CEU *Probability 2* problems

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▷ **Exercise 1 (The Fourier expansion of Brownian motion).** Let Z_n be iid standard normal variables, $n = 0, 1, \dots$, and

$$B(t) := \frac{t}{\sqrt{\pi}} \cdot Z_0 + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{\sin(mt)}{m} \cdot Z_m.$$

Prove that:

(a) For any $t \geq 0$ fixed, $B(t)$ is almost surely finite.

(b) Almost surely, $B(t)$ is finite for all $t \geq 0$.

(c) $\mathrm{Cov}(B(s), B(t)) = \min\{s, t\}$.

(d)* Can you show that $B(t)$ is a.s. continuous?

Hints:

- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.
- Taking the Fourier transform of the right hand side below, show and then use:

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{3x^2 - 6\pi x + 2\pi^2}{12}, \qquad 0 \leq x \leq 2\pi.$$

▷ **Exercise 2.** For any sequence $X_1, X_2 \dots$ of random variables, their **tail σ-field** is $\bigcap_{n=1}^{\infty} \sigma\{X_n, X_{n+1}, \dots\}$. On the other hand, the **exchangeable σ-field** consists of events that are invariant under finitely supported permutations of the variables. **Kolmogorov's 0-1 law** says that for independent variables, any event in their tail field has probability 0 or 1, while the **Hewitt-Savage 0-1 law** says that exchangeable events for independent variables have probability 0 or 1. Show that the tail field is contained in the exchangeable field, but not vice versa, hence HS01 is stronger than K01.

▷ **Exercise 3.** Let $\{B_t : t \geq 0\}$ be standard 1-dimensional Brownian motion, and $f : [0, \infty) \longrightarrow \mathbb{R}$ be any fixed continuous function.

(a) Prove that for any $\epsilon > 0$, we have $\mathbf{P}\big[\, |B_t - f(t)| < \epsilon \text{ for all } t \in [0,1] \,\big] > 0$.

(b) Prove that for any $K > 0$, we have $\mathbf{P}\big[\, |B_t - f(t)| < K \text{ for all } t \in [0,\infty) \,\big] = 0$.

▷ **Exercise 4.** Let $\{W_t : t \in [0,1]\}$ be standard 1-dimensional Brownian motion, and consider $B_t := W_t - tW_1$ for $t \in [0,1]$. It is called the **standard Brownian bridge**.

(a) Show that this is a Gaussian process with continuous paths, with $\mathrm{Cov}(B_s, B_t) = s(1-t)$ for $0 \leq s \leq t \leq 1$.

(b) Deduce that $\{B_t : t \in [0,1]\}$ and $\{B_{1-t} : t \in [0,1]\}$ have the same distribution.

(c)* For $\epsilon > 0$, let $\{W_t^\epsilon : t \in [0,1]\}$ be the process W_t conditioned on the event that $\{W_1 \in (-\epsilon, \epsilon)\}$. Show that the weak limit of $\{W_t^\epsilon : t \in [0,1]\}$ as $\epsilon \to 0$ is the Brownian bridge $\{B_t : t \in [0,1]\}$.

▷ **Exercise 5.** Let T be the Galton-Watson tree with offspring distribution $\xi \sim \mathsf{Geom}(1/2)$. Draw the tree into the plane with root ρ, add an extra vertex ρ' and an edge (ρ, ρ'), and walk around the tree, starting from ρ', going through each "corner" of the tree once, through each edge twice (once on each side). At each corner visited, consider the graph distance from ρ': let this be process be $\{X_t\}_{t=0}^{2n}$, which is positive everywhere except at $t = 0, 2n$, where n is the number of vertices of the original tree T.

Figure 1: The contour walk around a tree.

- **(a)** Using the memoryless property of $\mathsf{Geom}(1/2)$, show that $\{X_t\}$ is SRW on $\mathbb{Z}$.
- **(b)** Using martingale techniques, show that $\mathbf{P}[T \text{ has height } \geq n] = 1/n$.
- **(c)** Show that, conditioning T to have height at least n, with high probability the height will be around n and the total volume will be around n^2, where "around" means "up to constant factors".
- **(d)*** Any ideas how one could use 1-dimensional Brownian motion to define a "continuum random tree"?

▷ **Exercise 6.**

- **(a)** Show that $\dim_M \left(\{\frac{1}{n} : n = 1, 2, \dots\}\right) = 1/2$, where $\dim_M$ denotes Minkowski dimension.
- **(b)** Show that Hausdorff dimension has the countable stability property: $\dim_H \bigcup_i E_i = \sup_i \dim_H E_i$.

A bit of a diversion, but I cannot help myself. For the study of random walks and percolation on general locally finite rooted trees T, Russ Lyons (1990) defined an "average **branching number**"

$$\mathsf{br}(T) := \sup \left\{ \lambda \geq 1 : \inf_{\Pi} \sum_{e \in \Pi} \lambda^{-|e|} > 0 \right\}, \tag{1}$$

where the infimum is taken over all cutsets $\Pi \subset E(T)$ separating the root $o \in V(T)$ from infinity, and $|e|$ denotes the distance of the edge e from o.

