ON THE SECOND-ORDER REED–MULLER CODE\textsuperscript{1}

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Abstract

In this paper we shall give a recursion and a new explicit formula for some functions connected with
the weight distribution of the second-order Reed–Muller code. We define some new subcodes of it
and determine their information rates, respectively.

Keywords: Reed–Muller code, weight distribution.

1. Definitions and Lemmas

If \( u = (u_1, \cdots, u_m) \) and \( v = (v_1, \cdots, v_n) \) are two vectors then denote by \( |u|v| \)
the vector \((u_1, \cdots, u_m, v_1, \cdots, v_n)\) of length \(n + m\). We shall use Theorem 2 of
Ch.13.§3 in [2] which says the following:

\begin{align*}
\text{Theorem 1} \\
R(2, n + 1) = \{ |u|v| \ \text{where} \ u \in R(2, n), \ v \in R(1, n) \},
\end{align*}

where \( R(1, n) \) denotes the first-order Reed–Muller code of length \(2^n\).

Let \( EG(n, 2) \) be the Euclidean geometry of dimension \( n \) over \( GF(2) \) and let
\( H \) be a subset of \( EG(n, 2) \). Denote by \([H]\) the incidence vector of the subset \( H \) so
\([H]\) is a \((0 − 1)\) vector of dimension \(2^n\) indexed by the elements of \( EG(n, 2)\), for
which:

\[ [H]_\alpha = \begin{cases} 
1 & \text{if } \alpha \in H, \\
0 & \text{if } \alpha \notin H.
\end{cases} \]

We shall say that \( H \subset EG(n, 2) \) is a codeword of \( R(2, n) \) if and only if \([H]\) \( \in R(2, n)\).

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Definition 1 (of the codes $R_k(2, n)$) Let $V_1 < V_2 < \cdots < V_n$ be a nested sequence of subspaces of the geometry $EG(n, 2)$, for which $\dim V_i = k$. We define the code $R_k(2, n)$ ($k = 1, \ldots, n$) as the set of all codewords $H \subset EG(n, 2)$ of $R(2, n)$ which satisfy the condition:

$$| H \cap V_k | = 2^{k-1}.$$ 

It is clear that this definition depends only on the dimension $k$, because a regular linear transformation of the space $EG(n, 2)$ induces a bijection of the code $R(2, n)$ onto itself.

Remark 1 First of all the results of this paper enlarge the aspect of the very important code $R(2, n)$ though the necessity of examination of the codes defined above arose immediately in the theme ‘Geometry of numbers’. In the works [5] and [6] the author defined some new $N$-dimensional point-lattices with a ‘lot of O($2^{\frac{1}{2}} | \log^2 N + \log N|$) minima’. These constructions are based on the method of Barnes and Wall (see [4]) and the setting up of the second-order Reed–Muller code. Denote by $A_{2k-1}^{n,k}$ the number of codewords of the code $R_k(2, n)$ and let $A_{2k-1}^{k+1,k}$ be the number of codewords of weight $2^{k-1}$ in $R(2, k)$. By Theorem 1 we can determine the connection of the numbers $A_{2k-1}^{n,k}$ and $A_{2k-1}^{k+1,k}$, so we now prove Lemma 1:

Lemma 1 

$$A_{2k-1}^{n,k} = A_{2k-1}^{k+1,k} 2^{(n+1)-(k+1)}.$$ 

Proof. Let $k$ be a fix number for which $1 \leq k \leq n$. It is clear that the equality $A_{2k-1}^{k+1,k} = A_{2k-1}^{k+1,k}$ holds. Regard now the code $R_k(2, k+1)$. From Theorem 1 we get that a codeword in $R_k(2, k+1)$ has the form $|u| |u+v|$, where $u \in R_k(2, k)$, and $v \in R(1, k)$. But the number of codewords of $R(1, k)$ is equal to $2^{l+\binom{k+1}{1}}$ so we have the equality:

