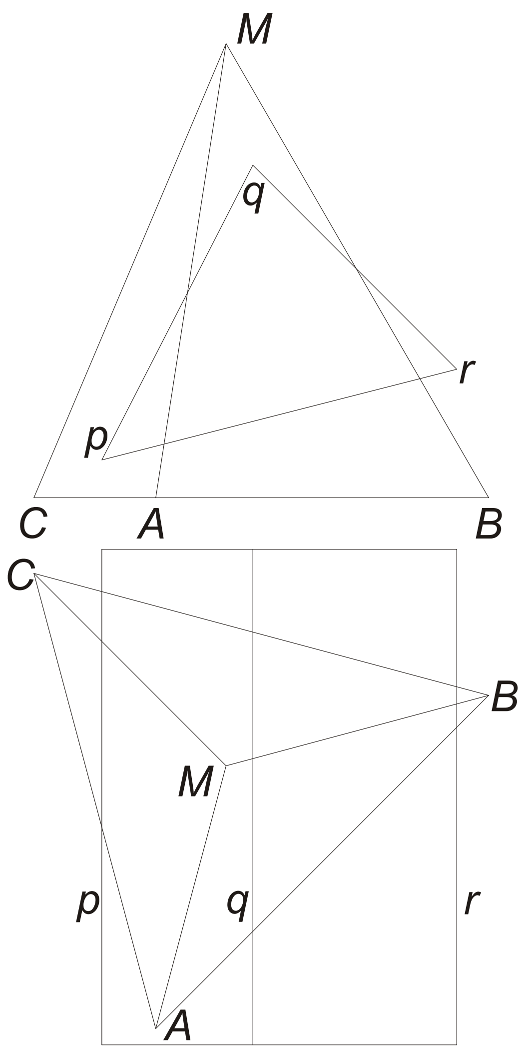
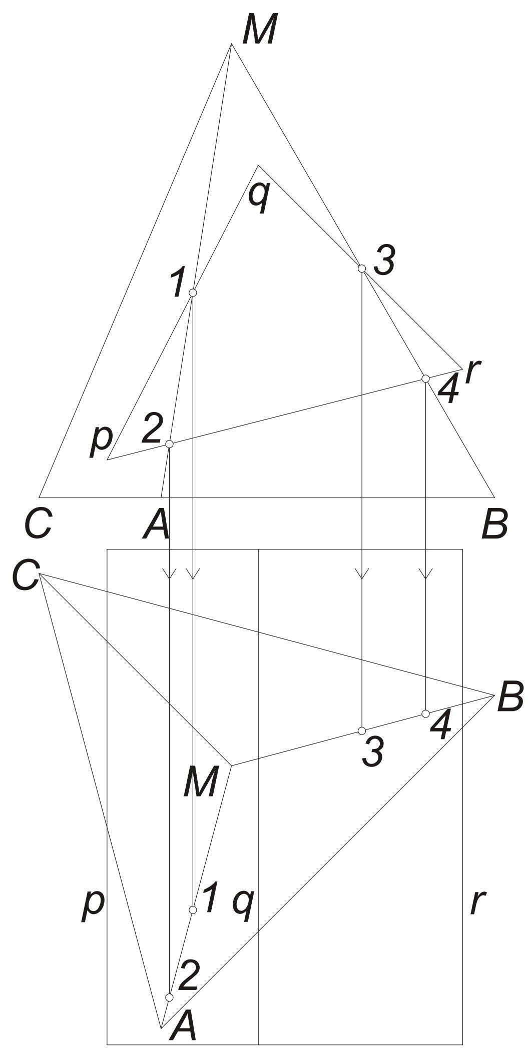
Intersection of two polyhedra

