Dynamical sensitivity of critical planar percolation, and the Incipient Infinite Cluster

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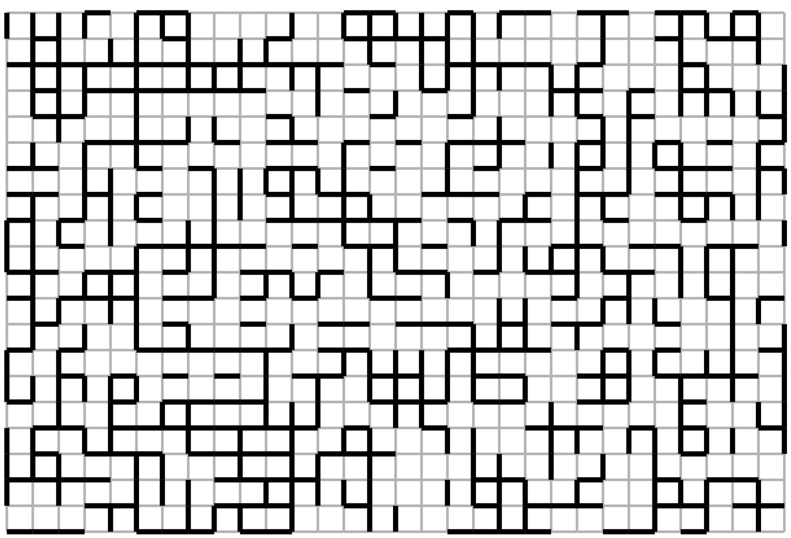
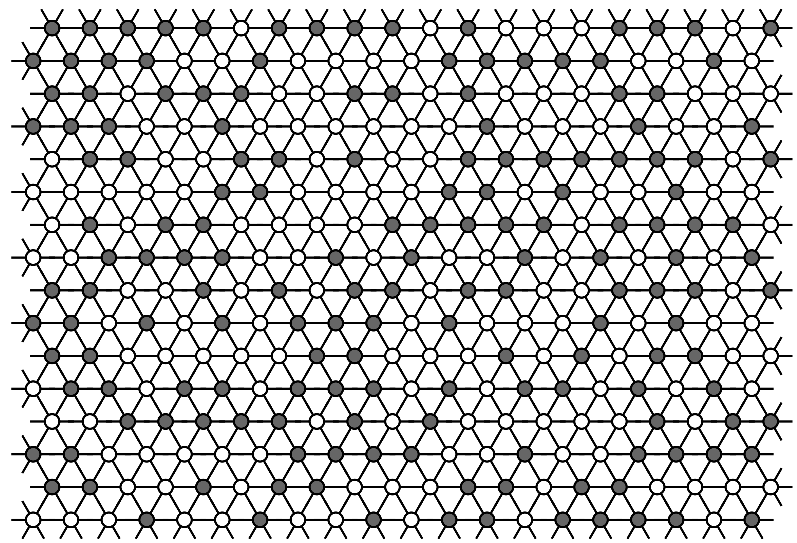
Plan of the talk

- Critical percolation: RSW, conformal invariance, SLE_6 exponents.
- Noise sensitivity of critical percolation.
- Dynamical percolation.
- Why is the Fourier spectrum useful?
- Exceptional times and the IIC.
- The Fourier spectrum of critical percolation. Strategy of proof.
- Further results and questions.

Bernoulli(p) site and bond percolation

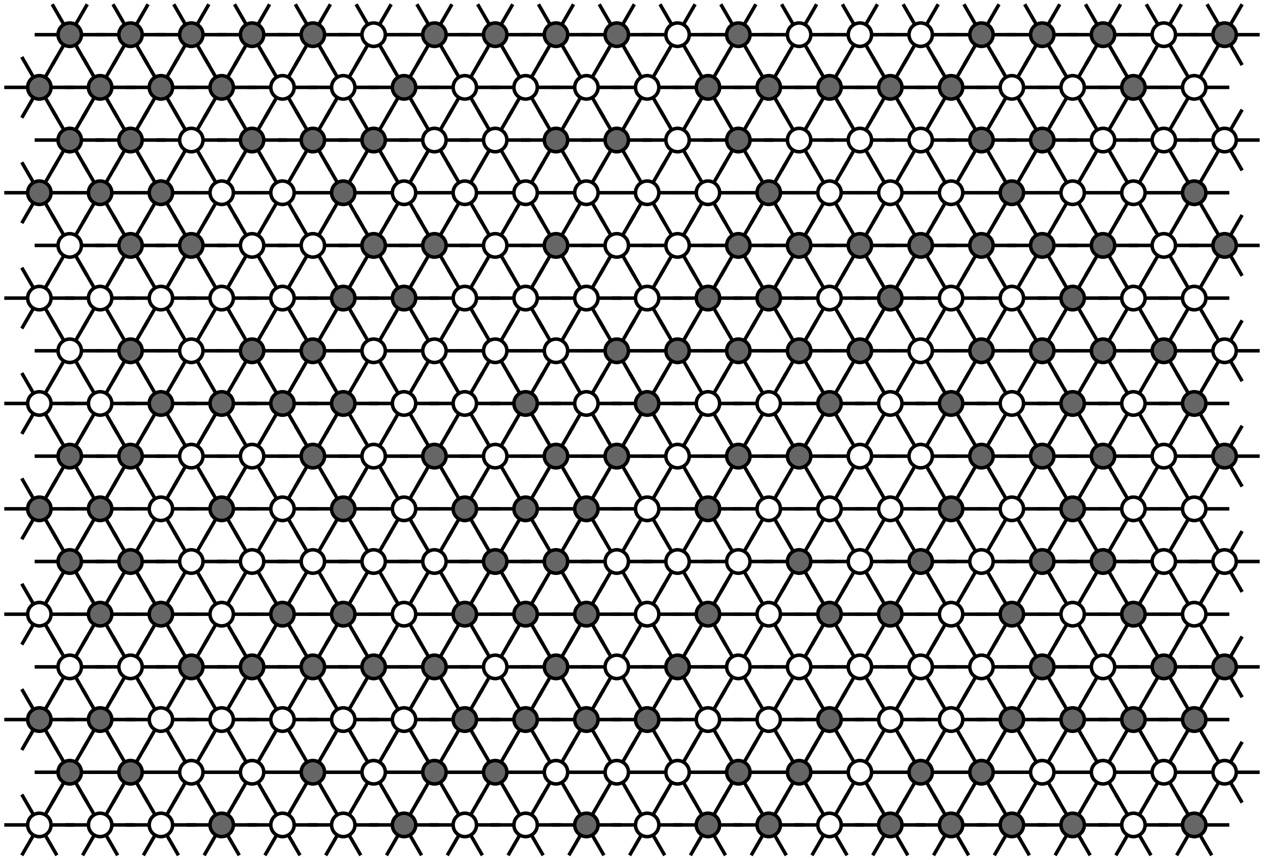
Given an (infinite) graph $G = (V, E)$ and $p \in [0, 1]$. Each site (or bond) is chosen open with probability p, closed with $1 - p$, independently of each other. Consider the open connected clusters.





Theorem (Harris 1960 and Kesten 1980).
$p_c(\mathbb{Z}^2, \text{bond}) = p_c(\Delta, \text{site}) = 1/2$, and $\theta(1/2) = 0$.
For $p > 1/2$, there is a.s. one infinite cluster.