$$A_{2k-1}^{k+1,k} = A_{2k-1}^{k+1,k} 2^{l+\binom{k+1}{1}}.$$ 

Similarly, the codewords in $R_k(2, k+2)$ have the form

$$|u| |u+v| + |u| u+v | + v |,$$

where $|u| u+v | \in R_k(2, k+2)$, and $v$ is an arbitrary element of $R(1, k+2)$. For this reason we get that:

$$A_{2k-1}^{k+2,k} = A_{2k-1}^{k+2,k} 2^{l+\binom{k+1}{1}} 2^{1+\binom{k+1}{1}}.$$ 

Since we can continue these conversions in this way, we proved the statement of our lemma:

$$A_{2k-1}^{n,k} = A_{2k-1}^{k+1,k} 2^{\sum_{j=k}^{n-1} \binom{j+1}{1}} = A_{2k-1}^{k+1,k} 2^{(n+1)-(k+1)}.$$
Remark 2 It is obvious that the condition of the definition can be replaced by

$$|H \cap V_k| = i,$$

where $i$ is a possible weight of a codeword in $R(2, k)$. Thus we have $i = 2^{k-1}$ or $i = 2^{k-1} \pm 2^{k-1-\delta}$ where $0 \leq \delta \leq \lfloor \frac{k}{2} \rfloor$. If $A^k_i$ is the number of codewords of weight $i$ in $R(2, k)$ then the number of codewords of the new code $R^n_{k,i}(2, n)$ is equal to

$$A^n_{i,k} = 2^{\binom{n+1}{2} - \binom{k+1}{2}} A^k_i.$$

From Theorem 8 of Ch.15.§2 in [2] we know the number $A^k_{2^{k-1}}$. This formula is the following:

$$A^k_{2^{k-1}} = 2^{1+\binom{k}{2}} - \sum_{\delta=1}^{\lfloor \frac{k}{2} \rfloor} 2^{\delta(k+1)} \frac{(2^k - 1)(2^{k-1} - 1) \cdots (2^{k-2\delta+1} - 1)}{(4^\delta - 1)(4^{\delta-1} - 1) \cdots (4 - 1)} = 2.$$ 

The expression is rather complicated, but it can be simplified by a deeper investigation of the generator function $g_k(x)$:

$$g_k(x) = \sum_{\delta=1}^{\lfloor \frac{k}{2} \rfloor} 2^{\delta(k-1)} \frac{(2^k - 1)(2^{k-1} - 1) \cdots (2^{k-2\delta+1} - 1)}{(4^\delta - 1)(4^{\delta-1} - 1) \cdots (4 - 1)} x^\delta.$$

So with this notation we get that

$$A^k_{2^{k-1}} = 2^{1+\binom{k}{2}} - g_k(4) - 2.$$

Lemma 2

$$g_k(1) = 2^{\binom{k}{2}} - 1.$$

Proof. If $k = 2, 3$ or 4, the equality holds trivially. At the same time the right hand side satisfies the following recurrence relation:

$$T_k = 2^{k-1} T_{k-1} + (2^{k-1} - 1),$$

where $T_k$ is equal to $2^{\binom{k}{2}} - 1$. We prove that this relation holds for the left hand side, too. Now, let $T_k$ be the following sum:

