PHASE TRANSITION OF THE ERDŐS-RÉNYI GRAPH

$G(n,p)$: n VERTICES, THERE ARE $\binom{n}{2}$ POSSIBLE EDGES; WE DRAW EACH POSSIBLE EDGE WITH PROBABILITY p INDEPENDENTLY.

HOW BIG IS THE LARGEST CONNECTED COMPONENT OF $G(n,p)$?

LARGEST: $\mathcal{C}_1$ SECOND LARGEST: $\mathcal{C}_2$

THEOREM: ASSUME $\boxed{n \gg 1}$ $\boxed{p \ll 1}$ $\boxed{n \cdot p = t}$

SUBCRITICAL

IF $\boxed{t<1}$: $|\mathcal{C}_1| \approx c(t) \cdot \log(n)$

IF $\boxed{t>1}$: $|\mathcal{C}_1| \approx \Theta(t) \cdot n$, $|\mathcal{C}_2| \approx c(t) \cdot \log(n)$

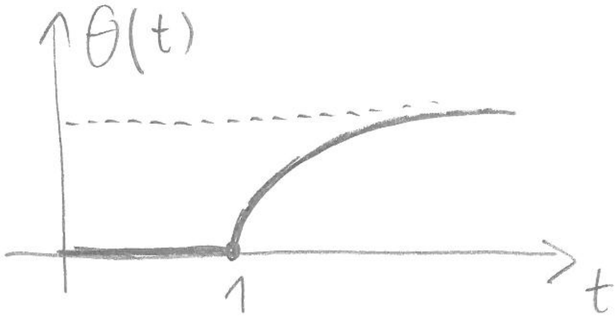
SUPERCRITICAL

IF $\boxed{t=1}$: $|\mathcal{C}_1| \asymp |\mathcal{C}_2| \asymp n^{2/3}$

CRITICAL

PHASE TRANSITION

SO: $\boxed{\dfrac{|\mathcal{C}_1|}{n} \approx \Theta(t)}$



$\Theta(t) = 0$ IF $\boxed{t \leq 1}$

$\Theta(t) > 0$ IF $\boxed{t > 1}$

GIANT COMPONENT

ANOTHER APPROACH:

IF $v \in V(G(n,p)) = \{1, 2, \ldots, n\}$, DENOTE BY $\mathcal{C}(v)$ THE CONNECTED COMPONENT OF v IN $G(n,p)$

LET $\boxed{n \gg 1}$ $\boxed{p \ll 1}$ $\boxed{n \cdot p = t}$

LET $\boxed{v_k^{(n)}(t) = \dfrac{1}{n} \sum_{v=1}^{n} \mathbb{1}[\,|\mathcal{C}(v)| = k \text{ IN } G(n,p)]}$

FRACTION OF VERTICES CONTAINED IN CONNECTED COMPONENTS OF SIZE k

THEN $v_k^{(n)}(0) = \begin{cases} 1 & \text{IF } k = 1 \\ 0 & \text{IF } k = 2, 3, 4, \ldots \end{cases}$

FOR ANY n, t: $\boxed{\sum_{k \geq 1} v_k^{(n)}(t) = 1}$

THEOREM: $G(n,p)$, $\boxed{n \to \infty}$ $\boxed{n \cdot p = t}$

$$\boxed{\lim_{n\to\infty} N_k^{(n)}(t) = N_k(t)}$$ (LAW OF LARGE NUMBERS)

WHERE

$\boxed{t<1}$: $\boxed{\underbrace{N_k(t) \leq C(t) \cdot e^{-\tilde{c}(t) \cdot k}}_{\text{EXPONENTIAL DECAY IN } k}}$ $\boxed{\sum_{k=1}^{\infty} N_k(t) = 1}$

$\boxed{t>1}$: $\boxed{-\|-}$ BUT $\boxed{\sum_{k=1}^{\infty} N_k(t) < 1}$

$\boxed{t=1}$: $\boxed{\underbrace{N_k(t) \asymp k^{-3/2}}_{\text{POLYNOMIAL DECAY}}}$ AND $\boxed{\sum_{k=1}^{\infty} N_k(t) = 1}$

