(\Sigma)$. Neither am I aware of a homotopical description of higher Chern classes.

Let E be a real rank-m vector bundle over the n-dimensional manifold M. The *Thom isomorphism theorem* claims the existence of an isomorphism $\Phi : H^q(M) \longrightarrow H_c^{m+q}(E)$ such that $U = \Phi(1) \in H^m(E)$, the so-called Thom class, has integral 1 over each fiber E_p. Note that the isomorphism of the cohomology groups follows also from the Poincaré duality theorem of the Introduction 0.2. Let us define $\hat{e}(E)$ by $\Phi(\hat{e}(E)) = U \wedge U$. Now a unicity theorem similar to the one about the Chern classes yields that for oriented bundles E we have $\hat{e}(E) = e(E)$. The significance of $\hat{e}(E)$ lies in the fact that for an arbitrary smooth section $s : M \longrightarrow E$ we have $\hat{e}(E) = s^*(U)$, which makes it possible to connect the local data at the zeros of a section to the global topological data of the Euler class. If the real rank of E and $\dim M$ coincide, we can define the indices $\iota(s,p) \in \{\pm 1\}$ for any section s that is transversal to the zero section s_0 at all its zeros. Now the main result is $I(\hat{e}(E)) = \sum_p \iota(s;p)$, which, together with the Poincaré-Hopf index theorem, gives the following generalization of the Gauss-Bonnet theorem for oriented compact manifolds:

$$\int_M e(TM) = \chi(M).$$

Let us finally examine the class $\mathrm{ch}_2(E)$ for complex bundles with structure group $SU(n)$. Since the Lie algebra $\mathfrak{su}_n$ consists of traceless matrices, we have $c_1(E) = 0$, and so $\mathrm{ch}_2(E) = -c_2(E)$. By definition, $\mathrm{ch}_2(E) = \left[\frac{1}{8\pi^2}\mathrm{Tr}(F_A \wedge F_A)\right] \in H^4(M;\mathbb{C})$, therefore we have

$$C_2(E) = \int_M c_2(E) = \frac{-1}{8\pi^2}\int_M \mathrm{Tr}(F_A \wedge F_A) \in \mathbb{Z}$$

for $SU(n)$-bundles.

One of the main applications of the theory is to (co-)bordism. For instance, $\mathbb{CP}^{2n}$ is not the boundary of any compact manifold N. For one computes $c_k(T\mathbb{CP}^n) = (-1)^k\binom{n+1}{k}c_1(H_n)^k$, from where the existence of a closed $2n$-form ω follows with $0 \neq \int_{\mathbb{CP}^n}\omega = \int_{\partial N}\omega = \int_N d\omega = 0$, a contradiction.

Another application is to enumerative algebraic geometry, such as Bezout's theorem on the number of intersection points of two algebraic curves.

For more on characteristic classes see [MS] first of all, and also [DFN III], [KN II], [MT], [BT].

5.2. The YM functional and its critical points

Given a Riemannian manifold M, which of the metric connexions has the least curvature? And which metric has a metric connexion with the global minimum of the curvature? If we want to measure the total average curvature, then the Gauss-Bonnet-Chern theorem says that all metrics and all connexions are the same: the integral is a topological invariant. That is, a real question is to identify the manifolds which can carry a flat Riemannian metric. For 2-manifolds with $\chi(M) = 0$ the only possibility is the torus. For flat Riemannian manifolds of arbitrary dimension see [KN I, Note 5]. In more generality, the above questions can be asked about complex vector bundles $E \longrightarrow M$ with Hermitian inner products. For this problem we will introduce the Yang-Mills functional, which is to measure the economicity of connexions. For more see [Se], [AB2], [DFN I-II], [MΦ], [N], [DK].

The *Yang-Mills functional* is defined on the set of unitary connexions $\mathcal{A}(E)$ as

$$\mathrm{YM}(A) = \frac{-1}{8\pi^2}\int_M \mathrm{Tr}(F_A \wedge *F_A),$$

where $*$ is the Hodge star operator, and we used the invariant inner product $\langle A, B\rangle = -\mathrm{Tr}(AB)$ on the compact Lie algebra of skew-hermitian matrices. (That is, we could consider a bundle with an arbitrary compact structure group G, and Lie algebra $\mathfrak{g}$.) Being the connexions unitary, $\mathrm{YM}(A) \in \mathbb{R}_{\geq 0}$, and it is clear that $\mathrm{YM}(A) = 0$ if and only if A is a flat connexion. Let $\mathcal{G}$ be the group of unitary bundle automorphisms of E — this is the *gauge group*, which acts on $\mathcal{A}$ by $A_\tau \mapsto \mathrm{Ad}(g^{-1})A_\tau - dg \cdot g$, and so $F_A \mapsto \mathrm{Ad}(g^{-1})F_A$, which means that the YM functional is $\mathcal{G}$-invariant. In Theorem 1.3 we identified $\mathcal{A}(E)$ with the space $\mathcal{C}(E)$ of all holomorphic structures, on which the *complex gauge group* $\mathcal{G}_\mathbb{C}$ of all bundle isomorphisms acts. So $\mathcal{A}/\mathcal{G}_\mathbb{C}$ is the space of isomorphism classes of holomorphic structures on E. Note that the definition of YM does not contain the Hermitian i.p. of E, only the Riemannian metric of M.