$$T_k = \sum_{\delta=1}^{\lfloor \frac{k}{2} \rfloor} 2^{\delta(k-1)} \frac{(2^k - 1) \cdots (2^{k-2\delta+1} - 1)}{(4^\delta - 1) \cdots (4 - 1)}.$$
Then
\[ T_k = (2^k - 1) - 1 \]
\[ = \frac{(2^k - 1)(2^k - 1) - (2^{k-1} - 1)}{4 - 1} + \sum_{\delta=2}^{1+\frac{1}{2}} 2^{\delta(\delta - 1)} \frac{(2^k - 1) \cdots (2^{k-\delta+1} - 1)}{(4^\delta - 1) \cdots (4 - 1)} \]
\[ = 2^{k-1} \frac{(2^k - 1)(2^k - 1) - (2^{k-1} - 1)}{4 - 1} + 2^2 \frac{(2^k - 1)(2^k - 1) - (2^{k-2} - 1)}{(4^2 - 1)(4 - 1)} \]
\[ - (2^{k-1} - 2^2) \frac{(2^k - 1)(2^k - 1) - (2^{k-2} - 1)}{4 - 1} + \sum_{\delta=3}^{1+\frac{1}{2}} 2^{\delta(\delta - 1)} \frac{(2^k - 1) \cdots (2^{k-\delta+1} - 1)}{(4^\delta - 1) \cdots (4 - 1)} \]
\[ = 2^{k-1} \left[ \frac{(2^k - 1)(2^k - 1) - (2^{k-2} - 1)}{4 - 1} + 2^2 \frac{(2^k - 1)(2^k - 1) - (2^{k-3} - 1)(2^k - 4)}{(4^2 - 1)(4 - 1)} \right] \]
\[ - (2^{k-1} - 2^4) \frac{(2^k - 1)(2^k - 1) - (2^{k-3} - 1)(2^k - 4)}{(4^2 - 1)(4 - 1)} \]
\[ + \sum_{\delta=3}^{1+\frac{1}{2}} 2^{\delta(\delta - 1)} \frac{(2^k - 1) \cdots (2^{k-\delta+1} - 1)}{(4^\delta - 1) \cdots (4 - 1)} = \cdots = 2^{k-1} T_{k-1}. \]

So we have the same recurrence relation for the two sides, therefore Lemma 2 is proved.

2. Recurrence Relation for the Numbers \( A_{2k-1}^{n,k} \)

First we introduce the 4-ary Gaussian binomial coefficients \([s]_\delta\):
\[ [s]_0 = 1, \]
\[ [s]_\delta = \frac{(4^s - 1)(4^{s-1} - 1) \cdots (4^{s-\delta+1} - 1)}{(4^\delta - 1)(4^{\delta-1} - 1) \cdots (4 - 1)}, \quad \delta = 1, 2, \ldots. \]

(Here \(s\) is a real number.) Denote by \([\delta]\) the following product:
\[ [\delta] = (4^\delta - 1)(4^{\delta-1} - 1) \cdots (4 - 1) \text{ for } [\delta] = 1, 2, \ldots. \]

The basic properties of these coefficients can be seen for example in [2]. With these notations we can write the expression of \( g_k(x) \) in the form:
\[ g_k(x) = \sum_{\delta=1}^{1+\frac{1}{2}} 2^{\delta(\delta - 1)} [\delta] \frac{[k]_\delta}{\frac{x^k}{\frac{x}{\delta}}} \times^\delta. \]
In this section we shall prove the following theorem:

**Theorem 2** The recurrence relations

1. \( g_{k+1}(4) = 2^{(k+2)} - 2^{k+1} - (2^{k+1} - 1)g_k(4); \)
2. \( 2^k A_{2^k-1}^{n,k} = (2^k - 1) \left[ 2^{(n+1)} - A_{2^k-2}^{n,k-1} \right] \)

are valid for each \( k \geq 4. \)

**Proof.** We define \( T_{k,\delta} = 0 \) for \( \delta < 1 \) and \( \delta > \left[ \frac{k}{2} \right], \) moreover in the case of \( 1 \leq \delta \leq \left[ \frac{k}{2} \right] \) let \( T_{k,\delta} \) be given by the following expression:

\[
T_{k,\delta} = 2^{\delta(\delta-1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right].
\]

