Stochastic models — Second problem set

Gábor Pete
http://www.math.bme.hu/~gabor

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Hand in 5 solutions out of the 13 problems below, by May 12. In the 5, you may include at most 1 from the first problem set (that you have not handed in earlier, of course). You may hand in partial solutions for partial credit.

▷ **Exercise 1.** For a subset A of the hypercube $\{0,1\}^n$, let $B(A,t) := \big\{x \in \{0,1\}^n : \mathrm{dist}(x,A) \leq t\big\}$, with the usual Hamming distance. Let $\epsilon, \lambda > 0$ be constants satisfying $\exp(-\lambda^2/2) = \epsilon$. Prove using Azuma-Hoeffding that

$$|A| \geq \epsilon\, 2^n \quad \Longrightarrow \quad \big|B(A, 2\lambda\sqrt{n})\big| \geq (1-\epsilon)\, 2^n\,.$$

That is, even small sets become huge if we enlarge them a little.

▷ **Exercise 2.** Prove the Bollobás-Thomason threshold theorem: for any sequence $\mathcal{A} = \mathcal{A}_n \subseteq \{0,1\}^{\binom{n}{2}}$ of upward closed events, let

$$p_t^{\mathcal{A}}(n) := \inf \big\{p : \mathbf{P}[\, G(n,p) \text{ satisfies } \mathcal{A}_n \,] \geq t\big\}.$$

Prove that for any ϵ there is $C_\epsilon < \infty$ such that $\big|p_{1-\epsilon}^{\mathcal{A}}(n) - p_\epsilon^{\mathcal{A}}(n)\big| \leq C_\epsilon \left(p_\epsilon^{\mathcal{A}}(n) \wedge (1 - p_{1-\epsilon}^{\mathcal{A}}(n))\right)$. (Hint: take many independent copies of low density to get success with good probability at a larger density.)

▷ **Exercise 3.** Find the order of magnitude of the threshold function $p_{1/2}(n)$ for the random graph $G(n,p)$ containing a copy of the cycle C_4.

▷ **Exercise 4.** Let $X_\lambda(n)$ be the number of isolated vertices in the random graph $G(n, \frac{\lambda \ln n}{n})$.

(a) Show that $\mathbf{E}X_\lambda(n) \sim n^{1-\lambda}$ as $n \to \infty$. Deduce that, for $\lambda > 1$ fixed, with probability tending to 1 there exist no isolated vertices. For $\lambda < 1$ fixed, using the 2nd Moment Method, show that there exist isolated vertices with probability tending to 1.

(b) Show that if $\alpha > 1 - \lambda > 0$, then the probability that there exists a union of components that has total size between n^α and $n - n^\alpha$ is going to 0. This is an indication that isolated vertices are indeed the main obstacles to connectivity.

▷ **Exercise 5.** Consider a Galton-Watson branching process with offspring distribution ξ, mean $\mathbf{E}\xi = \mu$. Let Z_n be the size of the nth level, with $Z_0 = 1$, the root.

(a) Show that Z_n/μ^n is a martingale.

(b) Deduce for $\mu < 1$ that $\mathbf{P}[\, Z_n > 0\,] \leq \exp(-c\,n)$ for some $c > 0$, and hence $\mathbf{P}[\, Z_n = 0 \text{ eventually}\,] = 1$.

(c) Deduce from a MG convergence thm that if $\mu = 1$ but $\mathbf{P}[\, \xi = 1\,] \neq 1$, then $\mathbf{P}[\, Z_n = 0 \text{ eventually}\,] = 1$.