Site percolation on triangular grid Δ
= face percolation on hexagonal grid:



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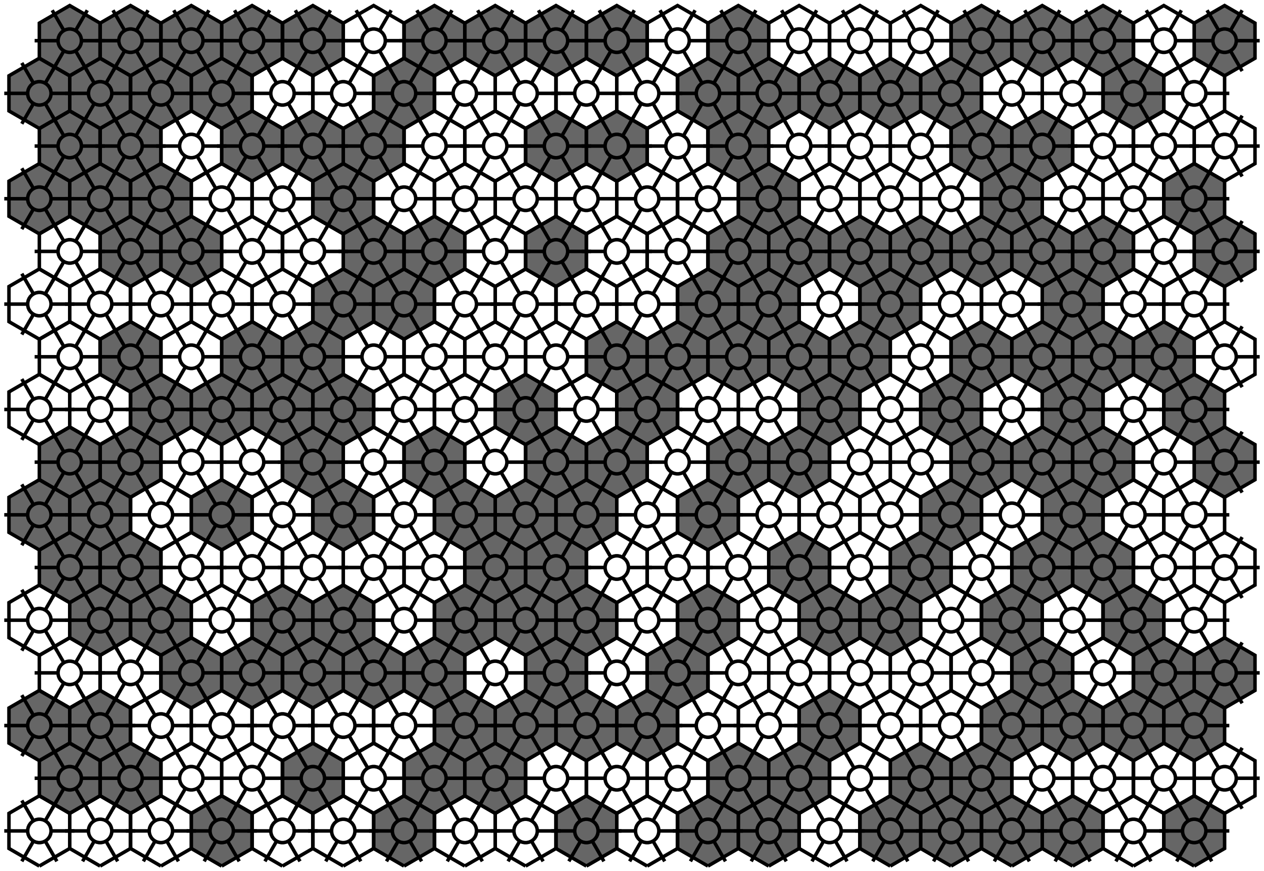

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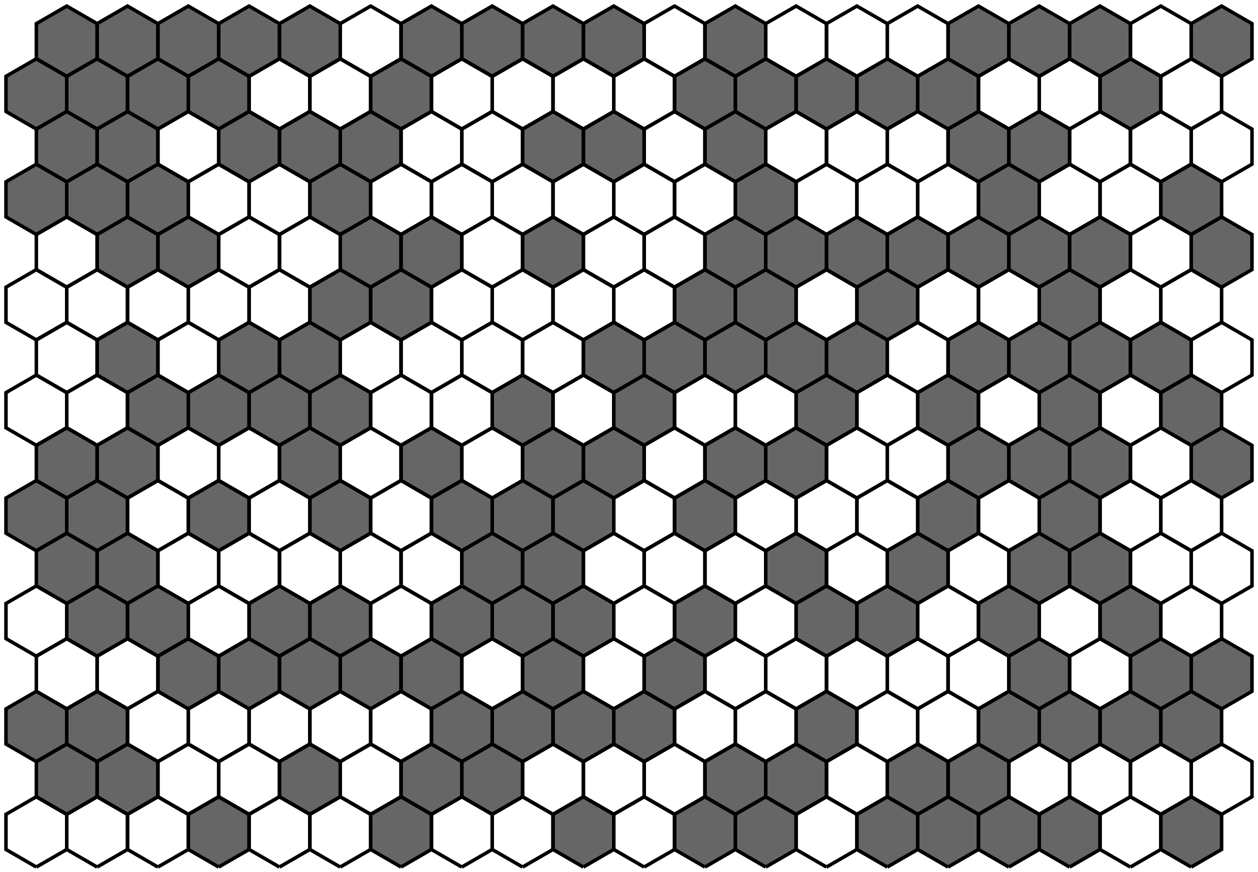
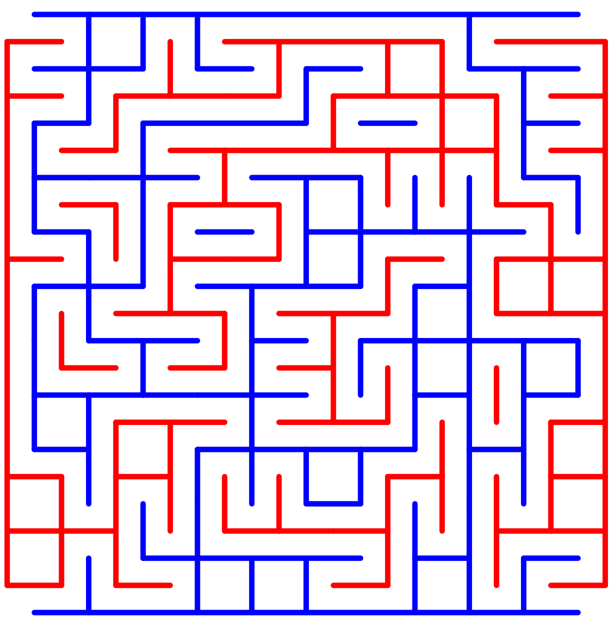
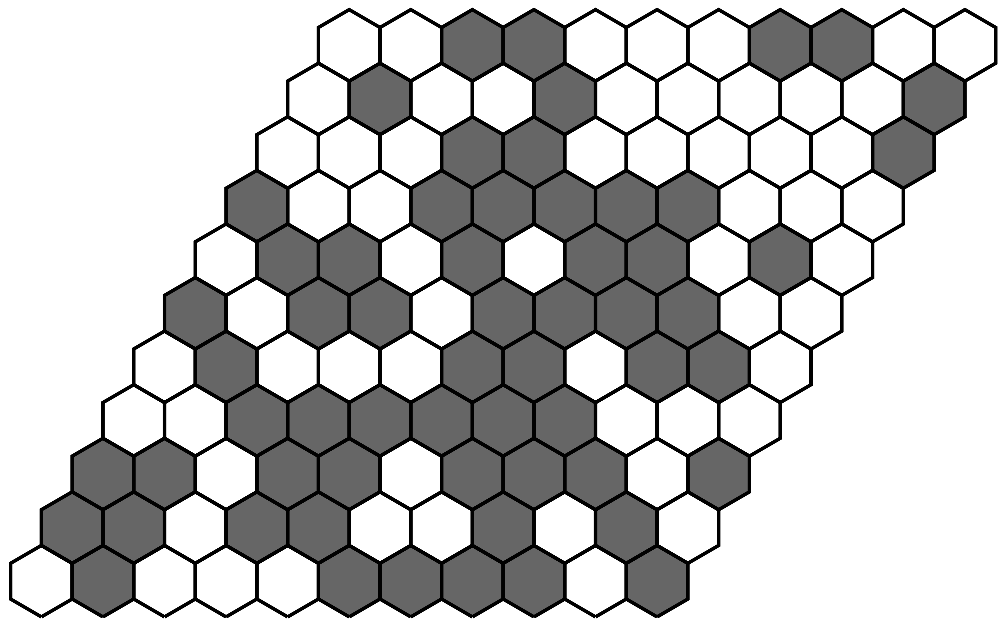