Now assume that \( k \geq 4 \) and \( 1 \leq \delta \leq \left[ \frac{k}{2} \right] - 1. \) Then we have the formulas:

\[
T_{k,\delta} = 2^{\delta(\delta-1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right],
\]

\[
T_{k,\delta+1} = 2^{\delta(\delta+1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right] + 2^{\delta(\delta-1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right],
\]

\[
T_{k+1,\delta+1} = 2^{\delta(\delta+1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right] + 2^{\delta(\delta-1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right] + 2^{\delta(\delta-1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right].
\]

At this time we know that

\[
(2^{2k-2\delta-1} - 2^{k-1}) T_{k,\delta} + T_{k,\delta+1} = 2^{\delta(\delta+1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right] + 2^{\delta(\delta-1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right] + 2^{\delta(\delta-1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right].
\]

But

\[
\frac{2^{2k-2\delta-1} - 2^{k-1}}{2^{\delta(4\delta+1)-1}} = 2^{k-2\delta-1} \frac{2^{2k-2\delta} - 1}{4^{\delta+1}-1} = 2^{k-2\delta-1} \frac{4^{\delta-1}-1}{4^{\delta+1}-1},
\]

so we have the equality:

\[
(2^{2k-2\delta-1} - 2^{k-1}) T_{k,\delta} + T_{k,\delta+1} = 2^{\delta(\delta+1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right] + 2^{\delta(\delta-1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right] + 2^{\delta(\delta-1)} \left[ \frac{k}{2} \right] \left[ \frac{k-1}{2} \right] \left[ \frac{k-2}{2} \delta \right] = T_{k+1,\delta+1}.
\]
From this relation we get that

\[ g_{k+1}(x) = \sum_{\delta=1}^{\left[\frac{k+1}{2}\right]} 2^{\delta(\delta-1)} [\delta \cdot \left[\frac{k}{\delta}\right] x^\delta = \sum_{\delta=1}^{\left[\frac{k+1}{2}\right]} T_{k+1,\delta} x^\delta \]

\[ = T_{k+1,1} + \sum_{\delta=2}^{\left[\frac{k+1}{2}\right]-1} T_{k+1,\delta} x^\delta = T_{k+1,1} + \sum_{\delta=1}^{\left[\frac{k+1}{2}-1\right]} T_{k+1,\delta+1} x^{\delta+1} \]

\[ = T_{k+1,1} + \sum_{\delta=1}^{\left[\frac{k+1}{2}\right]-1} \left(2^{2\delta-2\delta-1} - 2^{k-1}\right) T_{k,\delta} + T_{k,\delta+1} x^{\delta+1} \]

\[ = T_{k+1,1} x + 2^{k-1} x \sum_{\delta=1}^{\left[\frac{k+1}{2}\right]-1} T_{k,\delta} \cdot \left(\frac{x}{4}\right)^\delta \]

\[- 2^{k-1} x \sum_{\delta=1}^{\left[\frac{k+1}{2}\right]-1} T_{k,\delta} \cdot x^\delta + \sum_{\delta=2}^{\left[\frac{k+1}{2}\right]} T_{k,\delta} x^\delta. \]

If the number \( k \) is even then

\[ \left[\frac{k+1}{2}\right] - 1 = \frac{k}{2} - 1 = \left[\frac{k}{2}\right] - 1, \]

so we get that

\[ g_{k+1}(x) = T_{k+1,1} x + 2^{k-1} x \left( g_k \left(\frac{x}{4}\right) - T_{k,[\frac{k}{2}]} \cdot \left(\frac{x}{4}\right)^{\left[\frac{k}{2}\right]} \right) \]

\[- 2^{k-1} x \left( g_k(x) - T_{k,[\frac{k}{2}]} \cdot x^{\left[\frac{k}{2}\right]} \right) + g_k(x) - T_{k,1} x \]

\[ = 2^{k-1} x g_k \left(\frac{x}{4}\right) - (2^{k-1} x - 1) g_k(x) + x[T_{k+1,1} - T_{k,1}] \]

\[ = 2^{k-1} x g_k \left(\frac{x}{4}\right) - (2^{k-1} x - 1) g_k(x) + x(2^k - 1)2^{k-1}. \]