Intersection of a triangle-based pyramid with a triangle-based prism

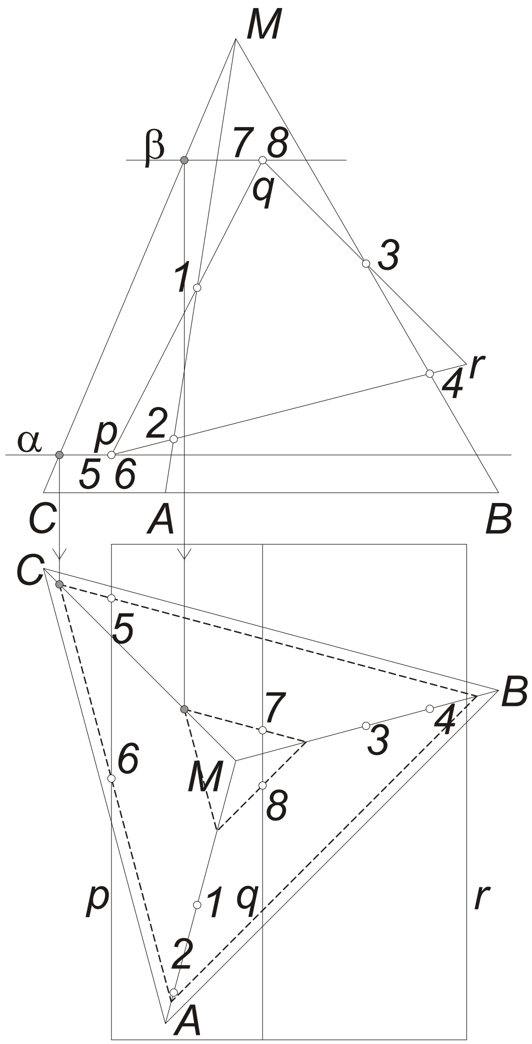
Exercise. *Given a regular, triangle-based pyramid $ABCM$ with its base ABC lying in a horizontal plane, and also a regular, triangle-based prism such that its lateral sidelines p, q, r are vertical projecting lines. Construct the intersection of the two polyhedra. Examine visibility assuming that the polyhedra are solids.*





We construct the *intersection polygon* of the pyramid and the prism. First, we construct the vertices of this polygon. Note that such a vertex is the intersection of an edge of one of the polyhedra with a face of the other one. Hence, we need to find the points where the edges of the prism intersect the pyramid, and also where the edges of the pyramid meet the prism.

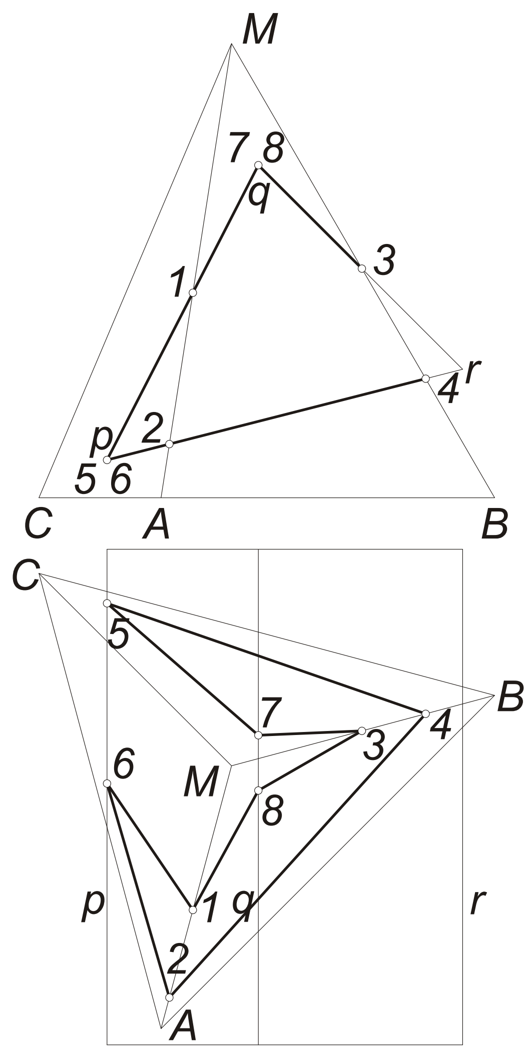
Since the lateral edges of the prism are vertical projecting lines, its lateral faces are vertical projecting planes (their vertical projections are straight lines). Thus, their intersection points with the edges of the pyramid are shown directly in the vertical projection. We obtain in this way the points $1, 2$ on AM, and $3, 4$ on BM. The horizontal projections of the points can be found using their lines of recall. We can see that CM and the edges of the base do not intersect the prism.