THUS IF $\boxed{t>1}$ THEN

$$1 = \lim_{n\to\infty} \sum_{k\geq 1} N_k^{(n)}(t) \neq \sum_{k\geq 1} \lim_{n\to\infty} N_k^{(n)}(t) =$$

$$= \sum_{k\geq 1} N_k(t) = 1 - \theta(t)$$

THE MISSING MASS IS CONTAINED IN $\mathcal{C}_1$ (THE GIANT COMPONENT)

TOPIC OF THIS COURSE:
FIND $N_k(t)$ USING VARIOUS METHODS.

METHOD 1: DIFFERENTIAL EQUATIONS

TRICK: TIME EVOLUTION OF THE E-R GRAPH:

n VERTICES, $\binom{n}{2}$ EDGES,

FOR EACH EDGE e, LET $\boxed{T_e \sim EXP\left(\frac{1}{n}\right)}$ EXPONENTIAL DISTRIBUTION

T_e IS THE TIME WHEN THE EDGE e APPEARS.

THEN IF $\boxed{t \in \mathbb{R}_+}$ FIXED, $\boxed{n \gg 1}$

1) $\mathbb{P}(T_e \leq t) = 1 - e^{-t/n} \approx \frac{t}{n}$

THUS AT TIME t, WE SEE $G(n,p)$
WHERE $\boxed{p = \mathbb{P}(T_e \leq t)}$ THUS $\boxed{n \cdot p \approx t}$

2) $\mathbb{P}(T_e \in [t, t+dt]) \approx \frac{dt}{n}$

(EDGE e APPEARS BETWEEN t AND $t+dt$ WITH PROBABILITY $\frac{dt}{n}$

IF v IS A VERTEX, WHEN DOES THE FIRST EDGE CONNECT TO v?

$\left(\min_{u \neq v} T_{\{v,u\}}\right) \sim EXP\left(\frac{n-1}{n}\right) \approx EXP(1)$

THUS $\mathbb{P}(|\mathcal{C}(w)|=1 \text{ AT TIME } t) \approx e^{-t}$

THUS $v_1^{(n)}(t) = \frac{1}{n}\sum_{w=1}^{n} \mathbb{1}[|\mathcal{C}(w)|=1 \text{ AT TIME } t] \approx e^{-t}$

SMOLUCHOWSKI COAG.

THUS $\lim_{n\to\infty} v_1^{(n)}(t) = v_1(t) = e^{-t}$

HOW ABOUT $v_2(t), v_3(t), \ldots$?

CLAIM: $$\boxed{\frac{d}{dt} v_k(t) = -k \cdot v_k(t) + \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)\cdot v_{k-l}(t)}$$

SKETCH PROOF:

WHAT IS $\mathbb{E}\left(v_k^{(n)}(t+dt) - v_k^{(n)}(t)\right) = (\star)$?

1. $v_k^{(n)}$ DECREASES by $\frac{k}{n}$ IF A COMPONENT OF SIZE k GETS CONNECTED TO ANOTHER COMPONENT.
2. $v_k^{(n)}$ INCREASES by $\frac{k}{n}$ IF COMPONENTS OF SIZE l AND $k-l$ GET CONNECTED TO EACH OTHER.

NUMBER OF EDGES THAT CAN CAUSE 1 IS EQAL TO $(v_k^{(n)}\cdot n)\cdot(n-k) \approx v_k^{(n)}\cdot n^2$

(IF k IS „SMALL")

NUMBER OF EDGES THAT CAN CAUSE ②:

(A) IF $\boxed{\ell \neq r-\ell}$ THEN $\left(N_\ell^{(m)}\cdot m\right)\cdot\left(N_{r-\ell}^{(m)}\cdot m\right) =$

$$= N_\ell^{(m)} \cdot N_{r-\ell}^{(m)} \cdot m^2$$

(B) IF $\boxed{\ell = r-\ell}$ THEN $\frac{1}{2}\cdot\left(N_\ell^{(m)}\cdot m\right)\cdot\left(N_\ell^{(m)}\cdot m - \ell\right) \approx$