Why is $p_c = 1/2$? Duality!

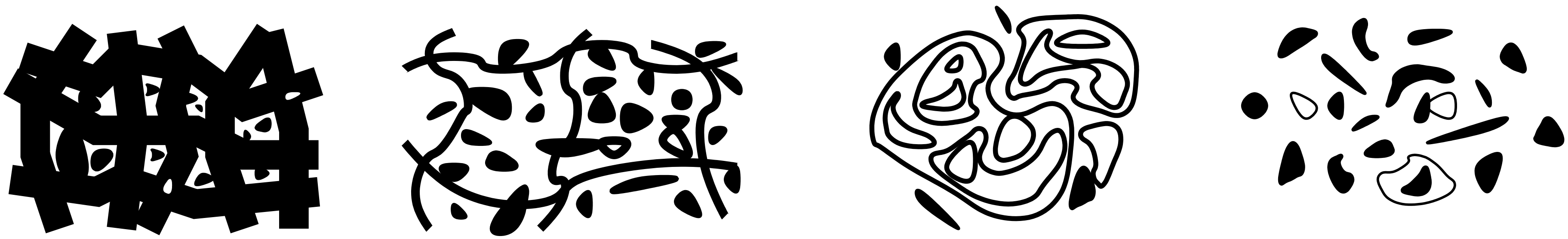
$\mathbb{Z}^2$ bond percolation at $p = 1/2$: in an $(n+1) \times n$ rectangle, left-right crossing has probability exactly $1/2$.
For site percolation on Δ, same on an $n \times n$ rhombus.





Crossing probabilities and criticality

$p \approx 0.8$ $\quad$ $p \approx 0.55$ $\quad$ $p = 0.5$ $\quad$ $p \approx 0.45$



Theorem (Russo 1978 and Seymour-Welsh 1978). In critical percolation on any planar lattice, for $L, n > 0$,

$$0 < a_L < \mathbf{P}[\,\text{left-right crossing in } n \times Ln\,] < b_L < 1.$$

Same holds for annulus-crossings. It follows easily that

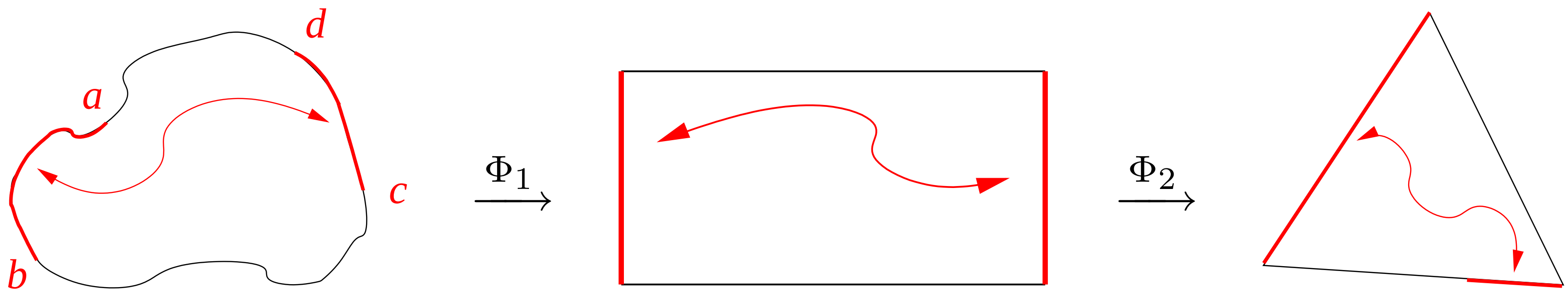
$$(r/R)^{\alpha} < \mathbf{P}[\,\partial B_r \longleftrightarrow \partial B_R\,] < (r/R)^{\beta}.$$