▷ **Exercise 7.** Let T be a locally finite infinite tree with root o.

- **(a)** Show that $\mathsf{br}(T)$ does not depend on the choice of the root o.
- **(b)** Show that the $d+1$-regular tree has $\mathsf{br}(\mathbb{T}_{d+1}) = d$.
- **(c)** Define the lower growth rate of T by $\underline{\mathsf{gr}}(T) := \liminf_n |T_n|^{1/n}$, where T_n is the set of vertices at distance exactly n from o. Show that $\mathsf{br}(T) \leq \underline{\mathsf{gr}}(T)$.

A clear motivation for definition (1) is given by the following interpretation. Let us denote the set of non-backtracking infinite rays starting from o by ∂T, the boundary of the tree, equipped with the metric $d(\xi, \eta) := e^{-|\xi \wedge \eta|}$, where $\xi \wedge \eta$ is the last common vertex of the two rays, and $|\xi \wedge \eta|$ is its distance from o. Then, basically by definition,

$$e^{\dim_H(\partial T, d)} = \mathsf{br}(T) \qquad \text{and} \qquad e^{\underline{\dim}_M(\partial T, d)} = \underline{\mathsf{gr}}(T)\,.$$

Since Hausdorff dimension has, over the past hundred years, proved a better notion than Minkowski dimension, the branching number ought to be a better way of measuring average branching than growth.

▷ **Exercise 8.** Find the branching number of the following two trees (see Figure 2):

(a) The quasi-transitive tree with degree 3 and degree 2 vertices alternating.

(b) The so-called 3-1-tree, which has 2^n vertices on each level n, with the left 2^{n-1} vertices each having one child, the right 2^{n-1} vertices each having three children; the root has two children.

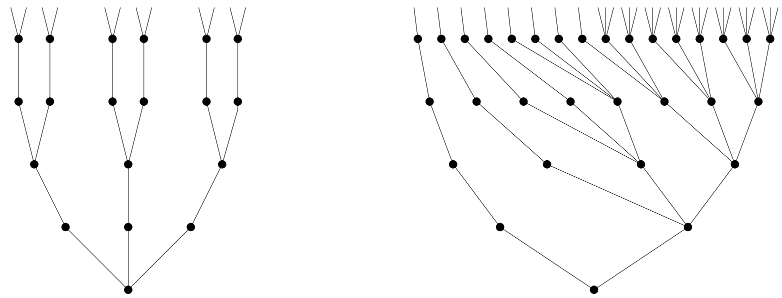

Figure 2: A quasi-transitive tree and the 3-1 tree.

▷ **Exercise 9.** If you know at this point what it means, prove that the 3-1 tree above is recurrent for simple random walk.

▷ **Exercise 10.**

(a) Let X and Y be independent standard normals. Show that X/Y has Cauchy distribution.

(b) Prove that the harmonic measure on the line $x = 1$ for 2-dim BM started at the origin is given by the Cauchy distribution.

▷ **Exercise 11.** Let X_i be iid variables with distribution $\mathbf{P}[\,X_i > t\,] = \mathbf{P}[\,X_i < -t\,] = t^{-2}/2$ for all $t \geq 1$. Find deterministic scaling factors a_n and a non-degenerate distribution Y such that $(X_1 + \cdots + X_n)/a_n \to Y$ in distribution.

▷ **Exercise 12.** The Hungarian Media Police has observed five possible TV-watching behaviours that people may have: (1) never watches the TV; (2) watches only state channels; (3) regularly watches the TV; (4) TV-addict; (5) brain-dead. The transitions between these states may be modelled by a Markov chain, with the following transition matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0.3 & 0 & 0.3 & 0.1 & 0.3 \\ 0 & 0 & 0.4 & 0.4 & 0.2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular, nobody *becomes* a state channel fan — one has to be born like that.

(a) If one starts as a state channel fan, what is the probability that they end up brain-dead?

(b) What is the expected time for a state channel fan to reach a terminal state: to quit TV completely, or to become brain-dead?

▷ **Exercise 13.** Let X be a $\mathrm{Poi}(\lambda)$ variable, $p \in (0, 1)$, and given X, let Y be $\mathrm{Binom}(X, p)$, while $Z = X - Y$. By noticing that the two-variable moment generating function $\phi_{Y,Z}(t, s) := \mathbf{E}[\,e^{tY+sZ}\,]$ decomposes as a product, show that Y and Z are *independent* $\mathrm{Poi}(\lambda p)$ and $\mathrm{Poi}(\lambda(1-p))$ variables, respectively.

▷ **Exercise 14.** Give symmetric weights $w(i, i+1)$ for $i = 0, 1, 2, \ldots$ such that the resulting continuous time random walk on $\mathbb{N}$, started from any vertex, almost surely reaches infinity in finite time.

▷ **Exercise 15.** Prove that any finite state Markov chain has a recurrent state. (Hint: consider the smallest subset U with the property that starting the chain from anywhere inside U will not take you out of U. And note that you are not supposed to use Banach-Alaoglu here: the idea is to give an elementary proof.)

