Applications of Stochastics — Renewal Equations Sketch

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Let $\xi_1, \xi_2, \dots$ be the i.i.d. lifetimes in a renewal process, with non-arithmetic distribution function $F(s) = \mathbf{P}[\,\xi \le s\,]$ and mean $\mathbf{E}\xi = \mu \in (0,\infty)$. Then $T_k := \sum_{i=1}^k \xi_i$ are the renewal times, $N_t := \min\{k : T_k \ge t\}$, and $U(t) := \mathbf{E}N_t$. The excess lifetime (or overshoot) is $\gamma_t := T_{N_t} - t$, the current lifetime is $\delta_t := t - T_{N_t - 1}$, and the total lifetime is $\beta_t := \gamma_t + \delta_t$.

▷ **Exercise 1.**

(a) Find the renewal equation $H(t) = h(t) + H * F(t)$ for $H(t) := \mathbf{P}[\,\beta_t > x\,]$, where $x \ge 0$ is fixed arbitrarily. (We actually did this in class.)

(b) Find the renewal equation for $H(t) := \mathbf{P}[\,\gamma_t > x\,]$.

(c) Using the Renewal Theorem, find the limit distributions of β_t and γ_t as $t \to \infty$.

(d) Identify the limit distribution of the total lifetime β_t as the size-biased version of ξ, and the limit distribution of the overshoot γ_t as the size-biased version $\widehat{\xi}$ multiplied with an independent $\mathsf{Unif}[0,1]$ variable. In order to avoid working with Stieltjes-integrals, you may assume that ξ has a density function.

Solution.

(b) Try to write a "recursion" for the distribution of γ_t, i.e., a renewal equation.

Let $H_x(t) := \mathbf{P}[\,\gamma_t > x\,]$. Condition on T_1.

$$\mathbf{P}\big[\,\gamma_t > x \;\big|\; T_1\,\big] = \begin{cases} H_x(t - T_1) & \text{if } T_1 < t\,, \\ 0 & \text{if } t < T_1 < t + x\,, \\ 1 & \text{if } t + x < T_1\,. \end{cases}$$

Now average over the possible values of T_1, i.e., take expectation. Recall that the cumulative distribution function of T_1 is $F(s)$.

$$\begin{aligned} \mathbf{P}[\,\gamma_t > x\,] = \mathbf{E}\Big[\,\mathbf{P}\big[\,\gamma_t > x \;\big|\; T_1\,\big]\,\Big] &= \int_0^t H_x(t-s)\,dF(s) \\ &\quad + 0 \cdot \mathbf{P}[\,t < T_1 < t + x\,] \\ &\quad + 1 \cdot \mathbf{P}[\,t + x < T_1\,]\,, \end{aligned}$$

or, with the Stieltjes-convolution notation:

$$H_x(t) = (H_x * F)(t) + 1 - F(t + x). \tag{1}$$

(c) The Renewal Theorem says that, if F has non-arithmetic distribution with finite mean μ, and the $h(t)$ term in the renewal equation is directly integrable (e.g., positive and integrable), then

$$\lim_{t\to\infty} H(t) = \frac{1}{\mu}\int_0^\infty h(s)\,ds.$$

In (1), we have $h(t) = 1 - F(t + x)$, which is positive, and

$$\int_0^\infty h(s)\,ds = \int_x^\infty 1 - F(s)\,ds.$$

This is less than the same integral from 0 to ∞, which is just $\mathbf{E}[\,T_1\,] = \mu < \infty$, so this is indeed directly integrable, and

$$x \mapsto \frac{1}{\mu}\int_x^\infty 1 - F(s)\,ds \in [0,1] \tag{2}$$

is indeed the tail of a distribution.

(d) What is the tail of the distribution of the size-biased version $\widehat{\xi}$ multiplied with an independent $U \sim \mathsf{Unif}[0,1]$ variable? By first conditioning on the value of $\widehat{\xi}$:

$$\mathbf{P}\big[\,\widehat{\xi}\cdot U > x\,\big] = \int_x^\infty \mathbf{P}[\,sU > x\,]\,dF_{\widehat{\xi}}(s) = \int_x^\infty 1 - \frac{x}{s}\,dF_{\widehat{\xi}}(s) = \frac{1}{\mu}\int_x^\infty (s-x)\,dF_\xi(s),$$

since $dF_{\widehat{\xi}}(s) = \frac{s}{\mu}dF_\xi(s)$ is the size-biasing. And this formula is actually the same as (2), by the following picture: