Weighting and ranking based on pairwise comparisons

Páros összehasonlítás alapú súlyozás és rangsorolás

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Slides can be downloaded from

http://www.sztaki.mta.hu/%7Ebozoki/slides

Multi Criteria Decision Making

Pairwise comparisons, Analytic Hierarchy Process (Saaty, 1977)

Incomplete pairwise comparison matrix

Efficiency (Pareto optimality)

Multi Criteria Decision Making

The aim of multiple criteria decision analysis

The aim is **to select the overall best one** from a finite set of *alternatives*, with respect to a finite set of **attributes (criteria)**,
or,
to rank the alternatives,
or,
to classify the alternatives.

Multiple criteria decision problems

Examples

- tenders, public procurements, privatizations
- evaluation of applications
- environmental studies
- ranking, classification

Multiple criteria decision problems

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Properties

- criteria contradict each other
- there is not a single best solution, that is optimal with respect to each criterion
- subjective factors influence the decision
- contradictive individual opinions have to be aggregated

Main tasks in multi criteria decision problems

- to assign weights of importance to the criteria
- to evaluate the alternatives
- to aggregate the evaluations with the weights of criteria
- sensitivity analysis

Decomposition of the goal: tree of criteria

- main criterion 1
  - criterion 1.1
  - criterion 1.2
  - criterion 1.3
  - criterion 1.4
  - criterion 1.5
- main criterion 2
  - criterion 2.1
  - criterion 2.2
- main criterion 3
  - criterion 3.1
    - subcriterion 3.1.1
    - subcriterion 3.1.2
  - criterion 3.2

Estimating weights from pairwise comparisons

’How many times criterion 1 is more important than criterion 2?’

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix},$$

is given, where for any $i, j = 1, \dots, n$ indices

$$a_{ij} > 0, \qquad a_{ij} = \frac{1}{a_{ji}}.$$

The aim is to find the $\mathbf{w} = (w_1, w_2, \dots, w_n)^\top \in \mathbb{R}^n_+$ weight vector such that ratios $\frac{w_i}{w_j}$ are *close enough* to a_{ij}s.

Evaluation of the alternatives

Alternatives are evaluated directly, or by using a function, or by pairwise comparisons as before.

’How many times alternative 1 is better than alternative 2 with respect to criterion 1.1?’

$$\mathbf{B} = \begin{pmatrix} 1 & b_{12} & b_{13} & \dots & b_{1m} \\ b_{21} & 1 & b_{23} & \dots & b_{2m} \\ b_{31} & b_{32} & 1 & \dots & b_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \dots & 1 \end{pmatrix}$$

Aggregation of the evaluations

- total scores are calculated as a weighted sum of the evaluations with respect to leaf nodes of the criteria tree (bottom up);
- partial sums are informative

Weighting methods

Eigenvector Method (Saaty): $\mathbf{A}\mathbf{w} = \lambda_{\max}\mathbf{w}$.

Logarithmic Least Squares Method (LLS):

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\log a_{ij} - \log \frac{w_i}{w_j} \right)^2$$

$$\sum_{i=1}^{n} w_i = 1, \qquad w_i > 0, \quad i = 1, 2, \ldots, n.$$

and 20+ other methods.

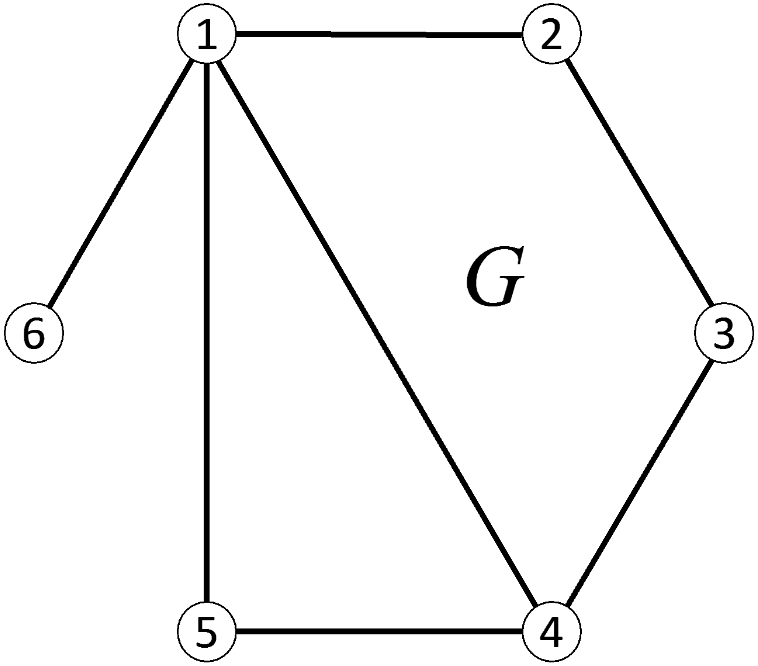
Incomplete pairwise comparison matrix

incomplete pairwise comparison matrix

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$

incomplete pairwise comparison matrix and its graph

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$\lambda_{\max}$-minimal completion

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & * & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & * \\ * & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & * & 1/a_{3n} & \dots & 1 \end{pmatrix}.$$

$\lambda_{\max}$-minimal completion

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & x_d \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/x_d & 1/a_{3n} & \dots & 1 \end{pmatrix},$$

where $x_1, x_2, \dots, x_d \in \mathbb{R}_+$.

$\lambda_{\max}$-minimal completion

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & x_d \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/x_d & 1/a_{3n} & \dots & 1 \end{pmatrix},$$

where $x_1, x_2, \dots, x_d \in \mathbb{R}_+$.

The aim is to solve

$$\min_{\mathbf{x}>\mathbf{0}} \lambda_{\max}(\mathbf{A}(\mathbf{x})).$$

$\lambda_{\max}$-minimal completion

Theorem (Bozóki, Fülöp, Rónyai, 2010): The optimal solution of the eigenvalue minimization problem

$$\min_{\mathbf{x}>\mathbf{0}} \lambda_{\max}(\mathbf{A}(\mathbf{x}))$$

is unique if and only if the graph G corresponding to the incomplete pairwise comparison matrix is *connected*.

$\lambda_{\max}$-minimal completion

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is unique if and only if the graph G corresponding to the incomplete pairwise comparison matrix is *connected*.

If graph G corresponding to the incomplete pairwise comparison matrix is connected, then by using the exponential parametrization $x_1 = e^{y_1}, x_2 = e^{y_2}, \ldots x_d = e^{y_d}$, the eigenvalue minimization problem is transformed into a strictly convex optimization problem.

The Logarithmic Least Squares (LLS) problem

$$\min \sum_{\substack{i,j:\\ a_{ij} \text{ is known}}} \left[\log a_{ij} - \log\left(\frac{w_i}{w_j}\right)\right]^2$$

$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

The most common normalizations are $\sum_{i=1}^{n} w_i = 1$, $\prod_{i=1}^{n} w_i = 1$ and $w_1 = 1$.

Theorem (Bozóki, Fülöp, Rónyai, 2010): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph G is connected. Then the optimal solution $\mathbf{w} = \exp \mathbf{y}$ of the logarithmic least squares problem is the unique solution of the following system of linear equations:

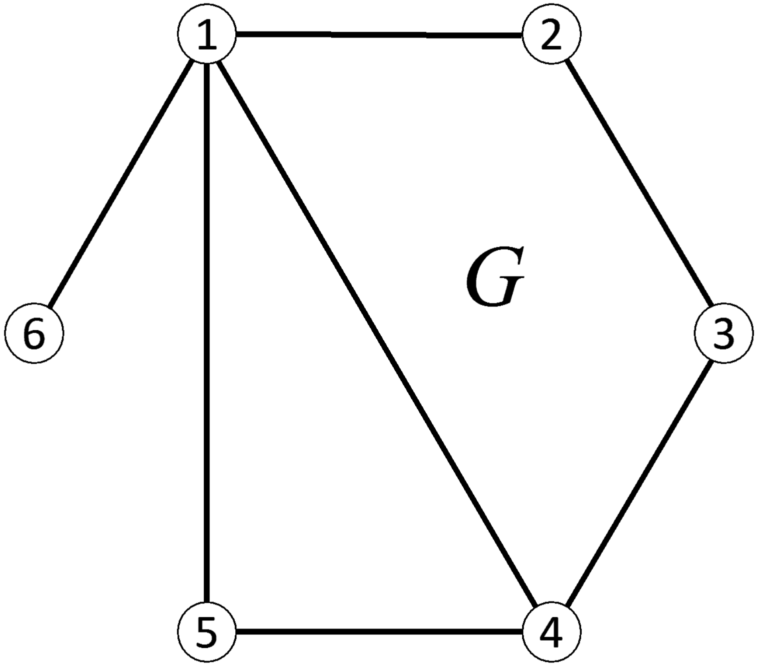
$$(\mathbf{Ly})_i = \sum_{k:e(i,k)\in E(G)} \log a_{ik} \qquad \text{for all } i = 1, 2, \ldots, n,$$

$$y_1 = 0$$

where $\mathbf{L}$ denotes the Laplacian matrix of G (ℓ_{ii} is the degree of node i and $\ell_{ij} = -1$ if nodes i and j are adjacent).

example

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1(=0) \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} \log(a_{12}\, a_{14}\, a_{15}\, a_{16}) \\ \log(a_{21}\, a_{23}) \\ \log(a_{32}\, a_{34}) \\ \log(a_{41}\, a_{43}\, a_{45}) \\ \log(a_{51}\, a_{54}) \\ \log a_{61} \end{pmatrix}$$