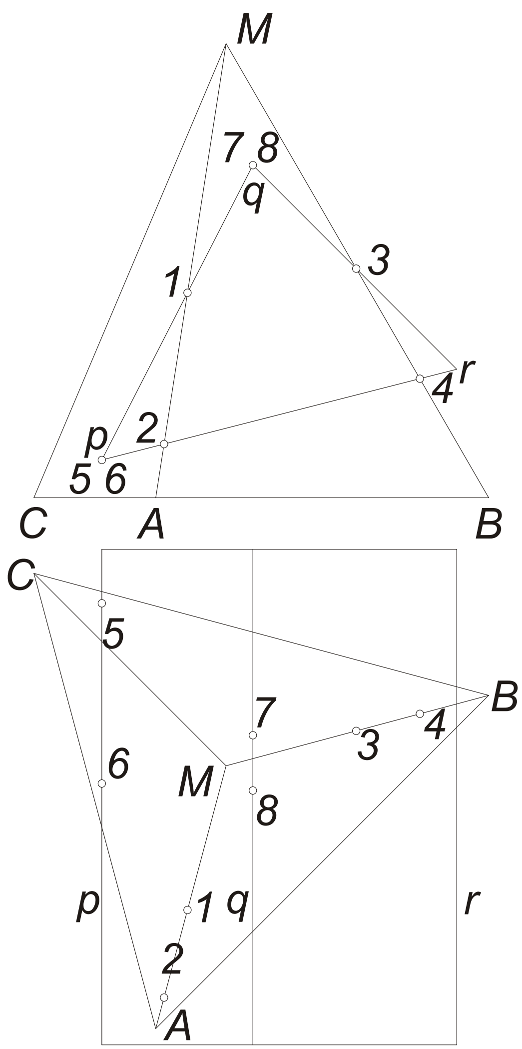
Now we find the intersection of p and q with the faces of the pyramid. (We can see that r, and the edges of the base and the top face of the prism do not intersect the other polyhedron.) To find these points, we draw the horizontal planes α and β, passing through p and q. Then these planes are parallel to ABC. In these planes, the sections of the pyramid are centrally similar to the base; more specifically, in this case they are regular triangles with their sides parallel to the sides of ABC.

Hence, it suffices to find where an edge of the pyramid, say CM intersects α and β. We construct the horizontal projections of these two points, which enable us to draw the two triangle-shaped sections of the pyramid (denoted by dashed lines in the figure). These sections determine the horizontal projections of the intersection points $5, 6, 7, 8$ on their lines of recall.

(In the vertical projection, we can observe that the point 2 is *not* on the plane α. Thus, its horizontal projection is not on the the section of the pyramid with α. This can be checked also by enlarging the figure.)



To determine the edges of the intersection polygon, we may apply the following principle, which can be used to determine the intersection of two (not necessarily convex) polyhedra with convex faces:
Two vertices are connected if, and only if, each of the two polyhedra has a face containing both vertices.
For example, 1 and 6 are connected, because both lie on the face CAM of the pyramid, and the face pq of the prism. Similarly, 3 and 7 can be connected, since both lie on the faces BCM and qr. On the other hand, 2 and 8 cannot be connected despite the fact that both are contained in the face ABM of the pyramid, as the prism has no face that contains bith vertices. After applying this principle, we may obtain that the vertices are connected in the order $1 - -2 - -6 - -4 - -5 - -7 - -3 - -8 - -1$, forming a single cycle. This 3-dimensional octagon is the intersection polygon of the two polyhedra.