Conformal invariance on Δ

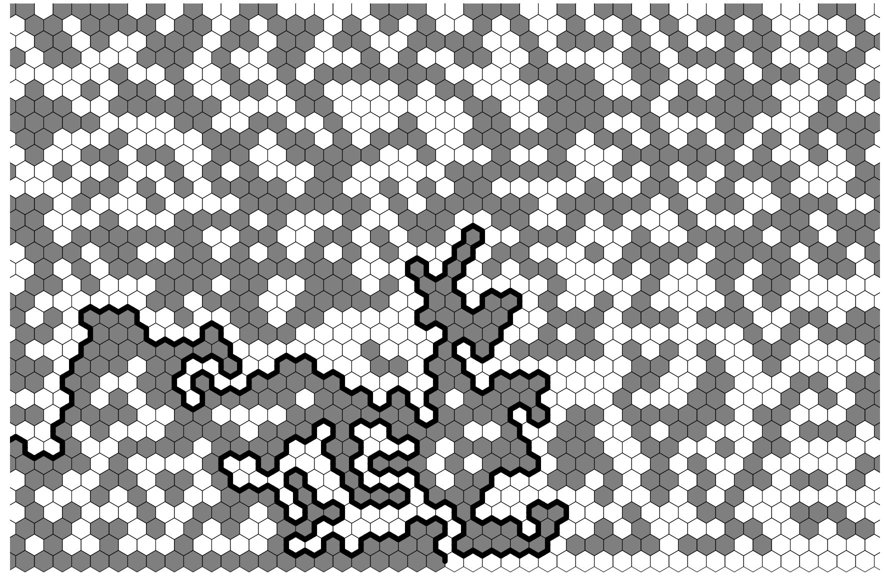
Theorem (Smirnov 2001). For p=1/2 bond percolation on Δ_ϵ, and $D \subset \mathbb{R}$ simply connected domain with four boundary points $\{a, b, c, d\}$,

$$\lim_{\epsilon \to 0} \mathbf{P}\Big[ab \longleftrightarrow cd \text{ inside the discrete approximation } D_\epsilon \Big]$$

exists, is strictly between 0 and 1, and is conformally invariant.



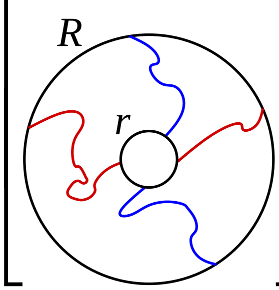
SLE_6 exponents



Given the conformal invariance, the exploration path converges to the Stochastic Loewner Evolution with $\kappa = 6$ (Schramm 2000).

Using the SLE_6 curve, several critical exponents can be computed (Lawler-Schramm-Werner, Smirnov-Werner 2001), e.g.:

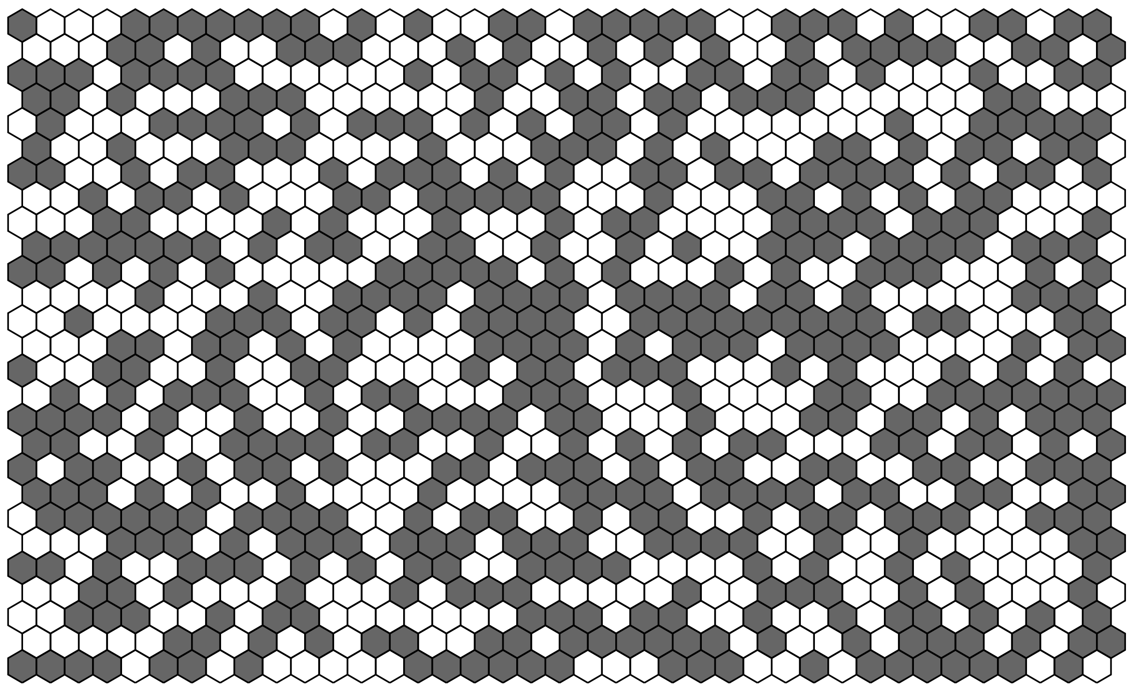
$$\alpha_4(r, R) := \mathbf{P} \left[\quad \right] = (r/R)^{5/4+o(1)},$$



while $\alpha_1(r, R) = (r/R)^{5/48+o(1)}$ and $\mathbf{P}_{p_c+\epsilon}[0 \longleftrightarrow \infty] = \epsilon^{5/36+o(1)}$.

Percolation and noise

Take an ω critical percolation configuration. Let ω^ϵ be a new configuration, where each site (or bond) is resampled with probability ϵ, independently. (The ϵ-noised version of ω.)



For how large an ϵ can we still predict from ω whether there is a left-right crossing in ω^ϵ?

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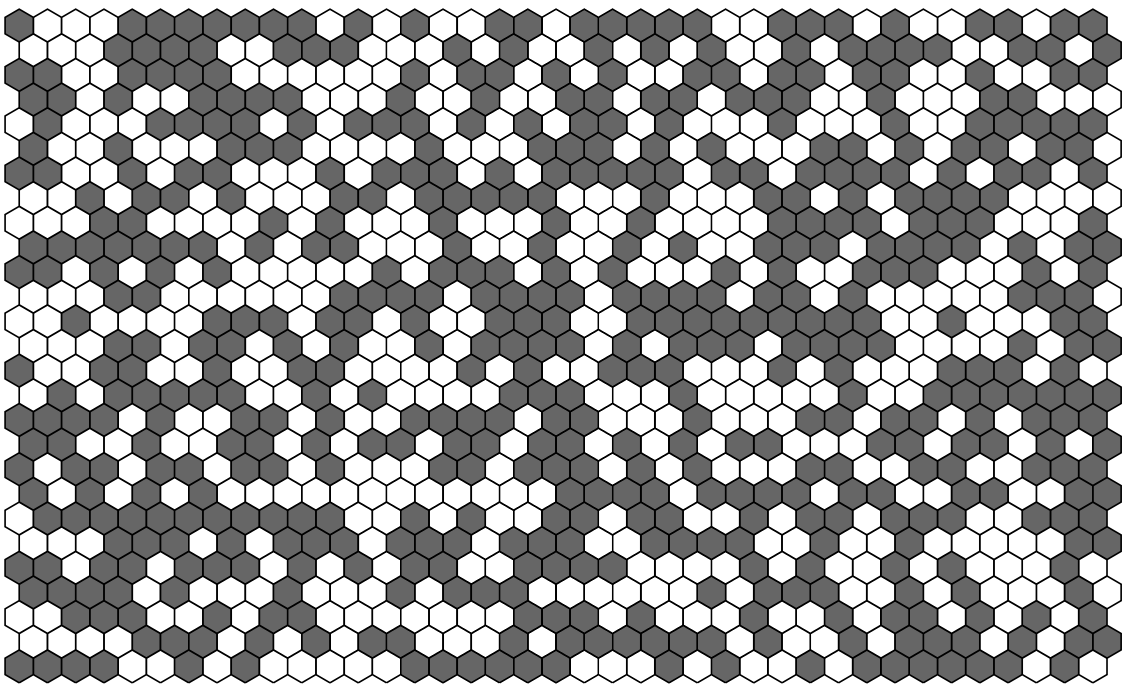


For how large an ϵ can we still predict from ω whether there is a left-right crossing in ω^ϵ?

