Stochastic Models at CEU — Second HW problem set

Gábor Pete
http://www.math.bme.hu/~gabor

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Solve 6 of the 15 problems below by April 11. Ask me for help if you get stuck with something.

The following was used as a piece of intuition for the evolving sets method:

▷ **Exercise 1.** If $\mathrm{Var}[X] \geq c\,(\mathbf{E}X)^2$ then $\mathbf{E}\big[\sqrt{X}\,\big] \leq (1-c')\sqrt{\mathbf{E}X}$, where $c' > 0$ depends only on $c > 0$.

▷ **Exercise 2.** Prove $O(n \log n)$ uniform mixing time for SRW on the hypercube $\{0,1\}^n$ using evolving sets and the standard one-dimensional Central Limit Theorem.

▷ **Exercise 3.** Consider percolation on $\{0,1\}^n$. Believe the result (due to Ajtai-Komlós-Szemerédi 1982) that for any $\epsilon > 0$, at $p = (1+\epsilon)/n$ there is a unique giant cluster. By exhibiting long "hanging paths" in this giant cluster, show that the mixing time inside the giant cluster is at least $c_\epsilon n^2$.

Folklore conjecture. The giant cluster at $p = (1+\epsilon)/n$ on the hypercube $\{0,1\}^n$ has $\mathrm{poly}(n)$ mixing time, probably $C_\epsilon\, n^2$.

▷ **Exercise 4.** A simple version of the Tetris game (with no player): on the discrete cycle of length K, unit squares with sticky corners are falling from the sky, at places $[i, i+1]$ chosen uniformly at random ($i = 0, 1, \ldots, K-1$, mod K). Let R_t be the size of the roof after t squares have fallen: those squares of the current configuration that could have been the last to fall. Show that $\lim_{t\to\infty} \mathbf{E}R_t = K/3$.

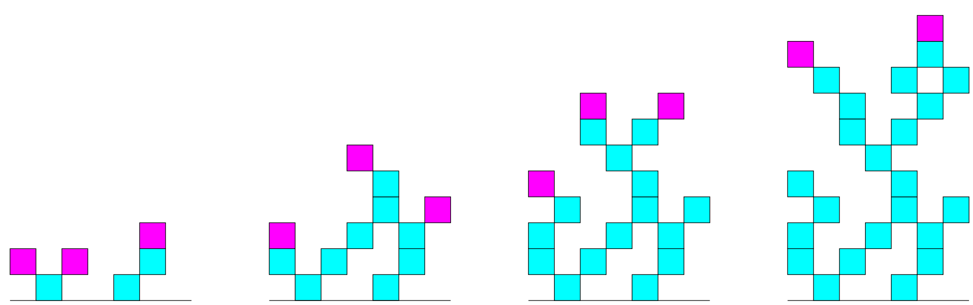

Figure 1: Sorry, this picture is on the segment, not on the cycle.

Remark. If there are two types of squares, particles and antiparticles that annihilate each other when falling on exactly on top of each other, this process is a SRW on a group, and the size of the roof has to do with the speed of the SRW. Here, for $K \geq 4$, the expected limiting size of the roof is already less than $0.32893K$, but this is far from trivial. What's the situation for $K = 3$?

Two exercises for the range $R_n = \{v \in V(G) : \exists k \in \{0, 1, \ldots, n\} \text{ with } X_k = v\}$ of SRW, which were mentioned in the discussion of the speed of random walk on the lamplighter groups $\mathbb{Z}_2 \wr \mathbb{Z}^d$:

▷ **Exercise 5.** For simple random walk $\mathbb{Z}^2$, show that $\mathbf{E}\,|R_n| \asymp \frac{n}{\log n}$.

▷ **Exercise 6.** For simple random walk on a transitive graph, $\lim_{n\to\infty} \frac{\mathbf{E}\,|R_n|}{n} = q := \mathbf{P}_o[\text{never return to } o]$.

▷ **Exercise 7.** Using the Carne-Varopolous bound, show that a $|B_n(x)| = o(n^2/\log n)$ volume growth in a bounded degree graph implies recurrence.