$\boxed{2 \mid r \text{ AND } \ell = r/2}$

$$\frac{1}{2}\cdot N_{\frac{k}{2}}^{(m)}\cdot N_{\frac{k}{2}}^{(m)}\cdot m^2$$

THUS

$$(\star) \approx -\left(N_r^{(m)}\cdot m^2\right)\cdot\frac{k}{m}\cdot\frac{dt}{n} + \sum_{\ell<\frac{k}{2}}\left(N_\ell^{(m)}\cdot N_{r-\ell}^{(m)}\cdot m^2\right)\cdot\frac{r}{m}\cdot\frac{dt}{n}$$

$$+ \mathbb{1}[r \text{ IS EVEN}]\cdot\left(\frac{1}{2}\cdot N_{\frac{r}{2}}^{(m)}\cdot N_{\frac{r}{2}}^{(m)}\cdot m^2\right)\cdot\frac{r}{m}\cdot\frac{dt}{n} =$$

$$= -k\cdot N_r\, dt + r\cdot\sum_{\ell<\frac{r}{2}} N_\ell\cdot N_{r-\ell}\, dt + \mathbb{1}[2 \mid r]\cdot\frac{1}{2}\cdot k\cdot N_{\frac{r}{2}}^2\, dt$$

$$= \left(-r\cdot N_r + \frac{r}{2}\sum_{\ell=1}^{r-1} N_\ell\cdot N_{r-\ell}\right)dt \quad \checkmark$$

(DIVIDE BY dt)

SOLUTION OF SMOLUCHOWSKI?

$\boxed{\dot{N}_1(t) = -N_1(t)}$ $\boxed{N_1(0) = 1}$ $\Rightarrow$ $\boxed{N_1(t) = e^{-t}}$ ✓

$$\dot{N}_2(t) = -2\cdot N_2(t) + \frac{2}{2}\cdot\left(N_1(t)\right)^2 = -2\cdot N_2(t) + e^{-2t}$$

$\boxed{N_2(0) = 0}$ $\rightarrow$ $\boxed{N_2(t) = e^{-2\cdot t}\cdot t}$

$N_3(t)$? $N_4(t)$? ETC...

<u>THM</u>: $\lim_{n\to\infty} N_k^{(n)}(t) = N_k(t) = \dfrac{k^{k-1}}{k!} \cdot e^{-k\cdot t} \cdot t^{k-1}$ (BOREL DISTRIBUTION)

<u>HW</u>: TRY TO CHECK THAT THIS $(N_k(t))_{k=1}^{\infty}$ SOLVES SMOLUCHOWSKI'S COAG. EQATIONS!

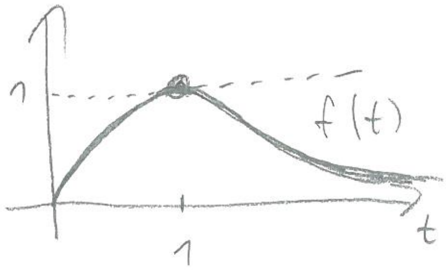
<u>NEXT CLASS</u>: PROOF OF THIS FORMULA, USING RANDOM WALKS!

<u>COROLLARY</u>:

STIRLING'S FORMULA: $\boxed{k! \asymp k^{k+\frac{1}{2}} \cdot e^{-k}}$

$$N_k(t) = \frac{1}{t} \cdot \frac{k^{k-1}}{k!} \cdot (e^{-t} \cdot t)^k \asymp k^{-3/2} \cdot (e^{1-t} \cdot t)^k$$

$\boxed{f(t) = e^{1-t} \cdot t}$ $\Longrightarrow$



IF $\boxed{t<1}$ OR $\boxed{t>1}$ THEN $f(t) < 1$, THUS

$\boxed{N_k(t) \le C(t) \cdot e^{-\tilde{C}(t)\cdot k}}$ EXPONENTIAL DECAY

IF $\boxed{t=1}$ THEN $f(t) = 1$, THUS

$\boxed{N_k(t) \asymp k^{-3/2}}$ POLYNOMIAL DECAY.