Noise sensitivity of percolation

Theorem (Benjamini, Kalai & Schramm 1998). If $\epsilon > 0$ is fixed, and f_n is the indicator function for a left-right percolation crossing in an $n \times n$ square, then as $n \to \infty$

$$\mathbf{E}\big[\, f_n(\omega)\, f_n(\omega^\epsilon)\,\big] - \mathbf{E}\big[\, f_n(\omega)\,\big]^2 \to 0.$$

This holds for all $\epsilon = \epsilon_n > c/\log n$.

Theorem (Steif & Schramm 2005). Same if $\epsilon_n > n^{-a}$ for some positive $a > 0$. If triangular lattice, may take any $a < 1/8$.

Theorem (Garban, P & Schramm 2008). Same holds if and only if $\epsilon_n\, \mathbf{E}\big[\, |\mathrm{pivotals}|\,\big] \to \infty$. For triangular lattice, this threshold is $\epsilon_n = n^{-3/4+o(1)}$.

Naive idea: how many pivotals are there?

A site (or bond) is pivotal in ω, if flipping it changes the existence of a left-right crossing. Expected number: $\mathbf{E}|\mathrm{Piv}_n| \asymp n^2\,\alpha_4(n) \quad (= n^{3/4+o(1)})$.

Second moment: $\mathbf{E}[\,|\mathrm{Piv}_n|^2\,] \leqslant C\,(\mathbf{E}|\mathrm{Piv}_n|)^2$.

From Chebyshev: $\mathbf{P}\big[\,|\mathrm{Piv}_n| > \lambda\mathbf{E}|\mathrm{Piv}_n|\,\big] < C/\lambda^2$ for all $\lambda > 0$-ra.

From Cauchy-Schwarz: $\exists\delta > 0$, s.t. $\mathbf{P}\big[\,|\mathrm{Piv}_n| > \delta\mathbf{E}|\mathrm{Piv}_n|\,\big] > \delta$.

Moreover: $\mathbf{P}\big[\,0 < |\mathrm{Piv}_n| < \lambda\mathbf{E}|\mathrm{Piv}_n|\,\big] \asymp \lambda^{11/9+o(1)}$, if $\lambda < 1$ (on Δ).

Cannot have many pivotals. $\Longrightarrow$ If $\epsilon_n\mathbf{E}[\,|\mathrm{Piv}_n|\,] \to 0$, then we don't hit any pivotals. $\Longrightarrow$ Asymptotically full correlation.

Cannot have few pivotals. $\Longrightarrow$ If $\epsilon_n\mathbf{E}[\,|\mathrm{Piv}_n|\,] \to \infty$, then we do hit many pivotals. But this $\not\Longrightarrow$ asymptotic independence!

Dynamical percolation

Each variable is resampled according to an independent Poisson(1) clock. This is a Markov process $\{\omega(t) : t \in [0, \infty)\}$, in which $\omega(t+s)$ is an ϵ-noised version of $\omega(t)$, with $\epsilon = 1 - \exp(-s)$.

An exceptional time is such a (random) t, at which an almost sure property of the static process fails for $\omega(t)$.

Main example: (Non-)existence of an infinite cluster in percolation.

Toy example: Brownian motion on the circle does sometimes hit a fix point, as opposed to its static version: a uniform random point.

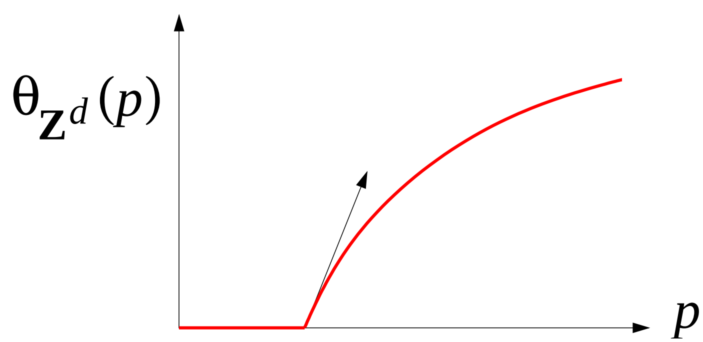
In this toy example, the set of exceptional times is a random Cantor set of Lebesgue measure zero (because of Fubini) and Hausdorff-dimension $1/2$.

Dynamical percolation results

Theorem (Häggström, Peres & Steif 1997).

- No exceptional times when $p \neq p_c$.
- No exceptional times when $p = p_c$ for bond percolation on $\mathbb{Z}^d$, $d \geqslant 19$.

The second fact is essentially due to:



Theorem (Steif & Schramm 2005).

- There are exceptional times (a.s.) for critical site percolation on the triangular lattice.
- They have Hausdorff dimension in $[1/6, 31/36]$.

Theorem (Garban, P & Schramm 2008).

- There are exceptional times also on $\mathbb{Z}^2$.
- On the triangular grid they have Hausdorff dimension $31/36$.
- On the triangular grid, there are exceptional times with an infinite white **and** an infinite black cluster simultaneously. (Dim$= 2/3$.)

What is the Fourier spectrum and why is it useful?

$f_n : \{\pm 1\}^{V_n} \leftrightarrow \{\pm 1\}$ indicator function of left-right crossing. $\mathbf{E}[\, f_n \,] = 0$.

$(N_\epsilon f)(\omega) := \mathbf{E}[\, f(\omega^\epsilon) \mid \omega \,]$ is the noise operator, acting on the space $L^2(\Omega, \mu)$, where $\Omega = \{\pm 1\}^{V_n}$, μ uniform measure, inner product $\mathbf{E}[\, fg \,]$.

Correlation: $\mathbf{E}[\, f(\omega^\epsilon) f(\omega) \,] - \mathbf{E}[\, f(\omega) \,]\mathbf{E}[\, f(\omega^\epsilon) \,] = \mathbf{E}[\, f(\omega) N_\epsilon f(\omega) \,] - \mathbf{E}[\, f(\omega) \,]^2$. So, we would like to diagonalize the noise operator N_ϵ.

Let χ_v be the function $\chi_v(\omega) = \omega(v)$, $\omega \in \Omega$.

For $S \subset V_n$, let $\chi_S := \prod_{v \in S} \chi_v$, the parity inside S. Then

$$N_\epsilon \chi_v = (1-\epsilon)\, \chi_v \, ; \qquad N_\epsilon \chi_S = (1-\epsilon)^{|S|}\, \chi_S .$$

Moreover, the family $\{\chi_S,\, S \subset V\}$ is an orthonormal basis of $L^2(\Omega, \mu)$.

