Applications of Stochastics — Exercise sheet 1

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Notation. The probability measure for the Erdős-Rényi random graph $G(n,p)$ is denoted by $\mathbf{P}_p$. For a random walk $\{X_t\}_{t\geq 0}$ on $\mathbb{Z}$, when started at $X_0 = \ell$, the probability measure and the corresponding expectation are denoted by $\mathbf{P}_\ell$ and $\mathbf{E}_\ell$.

For an increasing event $A \subset \{0,1\}^{\binom{n}{2}}$ for the Erdős-Rényi random graph $G(n,p)$, the critical (threshold) density will be denoted by $p_c(n) = p_c^A(n) := \min\{p : \mathbf{P}_p[\,A\,] \geq 1/2\}$.

The comparisons $\sim$, $\asymp$, $\ll$, $\gg$ are used as agreed in class.

"With high probability", abbreviated as "w.h.p.", means "with probability tending to 1".

▷ **Exercise 1.** Prove the easy direction of Strassen's theorem. Namely, let $(P, \leq)$ be a partially ordered set, $\mathcal{B}$ a sigma-algebra on P, and π a probability measure on $P \times P$ with the product sigma-algebra, with the property that $\pi\big(\{(x,y) \in P \times P : x \leq y\}\big) = 1$. Let the first marginal of π be $\mu(A) := \pi(A \times P)$ and the second marginal be $\nu(A) := \pi(P \times A)$ for any $A \in \mathcal{B}$. Then, ν stochastically dominates μ; i.e., for any increasing set $A \in \mathcal{B}$, we have $\mu(A) \leq \nu(A)$.

▷ **Exercise 2.** Find the order of magnitude of the critical density $p_c(n)$ for the random graph $G(n,p)$ containing a copy of the cycle C_4. (Hint: as in class, use the 1st and 2nd Moment Methods.)

▷ **Exercise 3.*** Let H be the following graph with 5 vertices and 7 edges: a complete graph K_4 with an extra edge from one of the four vertices to a fifth vertex. Find the order of magnitude of $p_c(n)$ for the random graph $G(n,p)$ containing a copy of this H. (Hint: the 1st Moment Method will give you $n^{-5/7}$, but the 2nd Moment Method now does not work! What goes wrong? What could be the right order of magnitude instead of $n^{-5/7}$?)

The critical density for the connectedness of $G(n,p)$ is $p_c(n) = (1+o(1))\frac{\ln n}{n}$, with a sharp threshold. The following exercise is not a proof of this, just a small indication for the value.

▷ **Exercise 4.** For $p = \frac{\lambda \ln n}{n}$, with $\lambda > 1$ fixed, show that, with probability tending to 1, there are no isolated vertices in $G(n,p)$. On the other hand, for $\lambda < 1$ fixed, there exist isolated vertices w.h.p.

▷ **Exercise 5.** Consider a Galton-Watson process with offspring distribution ξ, $\mathbf{E}\xi = \mu$. Let Z_n be the size of the nth level, with $Z_0 = 1$, the root. Recall that Z_n/μ^n is a martingale.

- **(a)** Assuming that $\mu > 1$ and $\mathbf{E}[\,\xi^2\,] < \infty$, first show that $\mathbf{E}\big[\,Z_n^2\,\big] \leq C(\mathbf{E}Z_n)^2$. (Hint: use the conditional variance formula $\mathbf{D}^2[Z_n] = \mathbf{E}\big[\,\mathbf{D}^2[Z_n \mid Z_{n-1}]\,\big] + \mathbf{D}^2\big[\mathbf{E}[\,Z_n \mid Z_{n-1}\,]\big]$.) Then, using the Second Moment Method, deduce that the GW process survives with positive probability.
- **(b)** Extend the above to the case $\mathbf{E}\xi = \infty$ or $\mathbf{D}\xi = \infty$ by a truncation $\xi \mathbf{1}_{\xi<K}$ for K large enough.

▷ **Exercise 6.** For the GW tree with offspring distribution Poisson$(1+\epsilon)$, show that the survival probability is asymptotically 2ϵ, as $\epsilon \to 0$.

The following exercise is already preparation for the next class, critical Galton-Watson trees.

▷ **Exercise 7.** Let T be the Galton-Watson tree with offspring distribution $\xi \sim \mathsf{Geom}(1/2) - 1$. Draw the tree into the plane with root ρ, add an extra vertex ρ' and an edge (ρ, ρ'), and walk around the tree, starting from ρ', going through each "corner" of the tree once, through each edge twice (once on each side). At each corner visited, consider the graph distance from ρ': let this be process be $\{X_t\}_{t=0}^{2n}$, which is positive everywhere except at $t = 0, 2n$, where n is the number of vertices of the original tree T.

Figure 1: The contour walk around a tree.

(a) Using the memoryless property of $\mathsf{Geom}(1/2)$, show that $\{X_t\}$ is a Simple Random Walk on $\mathbb{Z}$.

(b) Using that X_t is a bounded martingale, and that $\tau := \tau_0 \wedge \tau_n$ is almost surely finite (the minimum of the hitting times of 0 and n), show that $\mathbf{P}[\, T \text{ has height} \geq n\,] = 1/n$. Note that this also implies that $\mathbf{P}\big[\, T \text{ has height} \geq 100n \;\big|\; T \text{ has height} \geq n\,\big]$ is quite small.

(c) Show that $M_t := X_t^2 - t$ is a martingale. It is not bounded from below, but $M_{t\wedge\tau}$ is unlikely to get very small: show that there exists $c_n > 0$ such that $\mathbf{P}[\, \tau > t\,] < \exp(-c_n t)$.

(d) A version of the Optional Stopping Theorem says that the exponential decay for τ in the previous item implies that $\mathbf{E}M_\tau = \mathbf{E}M_0$. Use this to calculate $\mathbf{E}_\ell[\, \tau\,]$, for the walk started at $X_0 = \ell \in \{0, 1 \dots, n\}$.

(e) * Using the previous part, show that $\mathbf{E}_0[\, \tau \mid \tau_n < \tau_0\,] \asymp n^2$. Thus, conditioning the tree T to have height at least n, the expected total volume will be around n^2.