▷ **Exercise 8.** Recall Blackwell's proof for the Liouville property on $\mathbb{Z}^d$. By studying how badly the coupling may fail, show that any harmonic function f on $\mathbb{Z}^d$ with sublinear growth, i.e., with $\lim_{\|x\|_2\to\infty} f(x)/\|x\|_2 = 0$, must be constant.

▷ **Exercise 9.** Consider an irreducible Markov chain (V, P).

(a) Assume that $d_{\mathrm{TV}}\big(p_n(x,\cdot)), p_n(y,\cdot)\big) \to 0$ as $n \to \infty$, for any $x, y \in V$. Show that (V, P) has the Liouville property.

(b) Show that biased nearest-neighbor random walk on $\mathbb{Z}$ has the property of part (a), but nevertheless it does not have the strong Liouville property: it has non-constant positive harmonic functions.

The following exercise can be regarded as a discrete analogue of the classical theorem that functions satisfying the Mean Value Property are smooth. (Such a function cannot go up and down too much: whenever there is an edge contributing something to $\|\nabla f\|$, harmonicity carries this contribution far.)

▷ **Exercise 10** (Reverse Poincaré inequality)**.** Show that there is a constant $c = c(\Gamma, S) > 0$ such that for any harmonic function f on the Cayley graph $G(\Gamma, S)$,

$$c\,R\,\|\nabla f\|_{\ell^2(B_R)} \leq \|f\|_{\ell^2(B_{2R})}\,,$$

where B_R is the ball of radius R.

▷ **Exercise 11.** Can there exist a symmetric measure μ whose infinite support generates a finitely generated non-amenable group Γ such that the spectral radius is $\rho(\mu) = 1$?

▷ **Exercise 12.** A group having property (T) is "well-defined": if $\kappa(\Gamma, S_1) > 0$ then $\kappa(\Gamma, S_2) > 0$ for any pair of finite generating sets for Γ, S_1, S_2.

▷ **Exercise 13.**

(a) Give an $\mathrm{Aut}(\mathbb{Z}^2)$-invariant and $\mathbb{Z}^2$-ergodic percolation on $\mathbb{Z}^2$ with exactly two ∞ clusters.

(b) Give a deletion-tolerant version of part (a). (Hint: try deleting edges from the previous construction randomly with tiny probabilities.)

(c) Is there an ergodic deletion-tolerant $\mathbb{Z}^2$-invariant percolation on $\mathbb{Z}^2$ with infinitely many infinite clusters?

Similarly to generating functions in combinatorics, the partition functions of statistical physics contain a lot of information about the model. The first signs of this are the following:

▷ **Exercise 14.** Consider the Ising model with external magnetic field h on a finite graph $G(V, E)$:

$$H(\sigma, h) := -h \sum_{x\in V(G)} \sigma(x) + \sum_{(x,y)\in E(G)} \mathbf{1}_{\{\sigma(x)\neq\sigma(y)\}}\,.$$

(a) Show that the **expected total energy** is

$$\mathbf{E}_{\beta,h}[\,H\,] = -\frac{\partial}{\partial\beta}\ln Z_{\beta,h}\,,\ \text{with variance}\ \mathrm{Var}_{\beta,h}[H] = -\frac{\partial}{\partial\beta}\mathbf{E}_{\beta,h}[\,H\,]\,.$$

(b) The **average free energy** is defined by $f(\beta, h) := -(\beta|V|)^{-1}\ln Z_{\beta,h}$. Show that for the **total average magnetization** $M(\sigma) := |V|^{-1}\sum_{x\in V}\sigma(x)$, we have $\mathbf{E}_{\beta,h}[\,M\,] = -\frac{\partial}{\partial h}f(\beta, h)$.

▷ **Exercise 15.** On any transitive infinite graph, show that the Ising limit measures $\mathbf{P}^+_{\beta,h}$ and $\mathbf{P}^-_{\beta,h}$ are ergodic.