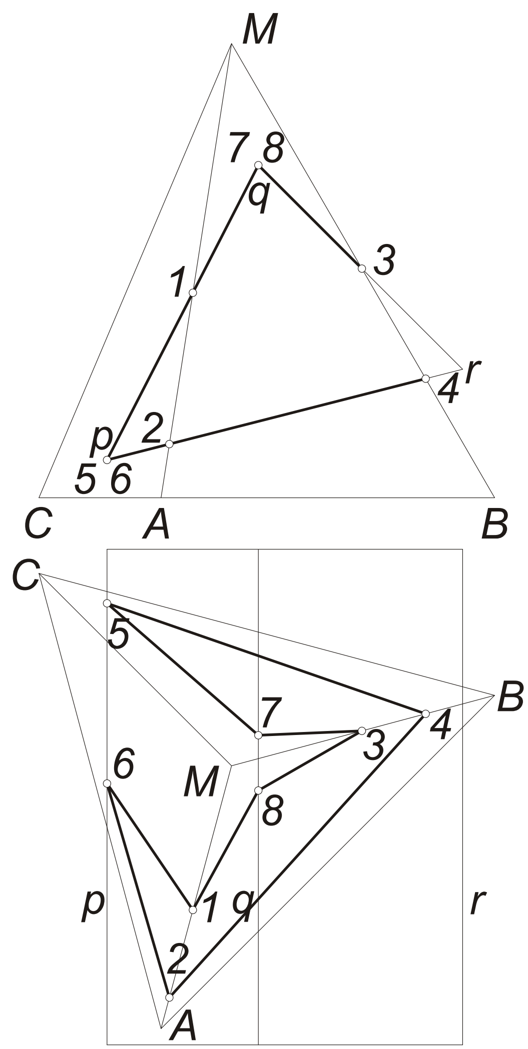
In more complicated cases we can make an *incidence chart*, which lists the faces of both polyhedra containing the vertices of the intersection polygon.

For instance, the point 1 is on the edge AM, and thus, it is incident to the faces CAM and ABM, which are joined at this edge; on the other hand, it is incident to pq (cf. the vertical projection).

Similarly, 5 is incident to the face BCM of the pyramid (cf. the horizontal projection), and to the edge p (cf. the vertical projection), and thus, it is incident to the faces rp and pq.

We determine these date for every other vertex as well, and organize the result in a chart:

	Pyramid	Prism
1:	CAM, ABM;	pq
2:	CAM, ABM;	rp
3:	ABM, BCM;	qr
4:	ABM, BCM;	rp
5:	BCM;	rp, pq
6:	CAM;	rp, pq
7:	BCM;	pq, qr
8:	ABM;	pq, qr



If we have completed the chart correctly, we can deduce the order of the vertices using that as well.

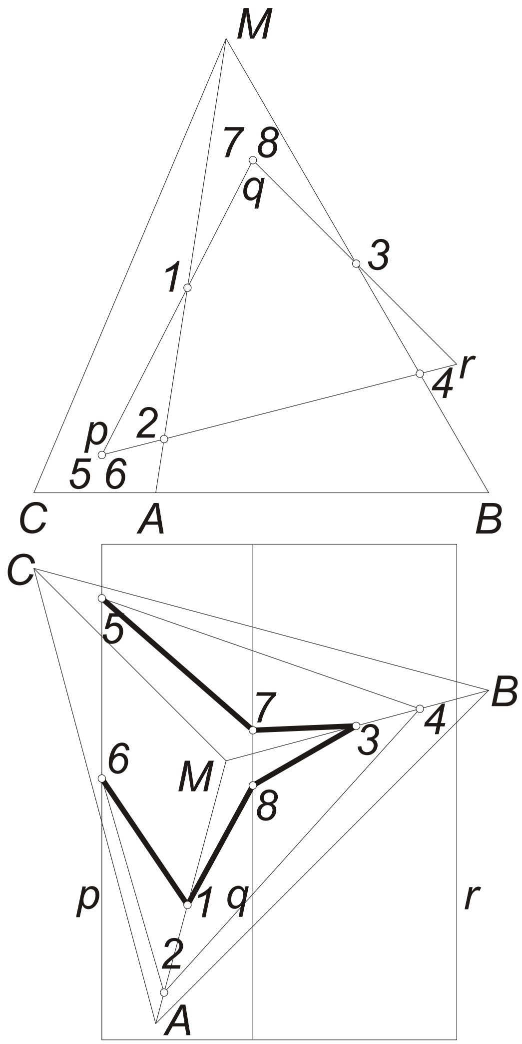
Let us start, for example, with the vertex 1. This point is incident to, e.g. the faces CAM and pq. Let us follow the segment in which these two faces intersect, and check which vertex we will arrive at: according to the chart, the only point incident to the same pair of faces is 6.

