Stochastic Models — Second HW problem set

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Solve 6 of the 14 problems below by May 24. Beware: not all problems are of the same difficulty! You can ask me for help if you get stuck with something.

▷ **Exercise 1.** Consider simple random walk on the dumbbell graph: take two copies of the complete graph K_n, add a loop at each vertex (so that the degrees become n), except at one distinguished vertex in each copy, which will be connected to each other by an edge. Show that $d(1) = 1/2$, but $\tau_{\text{mix}}^{\text{TV}} \asymp n^2$. That is, in the definition of $\tau_{\text{mix}}^{\text{TV}}$, the $1/4$ cannot be replaced by $1/2$.

▷ **Exercise 2.** Consider lazy SRW on the cycle C_n. Show that for any $t > 0$ there exists $\delta_0(t), \delta_1(t) > 0$, such that, for any n, we have $\delta_0(t) < d(tn^2) < 1 - \delta_1(t)$. Moreover, show that one can achieve $\lim_{t\to 0} \delta_0(t) = 1$.

▷ **Exercise 3.** Recall that $d(t) \le \sup_{x,y} \big\{ \mathbf{P}[\, X_t \neq Y_t\,] : \text{all couplings of RWs with } X_0 = x, Y_0 = y \big\}$.

For lazy SRW on the cycle C_n, show by coupling that $\lim_{t\to\infty} \delta_1(t) = 1$ can also be achieved in the previous exercise, hence the mixing time is $O(n^2)$.

The following three exercises together prove that the total variation mixing time of the 1/2-lazy random walk $X_0, X_1, \dots$ on the hypercube $\{0,1\}^n$ is $(1/2 + o(1))\, n \log n$.

▷ **Exercise 4.** Let Y_t be the number of missing coupons at time t in the coupon collector's problem. Show that $\mathbf{E}\, Y_{\alpha\, n \log n} \sim n^{1-\alpha}$ and $\mathbb{D}\, Y_{\alpha\, n\log n} = o(n^{1-\alpha})$. Using Markov's and Chebyshev's inequalities, deduce that $Y_{\alpha\, n \log n}/\sqrt{n} \to 0$ or ∞ in probability, for $\alpha > 1/2$ and $< 1/2$, respectively.

▷ **Exercise 5.** Show that $d_{\text{TV}}\big(\mathsf{N}(0,1), \mathsf{N}(x,1)\big) \to 0$ or 1, for $x \to 0$ and $x \to \infty$, respectively, where $\mathsf{N}(\mu, \sigma^2)$ is the normal distribution. Using this and the local version of the de Moivre–Laplace theorem, prove that $d_{\text{TV}}\big(\mathsf{Binom}(n, 1/2), \mathsf{Binom}(n - n^\beta, 1/2) + n^\beta\big) \to 0$ for any fixed $\beta < 1/2$, while $\to 1$ for $\beta > 1/2$.

▷ **Exercise 6.**

(a) For $X_0 = (0, 0, \dots, 0) \in \{0,1\}^n$, let the distribution of X_t be μ_t. What is it, conditioned on $\|X_t\|_1 = k$? And what is the distribution of $\|Z\|_1$, where Z has distribution π, uniform on $\{0,1\}^n$?

(b) Deduce from part (a) and the previous exercises that $d_{\text{TV}}\big(\mu_{\alpha\, n \log n}, \pi\big) \to 0$ or 1, for $\alpha > 1/2$ and $< 1/2$, respectively.

▷ **Exercise 7.** Consider a reversible Markov chain P on a finite state space V with stationary distribution π and absolute spectral gap g_{abs}. This exercise explains why $\tau_{\text{relax}} = 1/g_{\text{abs}}$ is called the relaxation time.

For $f : V \longrightarrow \mathbb{R}$, let $\text{Var}_\pi[f] := \mathbf{E}_\pi[f^2] - (\mathbf{E}_\pi f)^2 = \sum_x f(x)^2 \pi(x) - \big(\sum_x f(x)\pi(x)\big)^2$. Show that $g_{\text{abs}} > 0$ implies that $\lim_{t\to\infty} P^t f(x) = \mathbf{E}_\pi f$ for all $x \in V$. Moreover,

$$\text{Var}_\pi[P^t f] \le (1 - g_{\text{abs}})^{2t}\, \text{Var}_\pi[f]\,,$$

with equality at the eigenfunction corresponding to the λ_i giving $g_{\text{abs}} = 1 - |\lambda_i|$. Hence τ_{relax} is the time needed to reduce the standard deviation of any function to $1/e$ of its original standard deviation.

▷ **Exercise 8.** This exercise explains why it is hard to construct large expanders. A *covering map* $\varphi : G' \longrightarrow G$ between graphs is a surjective graph homomorphism that is locally an isomorphism: denoting by $N_G(v)$ the subgraph induced by $v \in G$ and all its neighbours, we require that each connected component of the subgraph of G' induced by the full inverse image $\varphi^{-1}(N_G(v))$ be isomorphic to $N_G(v)$.