Finally, if the number \( k \) is odd, we get immediately that

\[ g_{k+1}(x) = T_{k+1,1} x + 2^{k-1} x g_k \left(\frac{x}{4}\right) - 2^{k-1} x g_k(x) + g_k(x) - T_{k,1} x \]

\[ = 2^{k-1} x g_k \left(\frac{x}{4}\right) - (2^{k-1} x - 1) x g_k(x) + x[T_{k+1,1} - T_{k,1}] \]

\[ = 2^{k-1} x g_k \left(\frac{x}{4}\right) - (2^{k-1} x - 1) g_k(x) + x(2^k - 1)2^{k-1}. \]
Here we used the precise values of the numbers $T_{k+1,1}$ and $T_{k,1}$ which are $\frac{(2^{k+1}−1)(2^k−1)}{4−1}$ and $\frac{(2^k−1)(2^{k−1}−1)}{4−1}$, respectively. Substitute now the value 4 into this equation. Then we get the formula:

$$g_{k+1}(4) = 2^{2k+1} g_k(1) - (2^{k+1}−1)g_k(4) + (2^k−1)2^{k+1}$$

and the first statement of this theorem can be seen from Lemma 2:

$$g_{k+1}(4) = 2^{2k+1} + \frac{k(k−1)}{2} - 2^{2k+1} - (2^{k+1}−1)g_k(4) + (2^k−1)2^{k+1}$$

$$= 2^{2k+1} + \frac{k^2}{2} + 2k + 1 - 2^{k+1} - (2^{k+1}−1)g_k(4) = 2^{k+1} (\frac{k^2}{2} + 2k − 1) − (2^{k+1}−1)g_k(4)$$

Now apply the original formula of $A^k_{2^{k−1}}$. Then we have the equality:

$$A^k_{2^{k−1}} = 2^{1+k+\left(\frac{k}{2}\right)} - 2g_k(4) - 2$$

$$= 2^{k+1} + 2^{(k−1)+1} + 2k + 1 - 2g_k(4) - 2$$

$$= (2^{k+1}−2)g_{k−1}(4) + 1 = -2^k \left[ 2^{\left(\frac{k}{2}\right)+1} - 2g_{k−1}(4) - 2 \right]$$

$$+ 2^{(k+1)+1} + \left[ 2^{\left(\frac{k}{2}\right)+1} - 2g_{k−1}(4) - 2 \right] - 2^{\left(\frac{k}{2}\right)+1}$$

$$= 2^{(k+1)+1} - 2^{\left(\frac{k}{2}\right)+1} - (2^k−1)A_{2^{k−2}}^{k−1} = (2^k−1) \left[ 2^{\left(\frac{k}{2}\right)+1} - A_{2^{k−2}}^{k−1} \right]$$

By virtue of Lemma 2 this means that the following recursive formulas hold:

$$A^k_{2^{k−1}} = (2^k−1) \left[ 2^{\left(\frac{k}{2}\right)+1} - A_{2^{k−2}}^{k−1} \right],$$

$$2^k A_{2^{k−1}}^{n,k} = (2^k−1) \left[ 2^{(n+1)+1} - A_{2^{k−2}}^{n,k} \right].$$

So we proved the statement ii, of Theorem 2, too.

**Remark 3** Since we know the values of $A^k_{2^{k−1}}$ in the case of $k = 1, 2, 3$ and 4 (these are 2, 6, 70 and 870, respectively) the other values of the function $A^k_{2^{n−1}}$ can be computed easily by the formula of Theorem 2.