The Euler-Lagrange variation of the YM functional gives that the critical points of the functional are the *connexions of constant curvature*: $d_A * F_A = 0$. They are characterized by the property that $d_A\xi = 0$ for $\xi \in \Omega^0(M; \mathrm{End}(E))$ is equivalent to $T_\gamma \xi(x_1) = \xi(x_2)T_\gamma$, where T_γ is the parallel transport along the curve γ from x_1 to x_2. In a bundle with a constant curvature connexion A, there is a splitting of E into subbundles $E^{(\lambda)}$ according to the eigenvalues λ of the operator $*F_A$. In each such subbundle the curvature $*F_A$ acts as the scalar λ, so the first Chern class can easily be computed, and

$$\lambda = \frac{-2\pi i}{\mathrm{Vol}(M)} \frac{C_1(E^{(\lambda)})}{\dim(E^{(\lambda)})},$$

where the ratio $C_1(E^{(\lambda)})/\dim(E^{(\lambda)})$ is called the *slope* of $E^{(\lambda)}$.

Every connexion satisfies the Bianchi identity $d_A F_A = 0$, hence the so-called self-dual and anti-self-dual connexions $*F_A = \pm F_A$ satisfy the equation $d_A * F_A = 0$, too. So if they have finite energy $\mathrm{YM}(A) < \infty$, then they are critical points of the functional, and are called the instanton and anti-instanton solutions of the YM equation. Note that any critical point of the functional has $\mathrm{YM}(A) \geq |C_2(E)|$, with equality for the instanton and anti-instanton solutions. So they are the absolute minima of the YM functional (if they exist). Are there other critical points?

Theorem 5.1. *Let L be a holomorphic line bundle over the closed Riemann surface Σ with a given Riemannian metric. Then there is a Hermitian inner product h in L, unique up to a constant multiple, whose curvature (i.e. the curvature of the unique connexion compatible with both the holomorphic structure and the inner product) is constant. This inner product minimizes the YM functional on unitary connexions compatible with the holomorphic structure of L.* $\square$

Note that the constant curvature connexions are originally the critical points of the YM on the set $\mathcal{A}(L, h)$, with fixed Hermitian i.p. h, without a holomorphic structure, and now they have been found to be the minima of the 'orthogonal' sets.

Corollary 5.2. *If a holomorphic line bundle L on Σ has a non-zero holomorphic section, then $c_1(L) \geq 0$.* $\square$

5.3. Holomorphic line bundles and the Narasimhan-Seshadri theorem

Theorem 5.3. (Birkhoff - Grothendieck) *The holomorphic line bundles over a complex manifold M form a group under $\otimes$, which is isomorphic to $H^2(M; \mathbb{Z})$, and therefore parametrized by the first Chern class of the bundle. Every holomorphic vector bundle over $\mathbb{CP}^1$ is a direct sum of holomorphic line bundles, where the decomposition is basically unique.*

It would be nice to prove Theorem 5.1 for arbitrary holomorphic vector bundles, but this generalization simply would not be true, only for 'almost all' vector bundles. As we saw in the previous section, if there is a constant curvature connexion, then we can decompose the bundle into holomorphic subbundles, so we can restrict ourselves to indecomposable bundles.

A holomorphic bundle E is called *stable* if its every holomorphic subbundle has strictly smaller slope than E. Such a bundle is of course indecomposable.

Theorem 5.4. (Narasimhan - Seshadri) *A holomorphic bundle E over Σ is stable iff it is indecomposable and has an i.p. whose curvature is constant and scalar. In particular, a holomorphic $SU(n)$-bundle is stable iff it has a flat connexion.* $\square$

Let us finally see the connection to the symplectic invariant theory of Section 4.4. Let Σ be a Riemann surface, and E the trivial complex rank-2 vector bundle over it. Then the space $\mathcal{A}(E)$ is $\Omega^1(\Sigma; \mathfrak{su}_2)$, on which the bilinear YM functional $\mathrm{YM}(\alpha, \beta)$ is a symplectic form. The gauge group $\mathcal{G} = \{g : \Sigma \longrightarrow SU(2)\}$ leaves this YM form invariant, and the corresponding moment map is the curvature $F : \Omega^1(\Sigma; \mathfrak{su}_2) \longrightarrow \Omega^2(\Sigma; \mathfrak{su}_2)$. The factor space $F^{-1}(0)/\mathcal{G}$ is the set of isomorphism classes of flat connexions, and symplectic invariant theory says that this is the space of stable points. Thus we have arrived to the special case of Theorem 5.4.