$G(n,p)$: n VERTICES, $\binom{n}{2}$ POSSIBLE EDGES, EACH EDGE IS PRESENT INDEPENDENTLY WITH PROBABILITY p $\boxed{p = t/n}$ $\boxed{t \text{ FIXED}}$

$$v_r^{(n)}(t) = \frac{1}{n} \sum_{v=1}^{n} \mathbb{1}\left[|\mathcal{C}(v)| = r \text{ IN } G(n, \tfrac{t}{n}) \right]$$

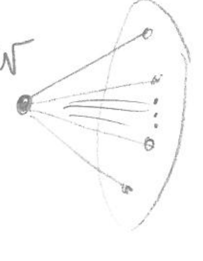
<u>WANT</u>: IF $\boxed{n \gg 1}$ THEN $v_r^{(n)}(t) \approx \frac{r^{r-1}}{r!} \cdot e^{-rt} \cdot t^{r-1}$ $\quad r = 1, 2, 3, \ldots$

SLIGHLY WEAKER STATEMENT: v IS A VERTEX:

$$\boxed{n \gg 1} \Rightarrow \mathbb{P}\left(|\mathcal{C}(v)| = r \text{ IN } G(n, \tfrac{t}{n}) \right) \approx \frac{r^{r-1}}{k!} \cdot e^{-r \cdot t} \cdot t^{r-1}$$

WE WILL PROVE THIS.

<u>EXPLORATION</u> <u>OF</u> <u>$\mathcal{C}(v)$</u>:



v $\quad n-1$ POSSIBLE NEIGHBOURS

N_1 = NUMBER OF NEIGHBOURS OF v IN $G(n, \frac{t}{n})$

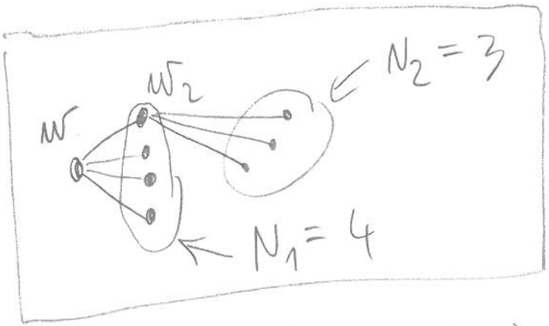
$$N_1 \sim BIN\left(n-1, \frac{t}{n}\right) \approx POI\left(\frac{n-1}{n} \cdot t\right) \approx POI(t)$$

THUS $\mathbb{P}(N_1 = r) \approx e^{-t} \cdot \frac{t^r}{r!}$

$$v_1^{(n)}(t) \approx \mathbb{P}(|\mathcal{C}(v)| = 1) = \mathbb{P}(N_1 = 0) \approx e^{-t} \checkmark$$

IF w HAS AT LEAST ONE NEIGHBOURS, LET w_2 BE THE „FIRST" NEIGHBOOR OF $w = w_1$

AND $\boxed{N_2 = \text{NUMBER OF „NEW" NEIGHBOURS OF } w_2 \text{ IN } G(n, \tfrac{t}{n})}$



GIVEN N_1,

$$N_2 \sim BIN\left(n-1-N_1, \frac{t}{n}\right) \approx$$

SINCE $\boxed{N_1 << n}$

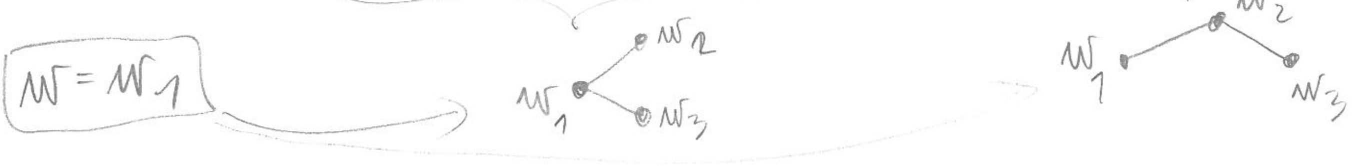
$$\approx POI\left(\left(\frac{n-1-N_1}{n}\right)\cdot t\right) \approx POI(t)$$