Any function $f \in L^2(\Omega, \mu)$ in this basis (Fourier-Walsh series):

$$\hat{f}(S) := \mathbf{E}[\, f \chi_S \,]\,; \qquad f = \sum_{S \subset V} \hat{f}(S)\, \chi_S\,.$$

The correlation:

$$\mathbf{E}[\, f N_\epsilon f \,] - \mathbf{E}[\, f \,]^2 = \sum_S \sum_{S'} \hat{f}(S)\, \hat{f}(S')\, \mathbf{E}\big[\, \chi_S\, N_\epsilon \chi_{S'} \,\big] - \mathbf{E}[\, f \chi_\emptyset \,]^2$$

$$= \sum_{\emptyset \neq S \subset V} \hat{f}(S)^2\, (1-\epsilon)^{|S|} = \sum_{k=1}^{|V_n|} (1-\epsilon)^k \sum_{|S|=k} \hat{f}(S)^2.$$

May consider the associated "spectral measure" $\nu_f(S) := \hat{f}(S)^2$. For the crossing function, $\mathbf{E}[\, f_n^2 \,] = 1$, so Parseval says this is a probability measure. A random sample $\mathscr{S}_n = \mathscr{S}(f_n) \subset V_n$ is a strange random set of bits.

If $\nu\big[|\mathscr{S}_n| < t k_n\big] \to 0$ as $t \to 0$, then $(1-\epsilon)^k \sim \exp(-\epsilon k)$ implies that for $\epsilon_n \gg 1/k_n$ we have $\mathbf{E}[\, f_n\, N_\epsilon f_n \,] - \mathbf{E}[\, f_n \,]^2 \to 0$, asymptotic independence.

Proving existence of exceptional times

Second Moment Method:

Let $Q_R := \{t \in [0,1] : 0 \longleftrightarrow R\}$ and $Z_R := Leb(Q_R)$.

$$\mathbf{P}\big[\, Q_R \neq \emptyset \,\big] = \mathbf{P}\big[\, Z_R > 0 \,\big] \geqslant \frac{\mathbf{E}[\, Z_R \,]^2}{\mathbf{E}\big[\, Z_R^2 \,\big]}\,.$$

$\mathbf{E}[\, Z_R \,] = \int_0^1 \mathbf{P}[\, 0 \longleftrightarrow_t R \,]\, dt = \mathbf{P}[\, 0 \longleftrightarrow R \,]$.

$\mathbf{E}\big[\, Z_R^2 \,\big] = \int_0^1 \int_0^1 \mathbf{P}[\, 0 \longleftrightarrow_t R,\, 0 \longleftrightarrow_s R \,]\, ds\, dt \asymp \int_0^1 \mathbf{E}\big[\, f_R(\omega_0)\, f_R(\omega_s) \,\big]\, ds$.

Thus we again want to estimate the correlation $\mathbf{E}\big[\, f_R(\omega_0)\, f_R(\omega_s) \,\big] = \mathbf{E}\big[\, f_R\, T_s f_R \,\big]$ from above, where

$$T_s f(\omega) := \mathbf{E}\big[\, f(\omega_s) \;\big|\; \omega_0 = \omega \,\big] = N_{1-\exp(-s)} f(\omega).$$

Hausdorff dimension of exceptional times

If f_R is the 0/1 indicator of the ℓ-arm event to radius R, with exponent $\mathbf{E}\big[\, f_R \,\big] = R^{-\xi_\ell + o(1)}$, then [GPS]:

$$\mathbf{E}\big[\, f_R(\omega_s)\, f_R(\omega_t) \,\big] / \mathbf{E}\big[\, f_R(\omega) \,\big]^2 \leqslant O(1)\, |t-s|^{-(4/3)\,\xi_\ell + o(1)},$$

as $|t-s| \to 0$. Now, by the Mass Distribution Principle, if

$$\sup_R \int_0^1 \int_0^1 \frac{\mathbf{E}\big[\, f_R(\omega_t)\, f_R(\omega_s) \,\big]}{\mathbf{E}\big[\, f_R(\omega) \,\big]^2\, |t-s|^{\gamma}}\, dt\, ds \, < \infty \,,$$

then $\dim(\mathscr{E}_\ell) \geqslant \gamma$ a.s. Hence $\dim(\mathscr{E}_\ell) \geqslant 1 - 4\xi_\ell/3$.

For $\mathbb{Z}^2$, we have "$\xi_1 + \xi_4 < \xi_5 = 2$", hence $1 - \frac{\xi_1}{2 - \xi_4} > 0$, so there exist exceptional times.

Local time measure for exceptional times [HPS]

$$\overline{M}_r(\omega_s) := \frac{\mathbb{1}\{0 \leftrightarrow r\}}{\mathbf{P}[\,0 \leftrightarrow r\,]}, \quad \overline{\mu}_r[a,b] := \int_a^b \overline{M}_r(\omega_s)\,ds, \quad \overline{\mu}[a,b] := \lim_{r\to\infty} \overline{\mu}_r[a,b].$$

This $\overline{M}_r(\omega)$ is a martingale w.r.t. the filtration $\overline{\mathscr{F}}_r$ of the percolation space generated by the variables $\mathbb{1}\{0 \leftrightarrow r\}$. Moreover, $\mathbf{E}\overline{\mu}_r[a,b] = b-a$, and, by the 2nd Moment Estimate, $\sup_r \mathbf{E}\big[\,\overline{\mu}_r[a,b]^2\,\big] < C_1$. So $\lim_r$ exists.

$$M_H(\omega) := \lim_{R\to\infty} \frac{\mathbf{P}[\,0 \leftrightarrow R \,|\, \omega^H\,]}{\mathbf{P}[\,0 \leftrightarrow R\,]} = \lim_{R\to\infty} \frac{\mathbf{P}[\,\omega^H \,|\, 0 \leftrightarrow R\,]}{\mathbf{P}[\,\omega^H\,]} = \frac{\mathrm{IIC}(\omega^H)}{\mathbf{P}[\,\omega^H\,]},$$

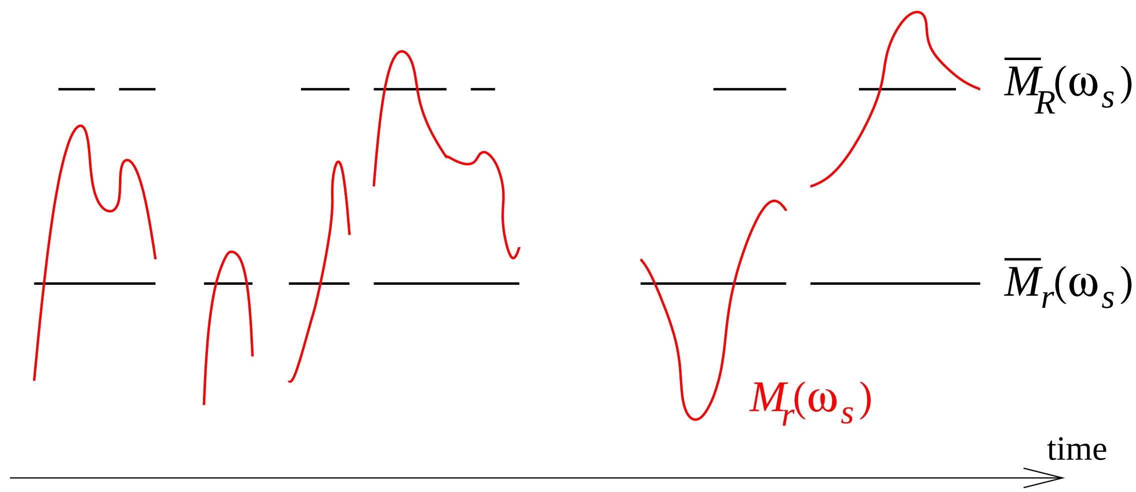
where $H \subset \Delta$, and IIC is Kesten's Incipient Infinite Cluster measure (1986).