5.4. Some physics

In electrodynamics, the pseudo-Riemannian base manifold M is the Minkowski space-time $\mathbb{R}^3 \times \mathbb{R}$, a connexion A in TM is the potential, the curvature F_A is the intensity of the electromagnetic field. The Bianchi identity and the Yang-Mills equation are the Maxwell equations.

In the Standard Model of quantum physics, we have a complex vector bundle over the space-time M with structural group $G = SU(3) \times SU(2) \times U(1)$. The sections of this vector bundle are the fermions, the connexions in the bundle are the bosons, and the curvature of a connexion is the Yang-Mills field intensity.

In the past few years there has been a revolution in 4-dimensional (symplectic) topology, mainly dew to Seiberg and Witten who showed the connection between the physics of Yang-Mills fields and geometry. Recently it turned out the theory of the Donaldson-, the Gromov-, and the Seiberg-Witten-invariants are equivalent, but there is still a lot to understand.

A second trail towards physics is mirror symmetry for Calabi-Yau manifolds, which comes from string theory.

For more see [MΦ], [N], [DK], [T], and [As].

6. Discrete Morse theory

As it has been shown by the recent work of *Robin Forman*, almost all of the Morse theory we have seen in this essay has a discrete counterpart for simplical and cell complexes. This discrete theory seems to have the same geometric depth, while some results are considerably easier to prove. Moreover, it also has some interesting corollaries to combinatorial problems. The basic paper by Forman is [Fo1]; for a nice survey see [Fo2].

Let K be the set of non-empty simplices of a finite simplicial complex M^n. We write $\alpha^{(p)} \in K^{(p)}$ for p-dimensional simplices, and $\alpha > \beta$ for the face relation. A function $f : K \longrightarrow \mathbb{R}$ is a *discrete Morse function* iff for every $\alpha^{(p)} \in K$

(i) $|\{\beta^{(p+1)} > \alpha : f(\beta) \leq f(\alpha)\}| \leq 1$, and
(ii) $|\{\gamma^{(p-1)} < \alpha : f(\gamma) \geq f(\alpha)\}| \leq 1$.

It is easy to see that for every simplex $\alpha^{(p)}$ at least one of the sets in (i) or (ii) is empty. A simplex $\alpha^{(p)}$ is *critical with index p* iff both sets are empty. Any discrete Morse function f defines a disjoint collection V of pairs of simplices:

$$V_f = \{(\alpha^{(p)}, \beta^{(p+1)}) : \alpha < \beta,\ f(\beta) \leq f(\alpha)\}.$$

This matching is called the *discrete gradient vector field of f*, and a simplex is critical iff it is not contained in any of the pairs of V. In general we call a matching of 'neighbouring simplices' a discrete vector field.

If V is a discrete vector field, then a V-path of dimension p is a sequence of simplices $\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \dots, \beta_{k-1}^{(p+1)}, \alpha_k^{(p)}$ such that $(\alpha_i, \beta_i) \in V$ for all $i = 0, 1, \dots, k$, and $\beta_i > \alpha_{i+1} \neq \alpha_i$. We say that a V-path is closed if $\alpha_k = \alpha_0$. The analogue of Smale's gradient like vector fields is much simpler here: a vector field V is the gradient vector field of a discrete Morse function iff there are no closed V-paths. Moreover, the discrete version of the cancellation theorem used in the h-cobordism Theorem 0.6 is rather obvious: if the simplices $\alpha^{(p)}$ and $\beta^{(p+1)}$ are critical for the Morse function f, and there is exactly one gradient path from $\partial\beta$ to α, then there is another Morse function g with the same critical simplices except that α and β are no longer critical. Moreover, the gradient vector field of g coincides with the gradient field of f except along the unique gradient path from $\partial\beta$ to α.

The basic results of Morse theory, Theorems 0.2 and 0.3 are also valid, with simplicial collapsing in place of the homotopy and diffeomorphic equivalences. Even the Smale-Witten-Floer-Schwarz-type representation of the cohomology of M by the Morse complex is not very difficult to prove: the main help that we did not have in the smooth category is the existence of the map Φ^∞, which is something like the limit of the gradient flow Φ at time $t = \infty$; these things will be defined now.

We can think of the gradient vector field V as a map from p-simplices to $(p+1)$-simplices. If we set the sign of $V(\alpha^{(p)})$ such that $\langle \alpha, \partial V(\alpha) \rangle = -1$ with the canonical inner product on oriented chains, then we have a linear extension $V : C_p(M; \mathbb{Z}) \longrightarrow C_{p+1}(M; \mathbb{Z})$. Now define the discrete-time flow $\Phi : C_p(M; \mathbb{Z}) \longrightarrow C_p(M; \mathbb{Z})$ by

$$\Phi = 1 + \partial V + V\partial.$$

This Φ is worth calling the flow of V, since in the smooth category Cartan's formula says $di(V) + i(V)d = \mathcal{L}_V$, and $\mathcal{L}_V \sim \Phi_V - 1$ approximately. If $\alpha^{(p)}$ is not a critical simplex, then $\Phi(\alpha)$ is a p-chain consisting entirely of oriented p-simplices on which f is less than $f(\alpha)$, so, loosely speaking, Φ decreases f — just like the gradient flow in the smooth case. But it is an important difference that the iterates of Φ stabilize in finite time: there is a positive integer N such that

$$\Phi^N = \Phi^{N+1} = \cdots = \Phi^\infty.$$

Let $C_p^\Phi \subseteq C_p(M;\mathbb{Z})$ be the set of Φ-invariant p-chains. As $\Phi\partial = \partial\Phi$, we have a chain complex

$$\mathcal{C}^\Phi :\ 0 \longrightarrow C_n^\Phi \xrightarrow{\partial} C_{n-1}^\Phi \xrightarrow{\partial} \cdots .$$

The main result is that $\Phi^\infty : C_*(M;\mathbb{Z}) \longrightarrow \mathcal{C}_*^\Phi$ induces an isomorphism on homology. The chain complex $\mathcal{C}_*^\Phi$ is called the *Morse complex*, since Φ^∞ establishes a 1-to-1 correspondence between $\mathcal{M}_p$ and C_p^Φ for all p, where $\mathcal{M}_p$ is the set of critical indices of dimension p. Thus we have the Morse complex

$$\mathcal{M} :\ 0 \longrightarrow \mathcal{M}_n \xrightarrow{\tilde\partial} \mathcal{M}_{n-1} \xrightarrow{\tilde\partial} \ldots ,$$

with $\tilde\partial = (\Phi^\infty)^{-1}\partial\Phi^\infty$, and

$$H_*(\mathcal{M}) = H_*(M;\mathbb{Z}).$$

Just as in Witten's theory, the boundary operator can be given directly: for any critical $\beta^{(p+1)}$,

$$\tilde\partial\beta = \sum_{\text{critical } \alpha^{(p)}} n(\alpha,\beta)\alpha,$$

where

$$n(\alpha,\beta) = \sum_{\tilde\alpha^{(p)}} \langle\partial\beta,\tilde\alpha\rangle \sum_{\gamma\in\Gamma(\tilde\alpha,\alpha)} m(\gamma),$$

where $\Gamma(\tilde\alpha,\alpha)$ is the set of gradient paths from $\tilde\alpha$ to α, and $m(\gamma) \in \{\pm 1\}$, according to an orientation matter just as in Witten's theory.

Finally, we have to note that inspite of the relative simplicity of discrete Morse theory, it implies the smooth theory.

In [Fo3] there is an application to evasive graph properties. Consider a game with two players, called the hider and the seeker. Let M be a subcomplex of the simplex Δ^n, known to both of the players, and the hider choses a simplex σ of Δ^n. The seeker can ask questions of the form 'is the vertex x_i in σ?', and wants to find out whether σ is in M, with the least possible number of questions. For any decision tree algorithm A, we denote by $Q(\sigma, A, M)$ the number of questions asked by the seeker before he/she reaches his/her goal. The *complexity of* M is defined by

$$c(M) = \inf_A \sup_\sigma Q(\sigma, A, M).$$

M is called *evasive* if $c(M) = n+1$, the worst possible value. It is known e.g. that if M is nonevasive then it is collapsible, and so contractible. If M is evasive, then for any A there exist simplices σ with $Q(\sigma, A, M) = n+1$; they are called the *evaders of* A. The main result of [Fo3] is that for any decision tree algorithm A, the number of evaders is $e(A) \geq b(M)$, the sum of the Betti numbers of the manifold.

A monotone graph property M on n vertices is a subcomplex of $\Delta^{\binom{n}{2}}$, such that the automorphism group of M acts transitively on the vertices of M. There is a conjecture by *Rivest*, *Vuillemin* and *Karp* that every such subcomplex M of a simplex Δ^m is evasive. It is known to be true for m a prime power. It would be nice to prove the conjecture using a Morse theory with group actions.

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