At the point 6 we can continue on the segment in which CAM and rp intersect. The other endpoint of this segment is 2. Reaching this point we can switch to the intersection of ABM and rp, which ends at 4, and so on.

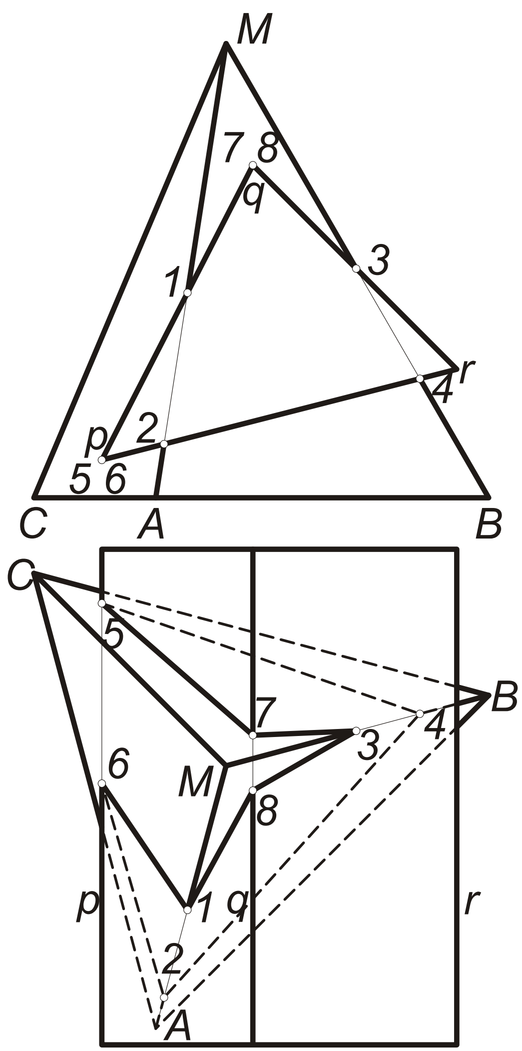
Finally, from 3, we can get to 8 on the intersection of ABM and qr, and from here, on the intersection of ABM and pq, we can get back to the starting point 1.

This is another way to obtain the cycle $1--2--6--4--5--7--3--8--1$.

	Pyramid	Prism
1:	CAM, ABM;	pq
2:	CAM, ABM;	rp
3:	ABM, BCM;	qr
4:	ABM, BCM;	rp
5:	BCM;	rp, pq
6:	CAM;	rp, pq
7:	BCM;	pq, qr
8:	ABM;	pq, qr



To determine the visibility of the edges of the intersection polygon, we may apply the following principle:
An edge of the intersection polygon is visible if, and only, both faces on which it lies are visible.
In our case we need to examine only the top view. In this view, among the faces of the pyramid, only the base invisible, and, among the faces of the prism, pq and qr are visible, whereas rp is not.
Since the intersection polygon is disjoint from the base of the pyramid, the edges on pq and qr are visible; namely, the edges 57, 73, 38, 81 and 16.



As for the remaining part of the figure, visibility can be determined in the usual way. Since we regard both polyhedra as solids, if a part of an edge of one of them is contained in the interior of the other one, this part can be ignored in the drawing as a nonexistent segment.