$$M_r(\omega_s) := M_{B_r}(\omega_s), \quad \mu_r[a,b] := \int_a^b M_r(\omega_s)\,ds, \quad \mu[a,b] := \lim_{r\to\infty} \mu_r[a,b].$$

Now $M_r(\omega)$ is a MG w.r.t. the full filtration $\mathscr{F}_r$ generated by $\omega(B_r)$, again $\mathbf{E}\mu_r[a,b] = b-a$, and $M_r(\omega) \leqslant C_2\,\overline{M}_r(\omega)$ because of quasi-multiplicativity:

$$\frac{\mathbf{P}[\,0 \leftrightarrow R\,|\,\omega^{B_r}\,]}{\mathbf{P}[\,0 \leftrightarrow R\,]} \asymp \frac{\mathbf{P}[\,0 \leftrightarrow R\,|\,\omega^{B_r}\,]}{\mathbf{P}[\,0 \leftrightarrow r\,]\mathbf{P}[\,r \leftrightarrow R\,]}$$
$$\leqslant \frac{\mathbf{P}[\,r \leftrightarrow R\,|\,\omega^{B_r}\,]\mathbb{1}\{0 \leftrightarrow r\}}{\mathbf{P}[\,0 \leftrightarrow r\,]\mathbf{P}[\,r \leftrightarrow R\,]} = \frac{\mathbb{1}\{0 \leftrightarrow r\}}{\mathbf{P}[\,0 \leftrightarrow r\,]}.$$

Hence, both local time measures exist, and are clearly supported inside $\mathscr{E}$.



Theorem (Hammond, P & Schramm 2008). The two measures coincide: $\overline{\mu} = \mu$ a.s. At a μ-typical time, the configuration has the distribution of IIC. But the configuration at the first exceptional time is different.

Question: is it true that supp $(\mu) = \mathscr{E}$?

Basic properties of the spectral sample

Inclusion formula: $\nu_f\big[\mathscr{S} \subset U\big] = \mathbf{E}\Big[\,\mathbf{E}\big[\, f \mid U\,\big]^2\Big]$.

Proof:

$$\mathbf{E}\big[\,\chi_S \mid U\,\big] = \begin{cases} \chi_S & S \subset U\,, \\ 0 & S \not\subset U\,. \end{cases}$$

Thus $\quad \mathbf{E}\Big[\,\mathbf{E}\big[\, f \mid U\,\big]^2\Big] = \mathbf{E}\Big[\Big(\sum_{S\subset U} \hat{f}(S)\,\chi_S\Big)^2\Big] = \sum_{S\subset U} \hat{f}(S)^2.$ $\blacksquare$

From this, for disjoint subsets A and B,

$$\begin{aligned}
\nu\big[\mathscr{S} \cap B \neq \emptyset,\ \mathscr{S} \cap A = \emptyset\big] &= \nu\big[\mathscr{S} \subseteq A^c\big] - \nu\big[\mathscr{S} \subseteq (A \cup B)^c\big] \\
&= \mathbf{E}\Big[\,\mathbf{E}[\, f \mid A^c\,]^2 - \mathbf{E}[\, f \mid (A \cup B)^c\,]^2\Big] \\
&= \mathbf{E}\Big[\,\big(\mathbf{E}[\, f \mid A^c\,] - \mathbf{E}[\, f \mid (A \cup B)^c\,]\big)^2\Big]\,.
\end{aligned}$$

For the spectral sample $\mathscr{S}_L$ of the $L \times L$ crossing:

With $A := \emptyset$ we get: $\nu\big[\mathscr{S}_L \cap B \neq \emptyset\big] \leqslant C\,\alpha_4(B, V_L)$;

with $A := B^c$ we get: $\nu\big[\emptyset \neq \mathscr{S}_L \subseteq B\big] \leqslant C\,\alpha_4(B, V_L)^2$.

If $B = \{x\}$: equality in both cases. Hence $\nu[x \in \mathscr{S}_L] = \mathbf{P}\big[\, x \in \mathrm{Piv}_L \,\big]$, and

$$\mathbf{E}_\nu[|\mathscr{S}_L|] = \mathbf{E}[\,|\mathrm{Piv_L}|\,] =: m_L \quad (= L^{3/4+o(1)}).$$

If B is a sub-square of side r, and $B' = B/3$, then

$$\begin{aligned}\mathbf{E}_\nu\Big[|\mathscr{S} \cap B'| \;\Big|\; \mathscr{S} \cap B \neq \emptyset\Big] &= \sum_{x \in B'} \frac{\nu[x \in \mathscr{S}]}{\nu[\mathscr{S} \cap B \neq \emptyset]} \geqslant \sum_{x \in B'} \frac{4\alpha_4(x, V_L)}{\alpha_4(B, V_L)} \\ &\asymp \sum_{x \in B'} \alpha_4(x, B) \asymp r^2 \alpha_4(r) \asymp m_r,\end{aligned}$$

as we would expect from a random fractal-like set. But we need something stronger: with good probability, and conditioned on other sub-squares.

Main results for the spectral sample (GPS)

If $r \in [1, L]$, then $\{|\mathscr{S}_L| < m_r\}$ is basically equivalent to being contained inside some $r \times r$ sub-square:

$$\mathbf{P}\big[\,|\mathscr{S}_L| < m_r\,\big] \asymp \alpha_4(r,L)^2 \left(\frac{L}{r}\right)^2.$$

In particular, on the triangular lattice Δ,

$$\mathbf{P}\big[\,|\mathscr{S}_L| < \lambda\, m_L\,\big] \asymp \lambda^{2/3+o(1)}.$$

The *scaling limit* of $\mathscr{S}_L$ is a conformally invariant Cantor-set with Hausdorff-dimension $3/4$.

The existence of the scaling limit follows from Schramm & Smirnov: *Percolation is black noise*, answering a question of Tsirelson.

The strategy of proof

Tile the $L \times L$ square with $(L/r)^2$ boxes of size r. Let $X = X_{r,L}$ be the number of boxes intersecting $\mathscr{S}_L$. We already know that

$$\mathbf{E}[\,X\,] \geqslant \alpha_4(r,L)(L/r)^2 \asymp (L/r)^{3/4+o(1)}.$$

1st step: X is smaller than $C\log(L/r)$ with only very small probability.

2nd step: In a non-empty r-box, with positive probability $|\mathscr{S}_L| \geqslant c\,m_r$.

If we could repeat this step for each of the X nonempty boxes, $\mathscr{S}_L$ would be large almost surely.