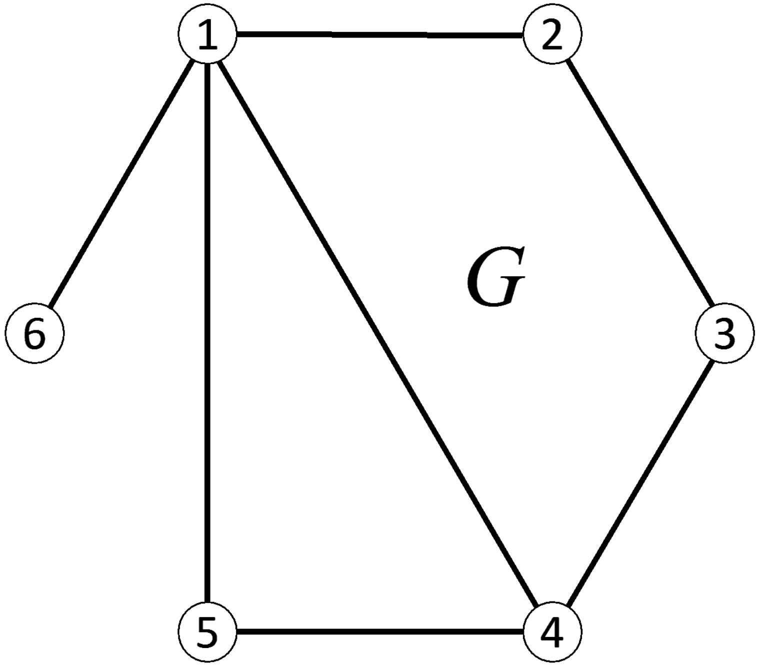
Pairwise Comparison Matrix Calculator

Weights from incomplete pairwise comparison matrices can be calculated at

pcmc.online

The spanning tree approach (Tsyganok, 2000, 2010)

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



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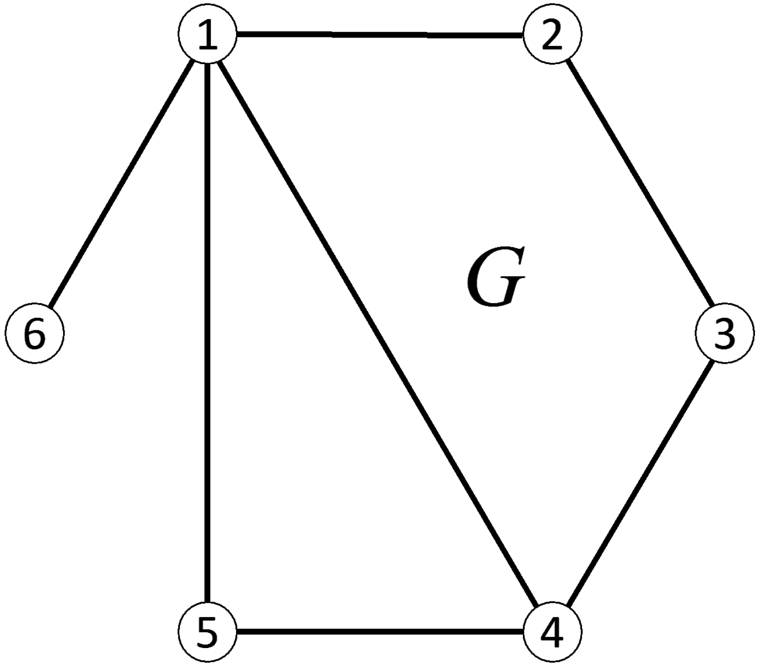
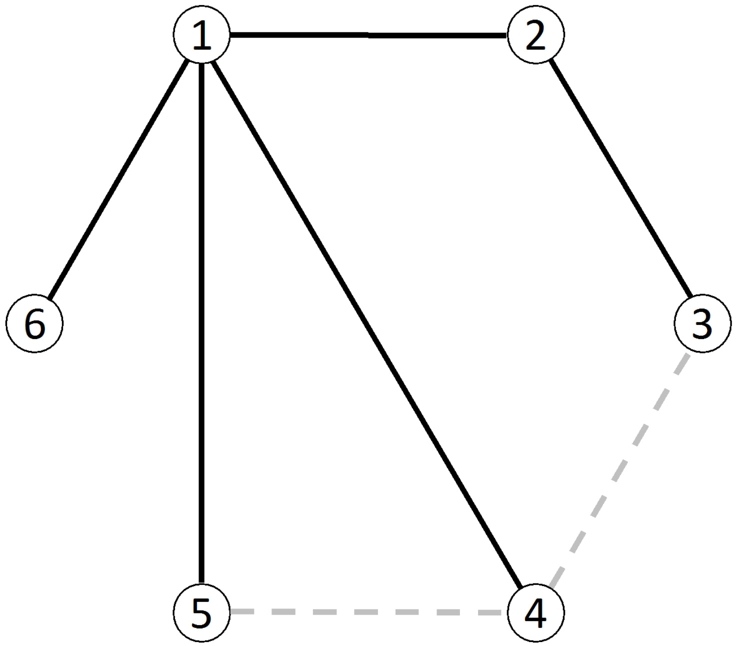
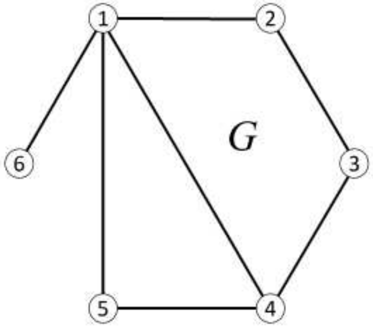
$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & & & \\ a_{41} & & & 1 & & \\ a_{51} & & & & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$

