Why the hydrodynamic limit of TASEP is Burgers' equation

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This is a somewhat informal account of the main ideas, based on hand-written notes of Márton Balázs.

1 Definitions

In the **Asymmetric Simple Exclusion Process** on the cycle $\mathbb{Z}_n$ or the line $\mathbb{Z}$, particles have independent Poisson clocks, and when the clock of a particle rings, it jumps to the right with probability p and to the left with probability $q = 1-p$, provided that there is no other particle at the target vertex. In the **Totally Asymmetric Simple Exclusion Process**, $p = 1$. This is what we will consider from now on, although simple modifications of the argument would also give the more general case.

On $\mathbb{Z}$, if we have infinitely many particles, then it needs a proof that this process is well-defined: in any small time interval $[t, t+\epsilon]$, there are infinitely many clocks ringing, there is no first ring, so, in order to decide if a certain jump can be made (whether the target vertex is empty), we might need to trace back the history of jumps infinitely long, which is impossible. The proof that this cascade of information coming from infinity does not happen relies on the fact that, almost surely, simultaneously for all $t, \epsilon > 0$, the particles whose clocks ring in the time interval $[t, t+\epsilon]$ are broken into finite segments by at least length 2 segments without any clocks ringing. This statement is not surprising, but not completely trivial to prove (because we need it simultaneously for continuum many t and ϵ), and I will now skip its verification.

Continuous time Markov processes are often described using their **infinitesimal generator**: if Ω is the state space, $\varphi : \Omega \longrightarrow \mathbb{R}$ a function, and $\{\omega(t)\}_{t\geq 0}$ is the Markov process, then

$$L\varphi(\omega) := \lim_{\epsilon\to 0} \frac{\mathbf{E}\big[\, \varphi(\omega(\epsilon)) \;\big|\; \omega(0) = \omega \,\big] - \varphi(\omega)}{\epsilon}\,.$$

Often, one can identify the action of L not on every function on X, but only on a suitably dense subspace of functions, and hope that this contains all the information needed.

In the case of TASEP, the state space is $\Omega = \{0,1\}^{\mathbb{Z}}$, where $\omega_i = 1$ in a configuration $\omega \in \Omega$ means that there is a particle at $i \in \mathbb{Z}$. We can restrict our attention to functions φ that depend only finitely many coordinates ω_i, say $i \in \{a, a+1, \dots, b\}$. By the basic properties of the Poisson point process, for very small ϵ, the probability that the clock of site i rings in $[0, \epsilon]$ is close to ϵ, and the probability that there are more than one sites in $\{a-1, a, \dots, b\}$ that ring is at most $C_{a,b}\, \epsilon^2$. Therefore,

$$L\varphi(\omega) = \sum_{i=a-1}^{b} \left[\varphi\big(\dots, \omega_{i-1}, \overset{\overset{i}{\smile}}{0}, \overset{\overset{i+1}{\smile}}{1}, \omega_{i+2}, \dots\big) - \varphi\big(\dots, \omega_{i-1}, \overset{\overset{i}{\smile}}{1}, \overset{\overset{i+1}{\smile}}{0}, \omega_{i+2}, \dots\big) \right] (1-\omega_{i+1})\, \omega_i\,.$$

2 Stationary measures

We want to show that the Bernoulli(p) product measures, for any density $p \in [0,1]$, are stationary. There are no other ergodic stationary distributions, but we will not prove that.

Let $\omega(0)$ be distributed according to Bernoulli(p) product measure. We want to show that the distribution of ω_t is the same. This is equivalent to $\mathbf{E}\varphi(\omega(t)) = \mathbf{E}\varphi(\omega(0))$ for all finitely supported φ, hence we just need to show that $\mathbf{E}_\omega L\varphi(\omega) = 0$, where the notation means that the expectation is taken over ω.

Note that the first term of $L\varphi(\omega)$ can be rewritten as $\sum_i \varphi(\omega)(1-\omega_i)\,\omega_{i+1}$, while the second term is just $-\sum_i \varphi(\omega)(1-\omega_{i+1})\,\omega_i$. Therefore,

$$\begin{aligned}
\mathbf{E}_\omega L\varphi(\omega) &= \mathbf{E}_\omega \sum_{i=a-1}^{b} \varphi(\omega)\Big[(1-\omega_i)\,\omega_{i+1} - (1-\omega_{i+1})\,\omega_i\Big] \\
&= \mathbf{E}_\omega \sum_{i=a-1}^{b} \varphi(\omega)\big[\omega_{i+1} - \omega_i\big] \\
&= \mathbf{E}_\omega \Big[\varphi(\omega)(\omega_{b+1} - \omega_{a-1})\Big] = \mathbf{E}_\omega\varphi(\omega)\cdot \mathbf{E}_\omega\big[\omega_{b+1} - \omega_{a-1}\big] = \mathbf{E}_\omega\varphi(\omega)\cdot(p-p) = 0\,,
\end{aligned}$$

where we used the independence of $\varphi(\omega)$ from ω_{b+1} and ω_{a-1} to get the factorization of the expectations in the last line.