▷ **Exercise 6.** Continuing the previous exercise:

(a) Assuming that $\mu > 1$ and $\mathbf{E}[\,\xi^2\,] < \infty$, first show that $\mathbf{E}\big[\, Z_n^2 \,\big] \leq C(\mathbf{E}Z_n)^2$. (Hint: use the conditional variance formula $\mathbf{D}^2[Z_n] = \mathbf{E}\big[\,\mathbf{D}^2[Z_n \mid Z_{n-1}]\,\big] + \mathbf{D}^2\big[\mathbf{E}[\, Z_n \mid Z_{n-1}\,]\big]$.) Then, using the Second Moment Method, deduce that the GW process survives with positive probability.

(b) Extend the above to the cases $\mathbf{E}\xi = \infty$ or $\mathbf{D}\xi = \infty$ by a truncation $\xi \mathbf{1}_{\xi < K}$ for K large enough.

▷ **Exercise 7.** For the GW tree with offspring distribution Poisson$(1+\epsilon)$, show that the survival probability is asymptotically 2ϵ, as $\epsilon \to 0$.

▷ **Exercise 8.** Using the exploration Markov chain for GW trees and a Doob transform, show that if we condition the GW tree with offspring distribution Poisson(λ) on extinction, where $\lambda > 1$, then we get a GW tree with offspring distribution Poisson(μ) with $\mu < 1$, where $\lambda e^{-\lambda} = \mu e^{-\mu}$.

▷ **Exercise 9.** If X is a non-negative random variable with finite expectation, then its size-biased version $\widehat{X}$ is defined by $\mathbf{P}[\widehat{X} \in A] = \mathbf{E}[\, X\,\mathbf{1}_{\{X\in A\}}\,]/\mathbf{E}X$.

(a) Show that the size-biased version of $\mathsf{Poi}(\lambda)$ is just $\mathsf{Poi}(\lambda)+1$.

(b) Show that the size-biased version of $\mathsf{Expon}(\lambda)$ is the sum of two independent $\mathsf{Expon}(\lambda)$'s.

(c) Take Poisson point process of intensity λ on $\mathbb{R}$. Condition on the interval $(-\epsilon, \epsilon)$ to contain at least one arrival. As $\epsilon \to 0$, what is the point process we obtain in the limit? What does this have to do with parts (a) and (b)?

▷ **Exercise 10.** Let $X_k(n)$ be the number of degree k vertices in the Erdős-Rényi graph $G(n, \lambda/n)$, where $\lambda > 0$ is fixed. Show that $X_k(n)/n$ converges in probability, as $n \to \infty$, to $\mathbf{P}[\,\mathsf{Poi}(\lambda) = k\,] = e^{-\lambda}\lambda^k/k!$. (Hint: calculate the 2nd moment or use Azuma-Hoeffding.)

▷ **Exercise 11.** Consider Pólya's urn process $(G_n, R_n)_{n\geq 0}$, started with $G_0 = g$ green and $R_0 = r$ red balls. Recall that $G_n/(G_n + R_n)$ is a bounded martingale, hence it converges almost surely to some $\gamma \in [0,1]$.

(a) Suppose that $r, g > 1$. Define $W_n = \log(G_n + R_n) - \log(G_n - 1)$. Show that $(W_n)_{n\geq 0}$ is a supermartingale, and deduce that the limit γ is in fact almost surely strictly in $(0,1)$.

(b) Extend the argument to the case when r and g can be 1.

▷ **Exercise 12.** Assume that $\pi : G' \longrightarrow G$ is a topological covering between infinite graphs, or in other words, G is a factor graph of G'. Show that $p_c(G') \leq p_c(G)$.

▷ **Exercise 13.** Consider the graph G with 6 vertices and 7 edges that looks like a figure 8 on a digital display. Consider the uniform measure on the 15 spanning trees of G, denoted by UST, and the uniform measure on the 7 connected subgraphs with 6 edges (one more than a spanning tree), denoted by $\mathsf{UST}+1$. Find an explicit monotone coupling between the two measures (i.e., with $\mathsf{UST} \subset \mathsf{UST}+1$).

Question. Is there such a monotone coupling for every finite graph? (Finding the answer might lead to a fantastic PhD thesis.)