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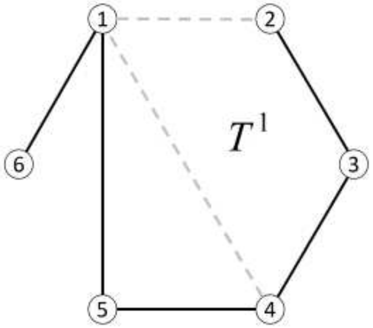
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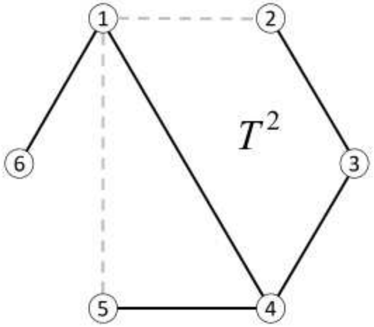
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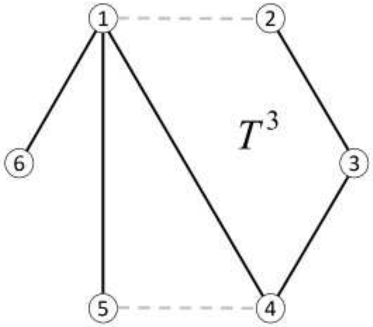
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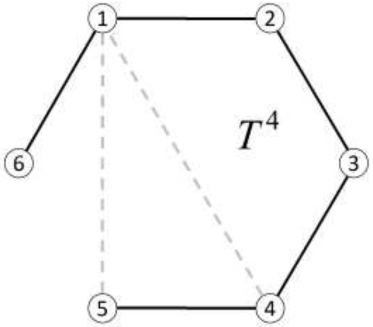
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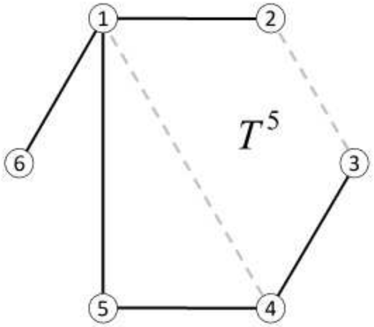
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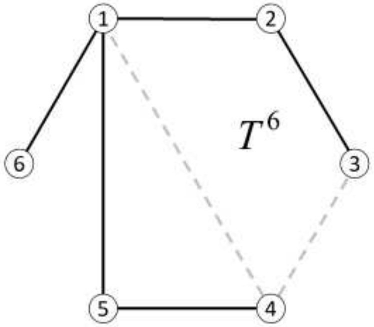
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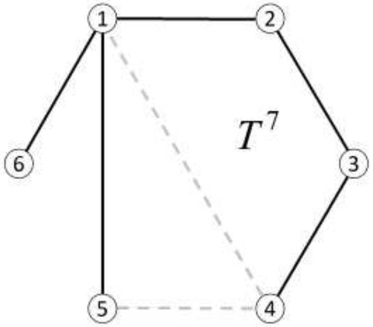
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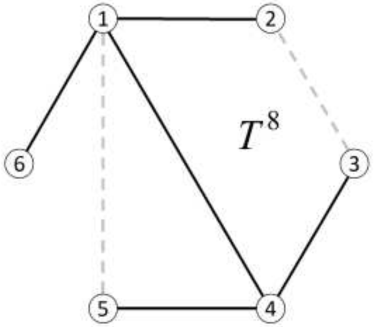
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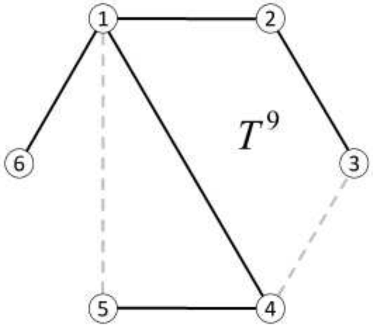
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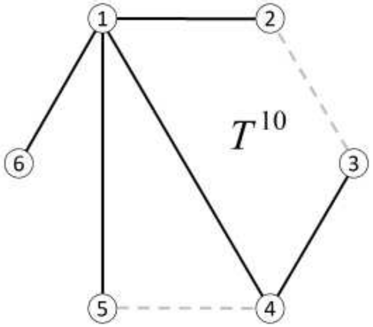
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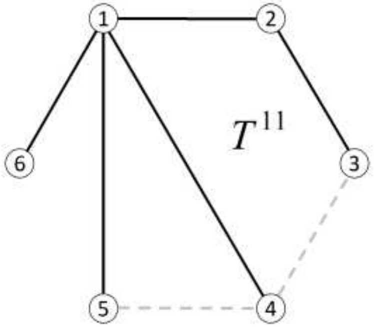
1
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T^{11}
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The spanning tree approach

Every spanning tree induces a weight vector.

Natural ways of aggregation: arithmetic mean, geometric mean etc.