### 3. Explicit Formula for the Numbers $A^k_{2^{k−1}}$

In this section we prove the following statement:
Theorem 3

\[ A_{2k-1}^{n,k} = \sum_{\delta=1}^{\left[\frac{k+1}{2}\right]} (2^k - 1) \cdots (2^{k-2\delta+2} - 1) 2^{\left(\frac{n+1}{2}\right)-(k+1)+\left(\frac{k-2\delta+1}{2}\right)+1}, \]

\[ A_{2n-1} = A_{2n-1}^{n,n} = \sum_{\delta=1}^{\left[\frac{n+1}{2}\right]} 2^{(n-2\delta+1)+1} (2^n - 1) \cdots (2^{n-2\delta+2} - 1). \]

Proof. Apply the recursion formulas for the function \( A_{2k-1}^{n,k} \). Then we get the undermentioned formula:

\[ A_{2k-1}^{n,k} = 2^{\left(\frac{n+1}{2}\right)+1} \left\{ \frac{2^k - 1}{2^k} - \frac{2^k - 1}{2^k} \frac{2^{k-1} - 1}{2^{k-1}} \right\} \]

\[ + \frac{(2^k - 1)(2^{k-1} - 1)(2^{k-2} - 1)}{2^k 2^{k-1} 2^{k-2}} - \frac{(2^k - 1)(2^{k-1} - 1)(2^{k-2} - 1)(2^{k-3} - 1)}{2^k 2^{k-1} 2^{k-2} 2^{k-3}} + \ldots \]

\[ + \frac{(2^k - 1) \cdots (2^{k-2l+2} - 1)}{2^k \cdots 2^{k-2l+2}} - \frac{(2^k - 1) \cdots (2^{k-2l+1} - 1)}{2^k \cdots 2^{k-2l+1}} \}

\[ + \frac{(2^k - 1) \cdots (2^{k-2l+1} - 1)}{2^k \cdots 2^{k-2l+1}} A_{2k-2l-1}^{n,k-2l} \]

where \( l = 1, \ldots, \left[\frac{k}{2}\right] \). So if \( k = 2l \) then

\[ A_{2^{2l-1}}^{n,2l} = 2^{\left(\frac{n+1}{2}\right)+1} \left\{ \sum_{\delta=1}^l \frac{2^{2l} - 1}{2^2 \cdots 2^{2l-2\delta+1}} \right\} \]

\[ + \frac{(2^l - 1) \cdots (2 - 1)}{2^2 \cdots 2} A_{2^{2l-1}}^{n,0}, \]

and if \( k = 2l + 1 \) then

\[ A_{2^{2l}}^{n,2l+1} = 2^{\left(\frac{n+1}{2}\right)+1} \left\{ \sum_{\delta=1}^l \frac{2^{2l+1} - 1}{2^{2l+1} \cdots 2^{2l-2\delta+2}} \right\} \]

\[ + \frac{(2^{2l+1} - 1) \cdots (2^2 - 1)}{2^{2l+1} \cdots 2^2} A_{2^{2l}}^{n,1}, \]

where we used the equalities:

\[ A_{2^{20}}^{n,1} = 2^{\left(\frac{n+1}{2}\right)-(\frac{2}{2})} \cdot A_{2^{20}}^2 = 2^{\left(\frac{n+1}{2}\right)} \quad \text{and} \quad A_{2^{2-1}}^{n,0} = 0. \]
Therefore we have got the formulas:

\[
A_{2^{2l}}^{n,2l} = 2^{(n+1)/2} \left\{ \sum_{\delta=1}^{l+1} \frac{(2^{2l} - 1) \cdots (2^{2l} - 2\delta + 2 - 1)}{2^{(2l+1)/2} - (2^{2l+1}/2)} \right\}
\]

and

\[
A_{2^{2l}}^{n,2l+1} = 2^{(n+1)/2} \left\{ \sum_{\delta=1}^{l+1} \frac{(2^{2l+1} - 1) \cdots (2^{2l+1} - 2\delta + 2 - 1)}{2^{(2l+2)/2} - (2^{2l+2}/2)} \right\},
\]

so

\[
A_{2^{2k-1}}^{n,k} = \sum_{\delta=1}^{[k+1]/2} (2^k - 1)(2^{k-1} - 1) \cdots (2^{k-2\delta + 2} - 1) 2^{(n+1)/2} - (k+1)/2 + (k-2\delta + 1)/2 + 1.
\]

In the case of \(k = n\) we get the simple explicit formula for the numbers \(A_{2^{n-1}}\), too.

