Applications of Stochastics — Isolated vertices in the Erdős-Rényi random graph $G(n,p)$

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Let $I = \sum_{i=1}^n I_i$ be the number of isolated vertices in $G(n,p)$, with $I_i := \mathbf{1}_{\{\text{vertex } i \text{ is isolated}\}}$. Clearly, $\mathbf{P}\big[\, I_i = 1 \,\big] = (1-p)^{n-1}$, hence $\mathbf{E}_{n,p}[\, I \,] = n(1-p)^{n-1}$. To understand the size of this expectation, here is a lemma, immensely useful in probabilistic combinatorics:

Lemma 1. *If $\epsilon_n \to 0$ and $\{a_n\}$ are two positive sequences with $\epsilon_n^2 a_n \to 0$, then*

$$(1-\epsilon_n)^{a_n} \sim \exp\big(-\epsilon_n a_n\big).$$

Proof. The Taylor series of $\exp(x)$ gives us that $\exp(-\epsilon_n) = 1 - \epsilon_n + O(\epsilon_n^2)$ if $\epsilon_n \to 0$, and in fact, $\exp\big(-\epsilon_n - O(\epsilon_n^2)\big) < 1 - \epsilon_n < \exp(-\epsilon_n)$. Therefore,

$$\exp\Big(\big(-\epsilon_n - O(\epsilon_n^2)\big)a_n\Big) < (1-\epsilon_n)^{a_n} < \exp\big(-\epsilon_n a_n\big),$$

and the lemma follows. $\square$

Let us fix $\epsilon \in \mathbb{R}$, and consider $p = p_n = (1+\epsilon)\frac{\log n}{n} \to 0$. Then

$$\mathbf{E}_{n,p}[\, I \,] \sim n \exp(-pn) = n \cdot n^{-(1+\epsilon)} \to \begin{cases} 0 & \text{if } \epsilon > 0\,, \\ \infty & \text{if } \epsilon < 0\,. \end{cases}$$

So, with probability tending to 1, we do not have isolated vertices when $\epsilon > 0$ is fixed. To get isolated vertices for $\epsilon < 0$ fixed, we will use the Second Moment Method:

$$\begin{aligned} \mathrm{Var}_{n,p}[I] = \sum_{i,j=1}^n \mathrm{Cov}[I_i, I_j] &= n\mathrm{Var}[I_1] + n(n-1)\mathrm{Cov}[I_1, I_2] \\ &= n\Big((1-p)^{n-1} - (1-p)^{2(n-1)}\Big) - n(n-1)\Big((1-p)^{2n-3} - (1-p)^{2(n-1)}\Big) \\ &< n\,(1-p)^{n-1} + n(n-1)\,(1-p)^{2n-3}\big(1-(1-p)^2\big) \\ &\sim n \cdot n^{-(1+\epsilon)} + n^2 \cdot n^{-2(1+\epsilon)} 2p \\ &< n^{-\epsilon} + 2\frac{\log n}{n^{1+2\epsilon}}\,; \end{aligned}$$

to get the second line, we used the observation that for two fixed vertices to be both isolated, we need $2n-3$ edges to be closed. To use Chebyshev's inequality, we now want that this is $\ll \mathbf{E}_{n,p}[\, I \,]^2 \sim n^{-2\epsilon}$. For $\epsilon < 0$ fixed, this is indeed the case, and we get, for any $\lambda < 1$ fixed, that

$$\mathbf{P}_{n,p}\big[\, I > \lambda n^{-\epsilon} \,\big] \to 1\,,$$

as we wanted.