Theorem (Lundy, Siraj, Greco, 2017): The geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in case of complete pairwise comparison matrices.

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Theorem (Bozóki, Tsyganok): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph is connected. Then the optimal solution of the logarithmic least squares problem is equal, up to a scalar multiplier, to the geometric mean of weight vectors calculated from all spanning trees.

proof

Let G be the connected graph associated to the (in)complete pairwise comparison matrix $\mathbf{A}$ and let $E(G)$ denote the set of edges. The edge between nodes i and j is denoted by $e(i,j)$.

The Laplacian matrix of graph G is denoted by $\mathbf{L}$. Let $T^1, T^2, \ldots, T^s, \ldots, T^S$ denote the spanning trees of G, where S denotes the number of spanning trees. $E(T^s)$ denotes the set of edges in T^s.

Let $\mathbf{w}^s, s = 1, 2, \ldots, S$, denote the weight vector calculated from spanning tree T^s. Weight vector $\mathbf{w}^s$ is unique up to a scalar multiplication. Assume without loss of generality that $w_1^s = 1$.

Let $\mathbf{y}^s := \log \mathbf{w}^s,\ s = 1, 2, \ldots, S$, where the logarithm is taken element-wise.

proof

Let $\mathbf{w}^{LLS}$ denote the optimal solution to the incomplete Logarithmic Least Squares problem (normalized by $w_1^{LLS} = 1$) and $\mathbf{y}^{LLS} := \log \mathbf{w}^{LLS}$, then

$$\left(\mathbf{L} \mathbf{y}^{LLS} \right)_i = \sum_{k: e(i,k) \in E(G)} b_{ik} \qquad \text{for all } i = 1, 2, \ldots, n,$$

where $b_{ik} = \log a_{ik}$ for all $(i,k) \in E(G)$.

$b_{ik} = -b_{ki}$ for all $(i,k) \in E(G)$.

In order to prove the theorem, it is sufficient to show that

$$\left(\mathbf{L} \frac{1}{S} \sum_{s=1}^{S} \mathbf{y}^s \right)_i = \sum_{k: e(i,k) \in E(G)} b_{ik} \qquad \text{for all } i = 1, 2, \ldots, n.$$

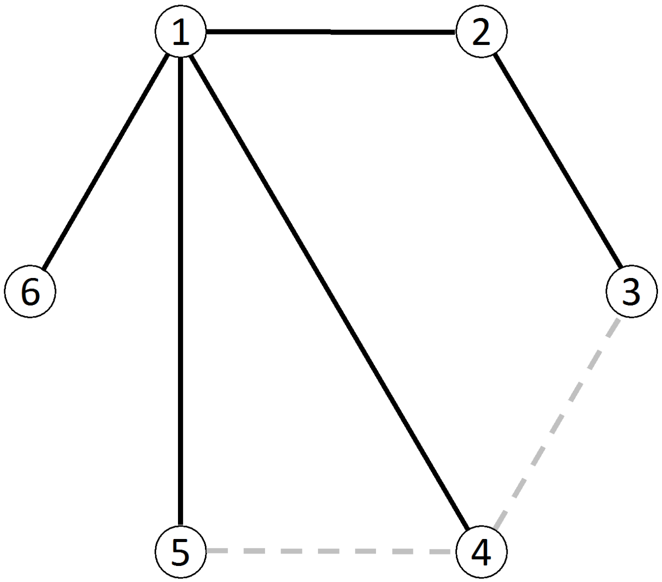
proof

Challenge: the Laplacian matrices of the spanning trees are different from the Laplacian of G.

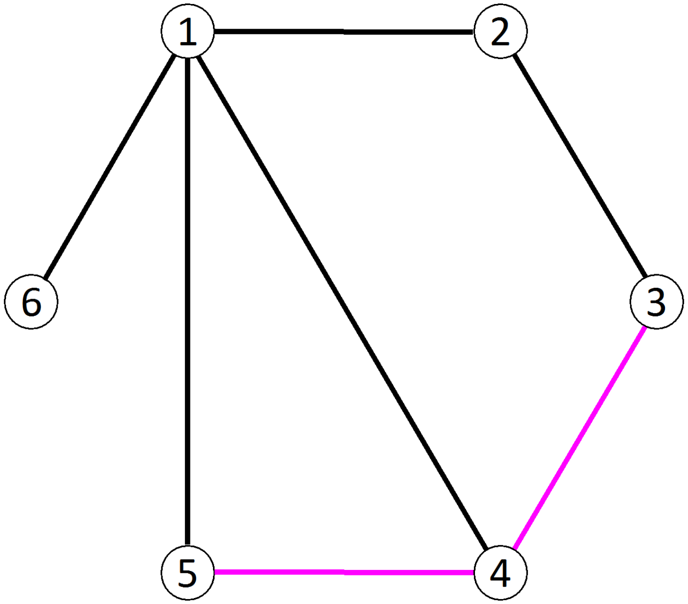
Consider an arbitrary spanning tree T^s. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i,j) \in E(T^s)$.
Introduce the incomplete pairwise comparison matrix $\mathbf{A}^s$ by $a_{ij}^s := a_{ij}$ for all $e(i,j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G) \backslash E(T^s)$. Again, $b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$.
Note that the Laplacian matrices of $\mathbf{A}$ and $\mathbf{A}^s$ are the same ($\mathbf{L}$).