### 4. The Information Rates of the New Codes

Since the code \(R_k(2, n)\) is not linear, the information rate \(R_k\) is defined by the quotient:

\[
R_k = \frac{\log_2 A_{2^{2k-1}}^{n,k}}{2^n}.
\]

This is equal to

\[
\frac{1}{2^n} \log_2 \left( \sum_{\delta=1}^{[k+1]/2} (2^k - 1)(2^{k-1} - 1) \cdots (2^{k-2\delta + 2} - 1) 2^{(n+1)/2} - (k+1)/2 + (k-2\delta + 1)/2 + 1 \right).
\]

We shall prove that this number is asymptotically equal to \(\frac{(n+1)/2}{2^n}\). More precisely we verify the statement:

**Theorem 4** For \(1 \leq k \leq n\) the following inequalities hold:

\[
\frac{(n+1)/2 - 1}{2^n} \leq R_k \leq \frac{(n+1)/2 + 1}{2^n}.
\]

**Proof.** Since the upper bound is the information rate of the second-order Reed–Muller code the second inequality trivially holds. On the other hand the value

\[
\sum_{\delta=1}^{[k+1]/2} (2^k - 1)(2^{k-1} - 1) \cdots (2^{k-2\delta + 2} - 1) 2^{(n+1)/2} - (k+1)/2 + (k-2\delta + 1)/2 + 1
\]
can be written in the following form:

\[
2^{(n+1)/2} + 1 \cdot \left[ \sum_{\delta=1}^{(k+1)/2} (2^k - 1)(2^{k-1} - 1) \cdots (2^{k-2\delta+2} - 1)2^{-(k+1)/2} + (k-2\delta+1) \right]
\]

\[
= 2^{(n+1)/2} + 1 \cdot \left[ \sum_{\delta=1}^{(k+1)/2} \left( 1 - \frac{1}{2^k} \right) \left( 1 - \frac{1}{2^{k-1}} \right) \cdots \left( 1 - \frac{1}{2^{k-2\delta+2}} \right) \frac{1}{2^{k-2\delta+1}} \right].
\]

Denote by \( L_k \) the sum in the bracket. It is easy to verify the following recursive formula for this number:

\[
L_k = \left( 1 - \frac{1}{2^k} \right) \left[ \frac{1}{2^{k-1}} + \left( 1 - \frac{1}{2^{k-1}} \right) L_{k-2} \right].
\]

If \( k \) is odd then \( L_1 = \frac{1}{2} \) and it can be seen by induction with respect to \( k \) that \( L_k \geq \frac{1}{2} \), because

\[
L_k \geq \left( 1 - \frac{1}{2^k} \right) \left[ \frac{1}{2^{k-1}} + \left( 1 - \frac{1}{2^{k-1}} \right) \frac{1}{2} \right] = \left( 1 - \frac{1}{2^k} \right) \left[ \frac{1}{2^{k-1}} + \frac{1}{2} - \frac{1}{2^k} \right] = \left( 1 - \frac{1}{2^k} \right) \left( \frac{1}{2} + \frac{1}{2^k} \right) = \frac{1}{2} + \frac{1}{2^{k+1}} - \frac{1}{2^{2k}} \geq \frac{1}{2}.
\]

If \( k \) is even, a similar calculation shows that \( L_k \) is greater than or equal to \( \frac{3}{8} \). This means that

\[
R_k \geq \log_2 2^{1+(n+1)/2} \cdot \frac{3}{8} \geq \log_2 2^{(n+1)/2} - \frac{1}{2^n}.
\]

So we have proved this theorem, too.

References