THUS N_1 AND N_2 ARE „INDEPENDENT"

$$v_2^{(n)}(t) \approx \mathbb{P}(|\mathcal{C}(w)| = 2) = \mathbb{P}(N_1 = 1, N_2 = 0) \approx e^{-t} \cdot t \cdot e^{-t} = e^{-2t} \cdot t \checkmark$$

SIMILARLY: N_1, N_2, N_3 I.I.D. $\sim POI(t)$:

$$\mathbb{P}(|\mathcal{C}(w)| = 3) = \mathbb{P}(N_1 = 2, N_2 = 0, N_3 = 0) + \mathbb{P}(N_1 = 1, N_2 = 1, N_3 = 0)$$



$$= \left(e^{-t} \cdot \frac{t^2}{2}\right) \cdot e^{-t} \cdot e^{-t} + (e^{-t} \cdot t) \cdot (e^{-t} \cdot t) \cdot e^{-t} = \frac{3}{2} \cdot e^{3t} \cdot t^2 \checkmark$$

G. OLDAC

$T = 1, 2, 3, \dots$

A_T = NUMBER OF ACTIVE VERTICES AFTER THE EXPLORATION OF VERTEX T

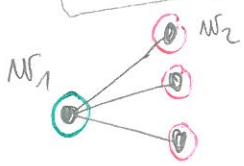
= EXPLORED = ACTIVE

EXAMPLE:

$T = 0$

$A_0 = 1$

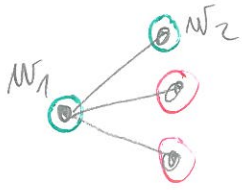
$T = 1$



$A_1 = 3$

$N_1 = 3$

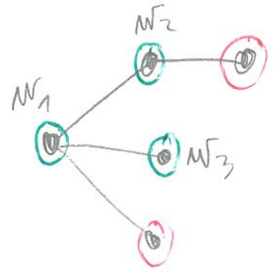
$T = 2$



$A_2 = 2$

$N_2 = 0$

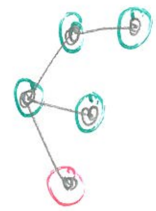
$T = 3$



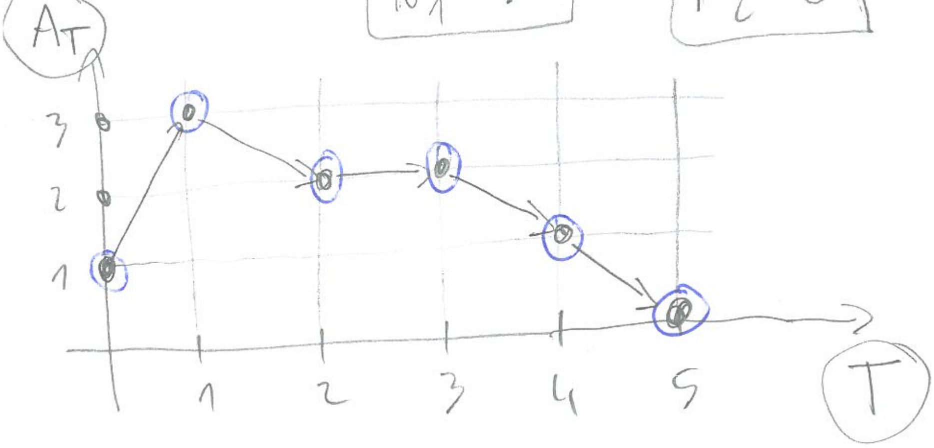
$A_3 = 2$

$N_3 = 1$

$T = 4$



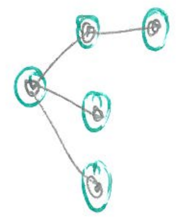
$A_4 = 1$ $N_4 = 0$



$A_5 = 0$

$N_5 = 0$

$T = 5$



EXPLORATION OF $\mathcal{C}(v)$ IS COMPLETE!