proof

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & & & \\ a_{41} & & & 1 & & \\ a_{51} & & & & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{32}a_{21}a_{14} & & \\ a_{41} & & a_{41}a_{12}a_{23} & 1 & a_{41}a_{15} & \\ a_{51} & & & a_{51}a_{14} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



proof

Consider an arbitrary spanning tree T^s. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i,j) \in E(T^s)$. Introduce the incomplete pairwise comparison matrix $\mathbf{A}^s$ by $a_{ij}^s := a_{ij}$ for all $e(i,j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G) \backslash E(T^s)$. Again,
$b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$.
Note that the Laplacian matrices of $\mathbf{A}$ and $\mathbf{A}^s$ are the same ($\mathbf{L}$).
Since weight vector $\mathbf{w}^s$ is generated by the matrix elements belonging to spanning tree T^s, it is the optimal solution of the LLS problem regarding $\mathbf{A}^s$, too. Equivalently, the following system of linear equations holds.

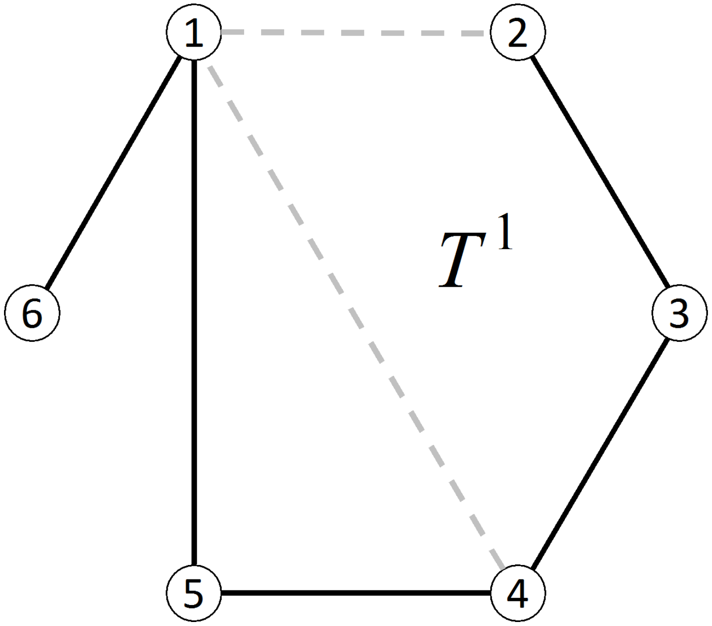
$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k) \in E(T^s)} b_{ik} + \sum_{k:e(i,k) \in E(G) \backslash E(T^s)} b_{ik}^s \quad \text{for all } i = 1, \ldots, n$$

proof

Lemma

$$\sum_{s=1}^{S}\left(\sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^{s}\right) = S\sum_{k:e(i,k)\in E(G)} b_{ik}$$

proof of the lemma



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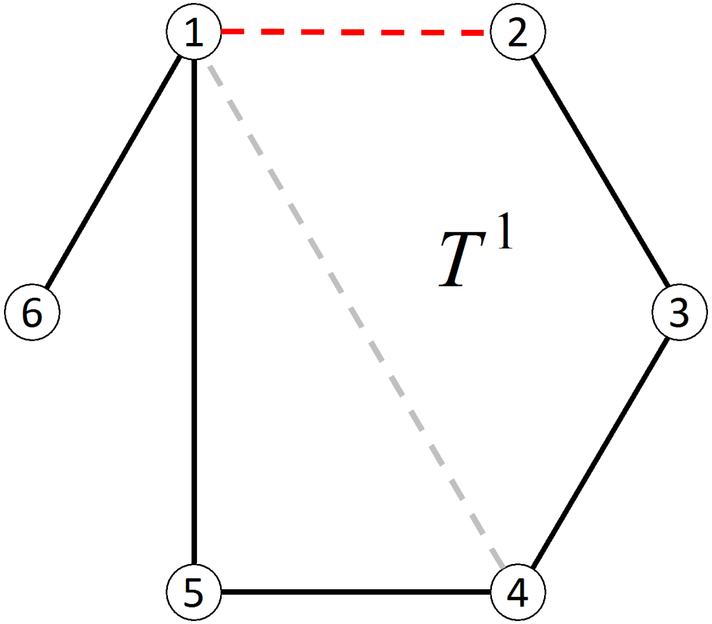