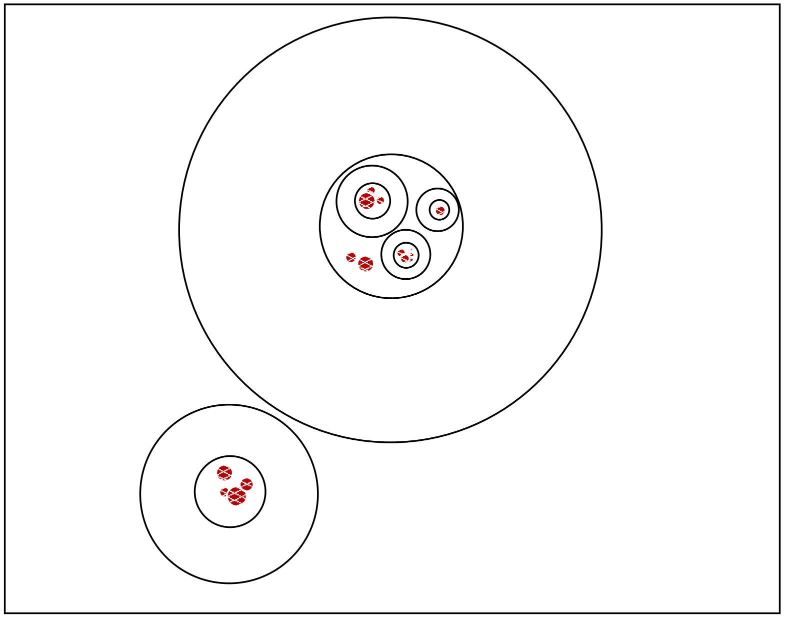
But we can prove Step 2 only in the presence of negative information about $\mathscr{S}_L$ everywhere else! (Partial independence.)

3rd step: Using a sampling trick and a strange large deviation result, 1+2 turns out to be enough.

Annulus structures

Proposition 1. $\nu[X \leqslant k] \leqslant k^{C\log k}\,(L/r)^2\,\alpha_4(r,L)^2$.

An annulus structure $\mathcal{A}$ compatible with a set S:

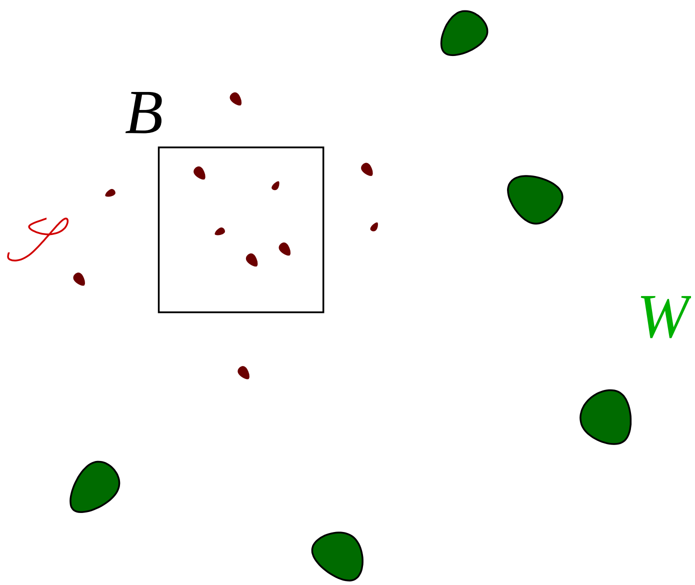


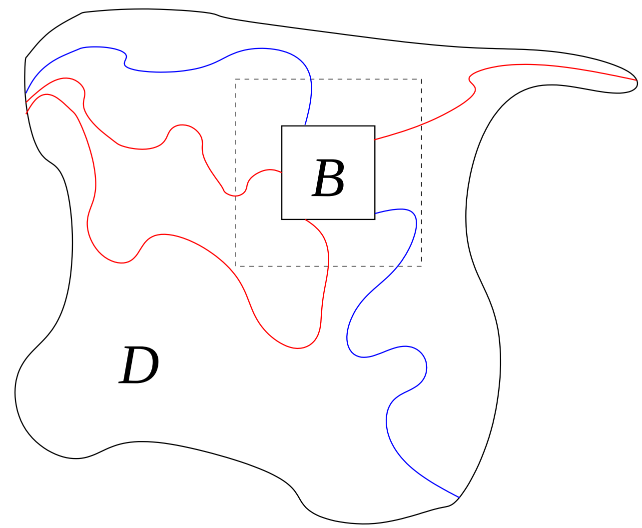
Lemma. $\nu\big[\mathscr{S}$ compatible with $\mathcal{A}\big] \leqslant \prod_{A\in\mathcal{A}} \alpha_4(A)^2$.

Thus, we need to construct a family of annulus structures that has some member compatible with any k-element set, but $\sum_{\mathcal{A}}\prod_{A\in\mathcal{A}}\alpha_4(A)^2$ is still small. This is done recursively.

Partial independence

Proposition 2. If B is an r-box in $[0,L]^2$, and $W \cap B = \emptyset$, then $\mathbf{P}\Big[\, |\mathscr{S} \cap B| > c\, r^2\, \alpha_4(r) \,\Big|\, \mathscr{S} \cap W = \emptyset \neq \mathscr{S} \cap B \Big] \geqslant c\,.$





Separation Lemma. If dist$(B, \partial D) >$ diam(B), then conditioned on the k-arm event in $D \setminus B$ with fixed endpoints on ∂D, then with a uniformly positive conditional probability the k arms are "well-separated" around B.

Large deviation lemma

Proposition 3. Suppose $X_i, Y_i \in \{0,1\}$, $i = 1, \ldots, n$, and that $\forall J \subset [n]$ and $\forall i \in [n] \setminus J$

$$\mathbf{P}\big[\, Y_i = 1 \,\big|\, \forall_{j \in J} Y_j = 0 \,\big] \geqslant c\,\mathbf{P}\big[\, X_i = 1 \,\big|\, \forall_{j \in J} Y_j = 0 \,\big].$$

Then

$$\mathbf{P}\Big[\forall_i Y_i = 0\Big] \leqslant c^{-1}\,\mathbf{E}\Big[\exp\Big(-(c/e)\sum_i X_i\Big)\Big].$$

We use this with $X_j := 1_{\mathscr{S} \cap B_j \neq \emptyset}$ and $Y_j := 1_{\mathscr{S} \cap B_j \cap Q \neq \emptyset}$ for a random Bernoulli set Q, independent from everything else, with density so that it meets with probability $1/2$ a fixed set of cardinality m_r in B_j.

Some related results and questions

Theorem (Garban, P & Schramm 2008). On Δ_η, with rate $\eta^{3/4+o(1)}$ clocks, the scaling limit of the dynamical percolation exists as a Markov process, and can be understood via the pivotals at the visible scales. It is *conformally covariant*: if the domain is changed by $\phi(z)$, then time is scaled locally by $|\phi'(z)|^{3/4}$.
Also, the scaling limit of near-critical percolation exists, and is conformally covariant. Consequently, the scaling limit of the Minimal Spanning Tree exists and is rotationally and scale invariant, but *not* conformally.

The Fourier work gives that in the scaling limit, the correlation of left-right crossing between times 0 and t is $t^{-2/3+o(1)}$. Is the probability of having the crossing all the way between time 0 and t is $\exp(-t^{2/3+o(1)})$?

Similar proofs for other Boolean functions?

Crossing function, but non-unif measure, e.g. Random Cluster models? Ising is expected to be stable, because of non-existence of pivotals ($\kappa < 4$).