$$A_0 = 1 \qquad A_T = A_{T-1} + N_T - 1$$

WHERE $N_1, N_2, N_3, \dots$ ARE I.I.D. $\mathrm{POI}(t)$

(RANDOM WALK)

$$J = \min\{T : A_T = 0\}$$ THEN $$J = |\mathcal{C}(v)|$$

WANT: $\mathbb{P}(J = r) = \frac{r^{r-1}}{r!} \cdot e^{-r \cdot t} \cdot t^{r-1}$

$$\mathbb{P}(J=r) = \mathbb{P}(A_1>0, A_2>0, \dots, A_{r-1}>0, A_r=0)$$

<u>CLAIM:</u> (KEMPERMAN'S FORMULA)

$$\boxed{\mathbb{P}(J=r) = \frac{1}{r}\cdot\mathbb{P}(A_r=0)}$$

THUS $\mathbb{P}(J=r) = \frac{1}{r}\cdot\mathbb{P}\left(1+(N_1-1)+\dots+(N_r-1)=0\right) =$

$$= \frac{1}{r}\cdot\mathbb{P}\Big(\underbrace{N_1+\dots+N_r}_{\mathrm{POI}(r\cdot t)} = r-1\Big) = \frac{1}{r}\cdot e^{-r\cdot t}\cdot\frac{(r\cdot t)^{r-1}}{(r-1)!} = \frac{r^{r-1}}{r!}\cdot e^{-rt}\cdot t^{r-1}$$

<u>PROOF</u>: $\boxed{Y_T = N_T - 1}$ $\quad T=1,2,\dots$

$y=(y_1,y_2,\dots,y_r)$: A POSSIBLE SEQUENCE OF INCREMENTS: $y_T\in\{-1,0,1,2,\dots\}=S$

$\Gamma=\{y\in S^r : y_1+\dots+y_r=-1\}$, THEN

$$\mathbb{P}(A_r=0) = \sum_{y\in\Gamma}\mathbb{P}(Y_1=y_1,\dots,Y_r=y_r) = \sum_{y\in\Gamma}\mathbb{P}(\underline{Y}=y)$$

$$\mathbb{P}(J=r) = \sum_{y\in\Gamma_0}\mathbb{P}(\underline{Y}=y)$$, WHERE

$$\Gamma_0 = \{y\in\Gamma : y_1\geq 0, y_2\geq 0,\dots, y_{r-1}\geq 0\}$$

IF $y \in \Gamma$, A CYCLIC REARRANGEMENT OF y

IS OF $\qquad y^i = (y_i, y_{i+1}, \dots, y_r, y_1, \dots, y_{i-1}) \in \Gamma$

ALSO

FACTS:

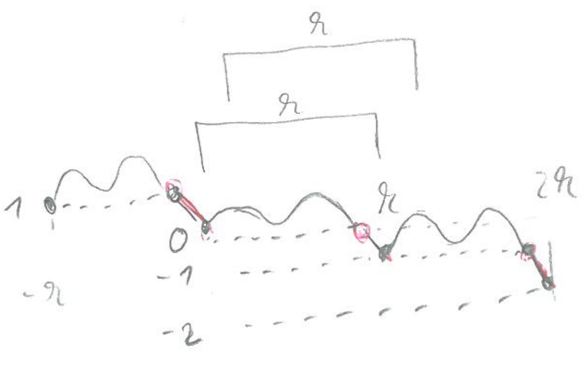
1. $y^0, y^1, \dots, y^{r-1}$ ARE ALL DIFFERENT ELEMENTS OF Γ

2. THERE IS EXACTLY ONE OF THESE WHICH IS IN Γ_0

3. $\mathbb{P}(\underline{Y} = y^0) = \mathbb{P}(\underline{Y} = y^1) = \dots = \mathbb{P}(\underline{Y} = y^{r-1})$