▷ **Exercise 16.** Prove that in any irreducible and aperiodic Markov chain $P = \big(p(x,y)\big)_{x,y\in V}$ on a finite state space V, there is some n such that $p_n(x,y) > 0$ for all $x, y \in V$.

Recall the notation

$$d(t) := \sup_{x\in V} d_{\mathrm{TV}}\big(p_t(x,\cdot), \pi(\cdot)\big) \quad \text{and} \quad \bar{d}(t) := \sup_{x,y\in V} d_{\mathrm{TV}}\big(p_t(x,\cdot), p_t(y,\cdot)\big)\,,$$

and $\tau_{\mathrm{mix}}^{\mathrm{TV}} := \inf\{t : d(t) < 1/4\}$.

▷ **Exercise 17.**

(a) Show that $d(t) \le \bar{d}(t) \le 2d(t)$.

(b) Using the coupling definition of TV-distance, show that $\bar{d}(t+s) \le \bar{d}(t)\,\bar{d}(s)$.

(c) Using the previous items, show that $d(\ell\tau_{\mathrm{mix}}^{\mathrm{TV}}) \le 2^{-\ell}$, hence $\tau_{\mathrm{mix}}^{\mathrm{TV}}$ indeed captures closeness to stationarity.

▷ **Exercise 18.** Consider simple random walk on the dumbbell graph: take two copies of the complete graph K_n, add a loop at each vertex (so that the degrees become n), except at one distinguished vertex in each copy, which will be connected to each other by an edge. Show that $d(1) = 1/2$, but $\tau_{\mathrm{mix}}^{\mathrm{TV}} \asymp n^2$. That is, in the definition of $\tau_{\mathrm{mix}}^{\mathrm{TV}}$, the 1/4 cannot be replaced by 1/2.

▷ **Exercise 19.** Consider lazy SRW on the cycle C_n. Show that for any $t > 0$ there exists $\delta_0(t), \delta_1(t) > 0$, with $\lim_{t\to 0} \delta_0(t) = 1$, such that, for any n, we have $\delta_0(t) < d(tn^2) < 1 - \delta_1(t)$. Conclude that there is no cutoff here in total variation. (It is also true that $\lim_{t\to\infty} \delta_1(t) = 1$ can be achieved, but this is not part of the exercise now.)

▷ **Exercise 20.** Show that $\lim_{\epsilon\to 0} d_{\mathrm{TV}}\big(\mathsf{N}(0,1), \mathsf{N}(\epsilon,1)\big) = 0$, where $\mathsf{N}(\mu,\sigma^2)$ is the normal distribution. Using this and the local version of the de Moivre–Laplace theorem, prove that $d_{\mathrm{TV}}\big(\mathsf{Binom}(n,1/2), \mathsf{Binom}(n,1/2)+n^{\beta}\big) \to 0$ for any fixed $\beta < 1/2$.

▷ **Exercise 21.** Let $M_0, M_1, M_2, \ldots$ be a martingale, and let $X_i = M_i - M_{i-1}$ be its difference sequence. Show that $\mathbf{E}[\,X_{i_1}\cdots X_{i_k}\,] = 0$ for any $k \ge 1$ and $i_1 < \cdots < i_k$. Hence the Azuma-Hoeffding inequality (from the class of March 8) can be used for MG differences.

▷ **Exercise 22.** Consider a reversible Markov chain P on a finite state space V with stationary distribution π and absolute spectral gap g_{abs}. This exercise explains why $\tau_{\mathrm{relax}} = 1/g_{\mathrm{abs}}$ is called the relaxation time.

(a) For $f : V \longrightarrow \mathbb{R}$, let $\mathrm{Var}_\pi[f] := \mathbf{E}_\pi[f^2] - (\mathbf{E}_\pi f)^2 = \sum_x f(x)^2\pi(x) - \big(\sum_x f(x)\pi(x)\big)^2$. Show that $g_{\mathrm{abs}} > 0$ implies that $\lim_{t\to\infty} P^t f(x) = \mathbf{E}_\pi f$ for all $x \in V$. Moreover,

$$\mathrm{Var}_\pi[P^t f] \le (1 - g_{\mathrm{abs}})^{2t}\,\mathrm{Var}_\pi[f]\,,$$

with equality at the eigenfunction corresponding to the λ_i giving $g_{\mathrm{abs}} = 1 - |\lambda_i|$. Hence τ_{relax} is the time needed to reduce the standard deviation of any function to $1/e$ of its original standard deviation.

(b) Using part (a), prove that there is a universal constant $C < \infty$ such that $\tau_{\mathrm{relax}} < C\,\tau_{\mathrm{mix}}^{\mathrm{TV}}$.

▷ **Exercise 23.** Consider the first digits of $1, 2, 4, \ldots, 2^n, \ldots$, in base 10. Do we ever see 7? And 8? Which is more frequent? (Hint: $\log_{10} 2$ is irrational.)