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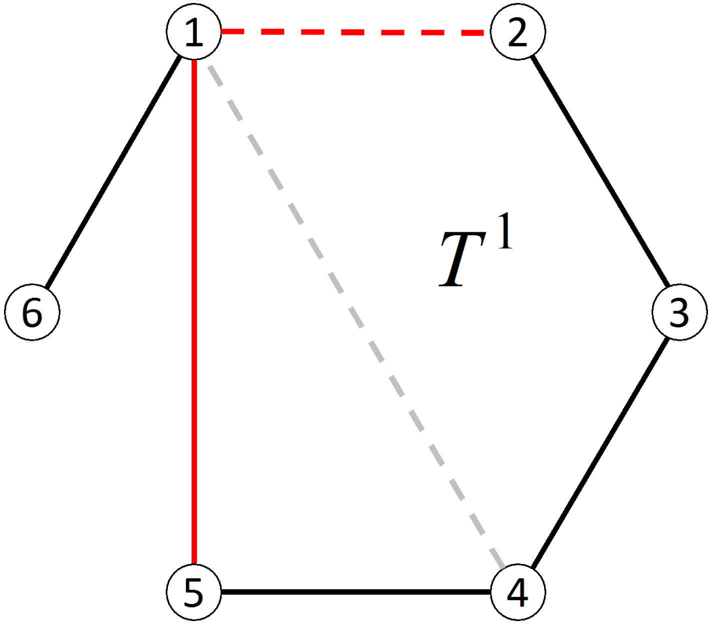
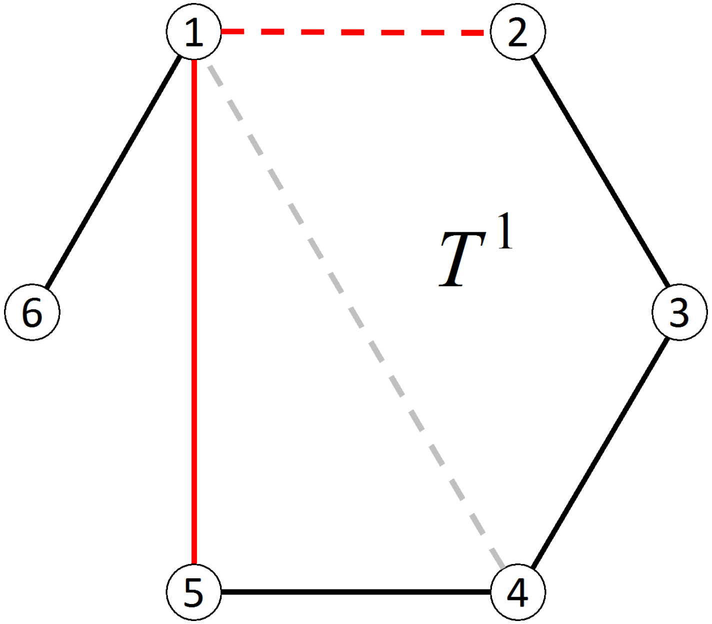
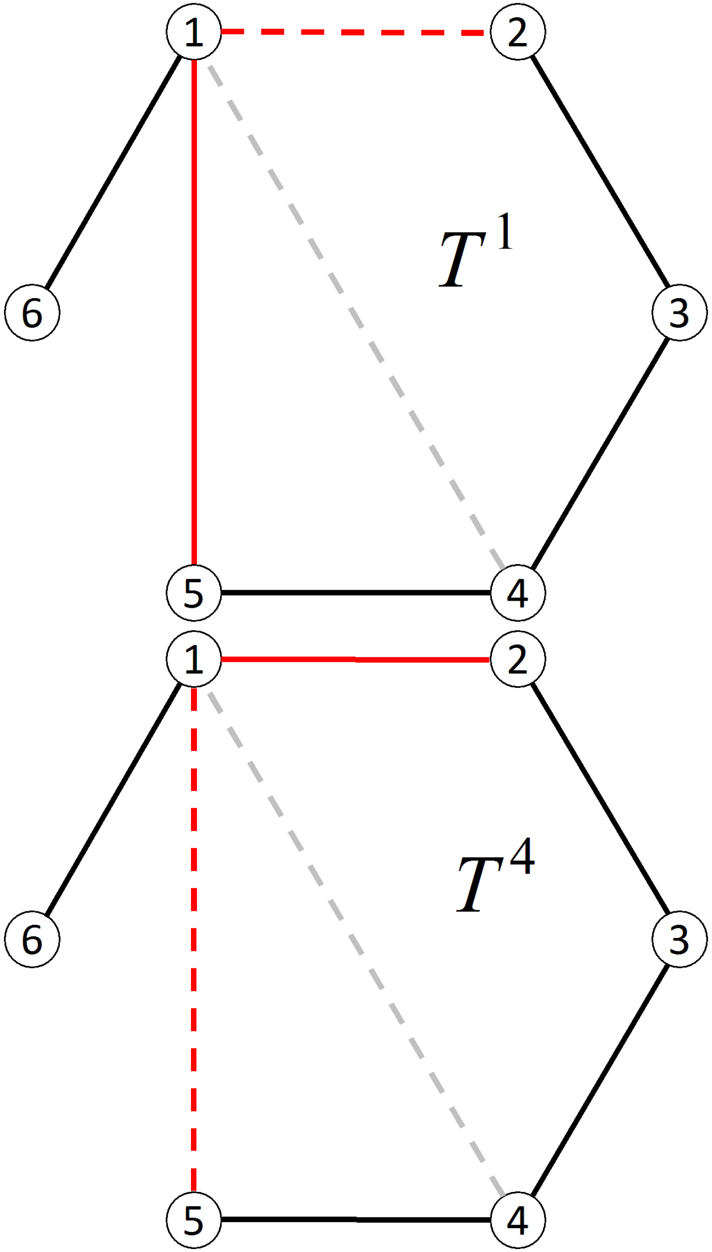