(a) If $G' \longrightarrow G$ is a covering map of infinite graphs, then the spectral radii satisfy $\rho(G') \le \rho(G)$, i.e., the larger graph is more non-amenable. In particular, if G is an infinite k-regular graph, then $\rho(G) \ge \rho(\mathbb{T}_k) = \frac{2\sqrt{k-1}}{k}$. (Hint: use the return probability definition of $\rho(G)$.)

(b) If $G' \longrightarrow G$ is a covering map of finite graphs, then $\lambda_2(G') \ge \lambda_2(G)$, i.e., the larger graph is a worse expander. (Hint: eigenfunctions on G can be "lifted" to G'.

Let G_n be a sequence of finite graphs. Pick a uniform random root ρ_n from $V(G_n)$, and take the ball $B_{G_n,\rho_n}(r)$ around it in the graph metric, with some fixed radius $r \in \mathbb{Z}_+$. We get a distribution $\mu_{n,r}$ on finite rooted graphs. We say that the sequence $\{G_n\}$ converges in the **Benjamini-Schramm sense** (also called **local weak convergence**) to a random rooted graph (G, ρ), if, for every r, the distributions $\mu_{n,r}$ converge weakly as $n \to \infty$ to the distribution of $B_{G,\rho}(r)$. The simplest case is when the limit is a transitive infinite graph G: the measures $\mu_{n,r}$ converge to the Dirac measure on a single graph, the r-ball of G. The following exercise generalizes the examples of boxes in $\mathbb{Z}^d$ and balls in the d-regular tree $\mathbb{T}_d$ that we saw on class:

▷ **Exercise 9.** Show that a transitive graph G has a sequence G_n of *subgraphs* converging to it in the local weak sense iff it is amenable.

▷ **Exercise 10.** The following is a simple model for a random d-regular bipartite (multi-)graph: take d independent uniform random permutations $\pi_i : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$, then take all the edges $\big\{(j, n + \pi(j)) : j \in \{1, \dots, n\},\ i \in \{1, \dots, d\}\big\}$. Show that this random graph converges to the d-regular tree $\mathbb{T}_d$ in the local weak sense. (Here the randomness for the measure $\mu_{n,r}$ comes from two sources: we take a random root ρ_n in the random graph G_n, and want to show convergence in this joint probability space.)

The phenomenon is the same as in the previous exercise, but the computation is a bit simpler:

▷ **Exercise 11.** Show that for any $\lambda \in \mathbb{R}_+$, the local weak limit of the Erdős-Rényi random graphs $G(n, \lambda/n)$ is the $\mathsf{PGW}(\lambda)$ tree: the Galton-Watson tree with $\mathsf{Poisson}(\lambda)$ offspring distribution, rooted as normally. (Again convergence in the joint probability space. It is also true, but would be a bit harder to show that $G(n, \lambda/n)$ is with high probability such that the neighbourhood statistics is close to the one in $\mathsf{PGW}(\lambda)$.)

▷ **Exercise 12.** Give an ergodic invariant site percolation on some transitive infinite graph such that if we take the union of the open sites in two independent copies, the resulting percolation is not ergodic.

An invariant percolation $\mathbf{P}$ on an infinite transitive graph G is called **mixing** if, for any two events A, B, and $\epsilon > 0$, there is a finite set $K \subset \mathrm{Aut}(G)$ such that if $\gamma \notin K$, then $\big|\mathbf{P}[\,A \cap \gamma(B)\,] - \mathbf{P}[\,A\,]\mathbf{P}[\,B\,]\big| < \epsilon$.

▷ **Exercise 13.** Consider the following site percolation on $\mathbb{Z}$: let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be fixed, $U \sim \mathsf{Unif}[0, 1]$ random, and let $\omega_n := 1$ if $U + n\alpha \pmod 1$ is in $[0, 1/2)$, and $\omega_n := 0$ otherwise. Show that this is $\mathbb{Z}$-invariant, ergodic, but not mixing.

▷ **Exercise 14.** Consider the Ising model with external magnetic field h on a finite graph $G(V, E)$:

$$H(\sigma, h) := -h \sum_{x \in V(G)} \sigma(x) + \sum_{(x,y) \in E(G)} \mathbf{1}_{\{\sigma(x) \neq \sigma(y)\}}\,.$$

(a) Show that the **expected total energy** is

$$\mathbf{E}_{\beta,h}[\,H\,] = -\frac{\partial}{\partial \beta} \ln Z_{\beta,h}\,, \text{ with variance } \mathrm{Var}_{\beta,h}[H] = -\frac{\partial}{\partial \beta} \mathbf{E}_{\beta,h}[\,H\,]\,.$$

(b) The **average free energy** is defined by $f(\beta, h) := -(\beta|V|)^{-1} \ln Z_{\beta,h}$. Show that for the **total average magnetization** $M(\sigma) := |V|^{-1} \sum_{x \in V} \sigma(x)$, we have $\mathbf{E}_{\beta,h}[\,M\,] = -\frac{\partial}{\partial h} f(\beta, h)$.