Applications of Stochastics — Exercise sheet 3

Gábor Pete
http://www.math.bme.hu/~gabor

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There were two gaps in my Erdős-Rényi giant cluster phase transition proof, for the critical case $G(n, \frac{1}{n})$. The following exercises do not completely fill them in, but provide very strong intuitive support. They prove the needed random walk ingredients for simple symmetric random walk on $\mathbb{Z}$ instead of i.i.d. $\mathsf{Poisson}(1) - 1$ jumps, plus they suggest some techniques that may be used for the case of $\mathsf{Poisson}(1) - 1$ jumps.

$\triangleright$ **Exercise 1.** Let $X_0, X_1, X_2, \dots$ be Simple Random Walk on $\mathbb{Z}$, i.e., with i.i.d. ± 1 jumps, each direction with probability $1/2$.

(a) For $x, y > 0$, the number of SRW trajectories that start at $X_0 = x$, end at $X_n = y$, and are 0 at some time in between, is equal to the number of trajectories from $X_0 = -x$ to $X_n = y$. (This is called the **reflection principle**.)

(b) Let $M_n := \max\{X_i : 0 \le i \le n\}$. Using part (a), show that, for any $t \ge 0$,

$$\mathbf{P}[\, M_n \ge t \,] \le 2\mathbf{P}[\, X_n \ge t \,]\,.$$

(c) Conclude from part (b) that there exists an absolute constant $c \in (0,1)$ such that

$$c < \mathbf{P}[\, M_n > \sqrt{n} \,] < 1 - c\,.$$

$\triangleright$ **Exercise 2.** Still for SRW on $\mathbb{Z}$, denoted by $X_0, X_1, X_2, \dots$, let τ_0 be the first time the walk hits 0, and let τ_k be the first time the walk hits k. For any sequence $i_0, \dots, i_n$ of integers in $\{1, 2, \dots, k-1\}$, satisfying $|i_{t+1} - i_t| = 1$ for all $t = 0, 1, \dots, n-1$, use Bayes' rule to show that

$$\mathbf{P}\big[\, X_{n+1} = i_n + 1 \;\big|\; X_0 = i_0, \dots, X_n = i_n, \tau_k < \tau_0 \,\big] = \frac{i_n + 1}{2 i_n}\,.$$

Therefore, the conditioned random walk is a Markov chain again, with transition probabilities just calculated.

$\triangleright$ **Exercise 3.** Still for SRW on $\mathbb{Z}$, using the previous two exercises, show that

$$\mathbf{P}_1[\, \tau_0 > n \,] \asymp \mathbf{P}_1[\, \tau_{\sqrt{n}} < \tau_0 \,] = 1/\sqrt{n}.$$

The second equality we know from the usual martingale argument, hence the first equality needs to be proved. Here are some hints:

(a) Notice that $\mathbf{P}_1[\, \tau_0 > n \mid \tau_{\sqrt{n}} < \tau_0 \,] > \mathbf{P}_0[\, M_n < \sqrt{n} \,] > c > 0$.

(b) Show that $X_i \,|\, \{\tau_0 > n\}$ stochastically dominates the unconditioned X_i, hence $\mathbf{P}_1[\, \tau_{\sqrt{n}} < \tau_0 \mid \tau_0 > n \,] > \mathbf{P}_1[\, M_n > \sqrt{n} \,] > c > 0$.

▷ **Exercise 4.*** Let $(X_i)_{i\geq 0}$ be a random walk on $\mathbb{Z}$, with i.i.d. increments ξ_i that have zero mean and an exponential tail: there exist $K \in \mathbb{N}$ and $0 < q < 1$ such that $\mathbf{P}[\,\xi \geq k+1\,] \leq q\,\mathbf{P}[\,\xi \geq k\,]$ for all $k \geq K$.

Starting from $X_0 = \ell \in \{1, 2, \dots, n-1\}$, let τ_0 be the first time the walk is at most 0, and let τ_n be the first time the walk is at least n. Show that, for any $0 < \ell < n$,

$$\mathbf{P}_\ell[\,\tau_n < \tau_0\,] \asymp \frac{\ell}{n}\,.$$

For this, first prove that $X_{\tau_n} - n$, conditioned on $\tau_n < \tau_0$, has an exponential tail, independently of n.

▷ **Exercise 5.**

(a) Let $X_0, X_1, \dots$ be a random walk on the integers with i.i.d. $\mathsf{Poisson}(1) - 1$ jumps. Using $\mathbf{P}[\,X_{2k} = 0\,] \asymp 1/\sqrt{k}$, which follows from the Local Central Limit Theorem (see, e.g., Theorem 3.5.2 in Durrett's book *Probability: theory and examples*), together with the First and Second Moment Methods, prove that there exist *some* $\kappa > 0$ such that $\mathbf{P}\big[\,\#\{1 \leq i \leq n : X_i = 0\} > \kappa\sqrt{n}\,\big] > \kappa$, for all n.

(b)* For SRW on $\mathbb{Z}$, calculate explicitly the probability $\mathbf{P}[\,X_{2k} = 0\,]$, then use Stirling's formula to show that the number of zeroes up to time n, divided by $\sqrt{n}$ has a limit distribution: the absolute value of a standard normal variable!