1 + 2 + 3 $\Rightarrow$ $\boxed{\sum_{y \in \Gamma} \mathbb{P}(\underline{Y} = y) = r \cdot \sum_{y \in \Gamma_0} \mathbb{P}(\underline{Y} = y)}$ ✓

WHY 1, 2? EXTEND PERIODICALLY:



USE THAT

$y_T \in \{-1, 0, 1, \dots\}$

$\theta(t) \approx \frac{|\mathcal{C}_1|}{n}$ IN $G(n, \frac{t}{n})$: GIANT COMPONENT DENSITY

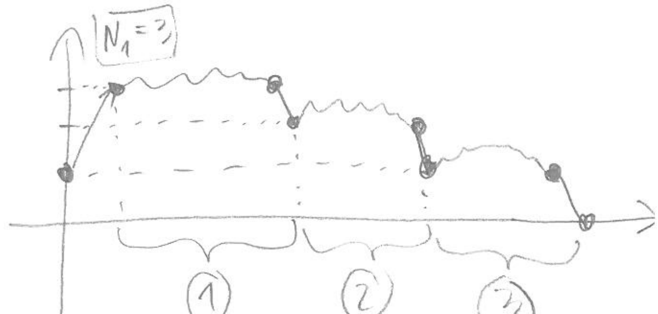
$\boxed{\theta = \mathbb{P}(J = +\infty)}$ $\boxed{\theta > 0}$ CAN ONLY HAPPEN IF $\boxed{t > 1}$, BECAUSE THEN $\mathbb{E}(N_T - 1) > 0$

AND THUS $A_T = 1 + (N_1 - 1) + \ldots + (N_T - 1)$, SO

$\boxed{\mathbb{E}(A_T) = 1 + T \cdot (t-1)}$ $\Rightarrow$ THE RANDOM WALK $A_1, A_2, \ldots$ HAS POSITIVE DRIFT, SO IT MAY NEVER REACH 0 AND THUS $J = +\infty$.

$$1 - \theta = \mathbb{P}(J < \infty) = \sum_{i=0}^{\infty} \mathbb{P}(J < \infty \mid N_1 = i) \cdot \mathbb{P}(N_1 = i) =$$

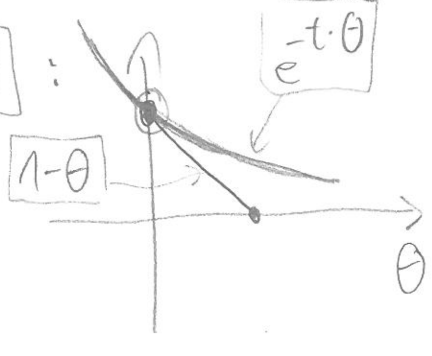
$$= \sum_{i=0}^{\infty} (1-\theta)^i \cdot \left(e^{-t} \cdot \frac{t^i}{i!}\right)$$ WHY?



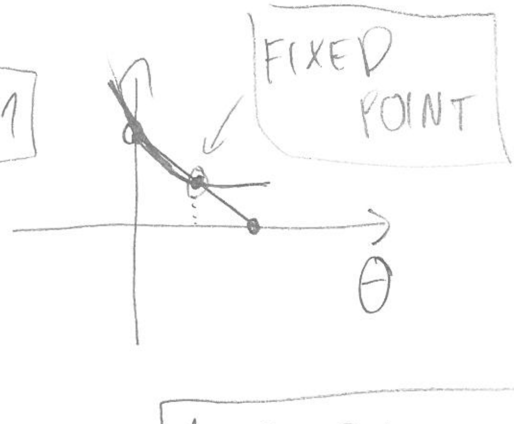
$$= e^{-t} \cdot e^{(1-\theta) \cdot t} = e^{-t \cdot \theta}$$

$\boxed{1 - \theta = e^{-t \cdot \theta}}$ $\Leftarrow$ FIXED POINT EQUATION

IF $\boxed{t < 1}$:



IF $\boxed{t > 1}$



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