3 Hydrodynamic limit

Our goal is to look at the movement of the collection of particles from far away, so that in the limit it becomes the movement of a fluid, and to find some equation(s) that determine the evolution of the density of this fluid. Since the microscopic particles move randomly, the macroscopic density evolution could a priori be random, as well. Nevertheless, for the present case, it will turn out to be deterministic.

The **density profile** will be $\rho : \mathbb{R}\times\mathbb{R}_{\geq 0} \longrightarrow [0,1]$; the first coordinate $X \in \mathbb{R}$ stands for space, the second coordinate $T \in \mathbb{R}_{\geq 0}$ stands for time. The starting density $\rho(\cdot, 0)$, a continuous function, will be approximated on the lattice $\epsilon\mathbb{Z}$, using randomly placed tiny ϵ-particles: we set $\omega^\epsilon_{\lfloor X/\epsilon\rfloor}(0)$ to be 1 with probability $\rho(X,0)$, and 0 otherwise, independently from other choices. Due to the Law of Large Numbers, if $\delta > 0$ is so small that $\rho(\cdot,0)$ is basically constant on $[X, X+\delta]$, and we take $\epsilon > 0$ much smaller than δ, then the density of ϵ-particles in $[X, X+\delta]\cap\epsilon\mathbb{Z}$ will be close to $\rho(X,0)$ with large probability. That is, this initial random microscopic configuration is a good approximation to the initial macroscopic density profile.

Now, we need to scale not only space, but also need to speed up time, since it should not take for an ϵ-particle a constant order of time to try and jump to the neighbouring ϵ-position, only ϵ. Therefore, we will use the time scaling $t = T/\epsilon$ to get the microscopic time t, and hope that the random configuration $\omega^\epsilon_{\lfloor X/\epsilon\rfloor}(T/\epsilon)$ approximates well a nice macroscopic profile $\rho(X,T)$.

Using the above calculations for $L\varphi$, now for the function $\varphi(\omega) = \omega_i$ and any configuration $\omega(t)$, we get

$$L\omega_i(t) = \omega_{i-1}(t)\,\big(1-\omega_i(t)\big) - \omega_i(t)\,\big(1-\omega_{i+1}(t)\big)\,.$$

We will use this for $\omega_i(t) = \omega^\epsilon_{\lfloor X/\epsilon\rfloor}(T/\epsilon)$, and want to take expectation over the starting configuration and the randomness in the dynamics: $\frac{d}{dt}\mathbf{E}\omega_i(t) = \mathbf{E}L\omega_i(t)$. As we already used above, the continuity of $\rho(\cdot,0)$ implies that the ϵ-particles are locally distributed according to a Bernoulli product measure of some almost constant density, which is almost stationary. Hence it is reasonable to assume that the states of neighbouring positions remain almost independent from each other as time evolves, and hence, using $\frac{d}{dt} = \epsilon \frac{d}{dT}$,

$$\frac{d}{dT}\mathbf{E}\omega^{\epsilon}_{\lfloor X/\epsilon \rfloor}(T/\epsilon) = \frac{1}{\epsilon}\mathbf{E}L\omega^{\epsilon}_{\lfloor X/\epsilon \rfloor}(T/\epsilon) \sim \frac{\rho(X-\epsilon,T)\big(1-\rho(X,T)\big) - \rho(X,T)\big(1-\rho(X+\epsilon,T)\big)}{\epsilon} .$$

Letting $\epsilon \to 0$, the left hand side should converge to $\frac{d}{dT}\rho(X,T)$, while some simple calculus (involving adding and subtracting terms with $\rho(X-\epsilon,T)$, $\rho(X,T)$ and $\rho(X+\epsilon,T)$) shows that the right hand side should converge to $-\frac{d}{dX}H(\rho(X,T))$, where $H(\rho) = \rho(1-\rho)$.

That is, assuming that the microscopic densities converge to macroscopic density profiles as they should, this limit profile satisfies the **inviscid Burgers' equation**

$$\frac{d}{dT}\rho(X,T) + \frac{d}{dX}H(\rho(X,T)) = 0, \quad \text{where } H(\rho) = \rho(1-\rho) .$$

There should be enough independence in the system that a Law of Large Numbers type argument implies that this convergence indeed takes place, not only in expectation, but also in probability: for any smooth test function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ of bounded support, and any $\delta > 0$ and $T \geq 0$, we have

$$\lim_{\epsilon \to 0} \mathbf{P}\Big[\Big| \sum_{i \in \mathbb{Z}} \omega^{\epsilon}_i(T/\epsilon)\, \varphi(i\epsilon) - \int_{\mathbb{R}} \rho(T,X)\, \varphi(X)\, dX \Big| > \delta \Big] = 0 .$$