proof of the lemma



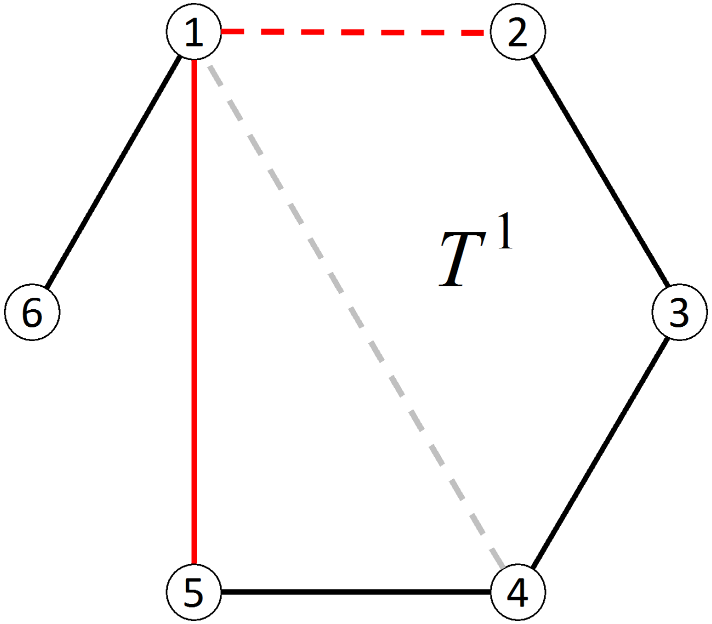
$$b_{12}^1 = b_{15} + b_{54} + b_{43} + b_{32}$$

proof of the lemma

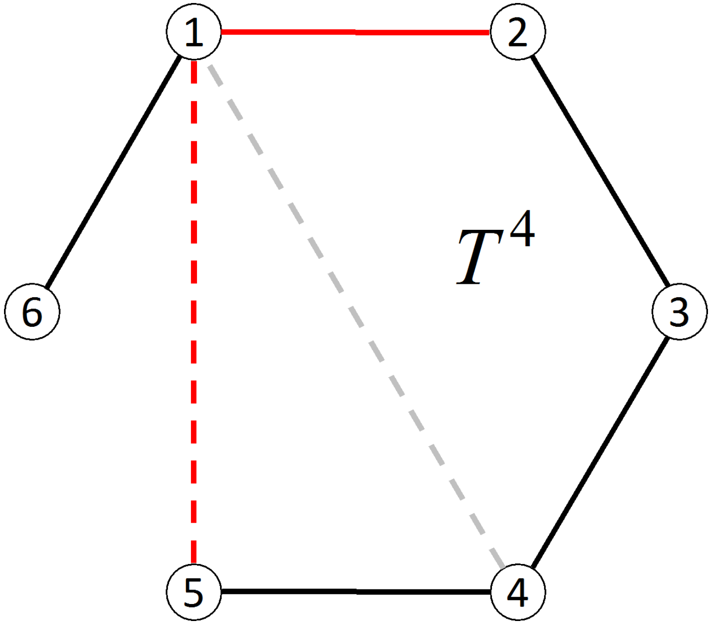


$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

proof of the lemma

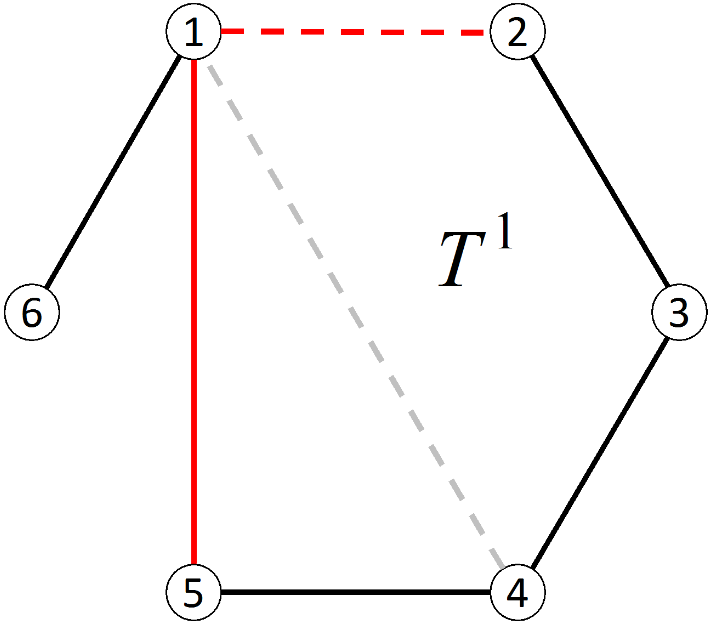


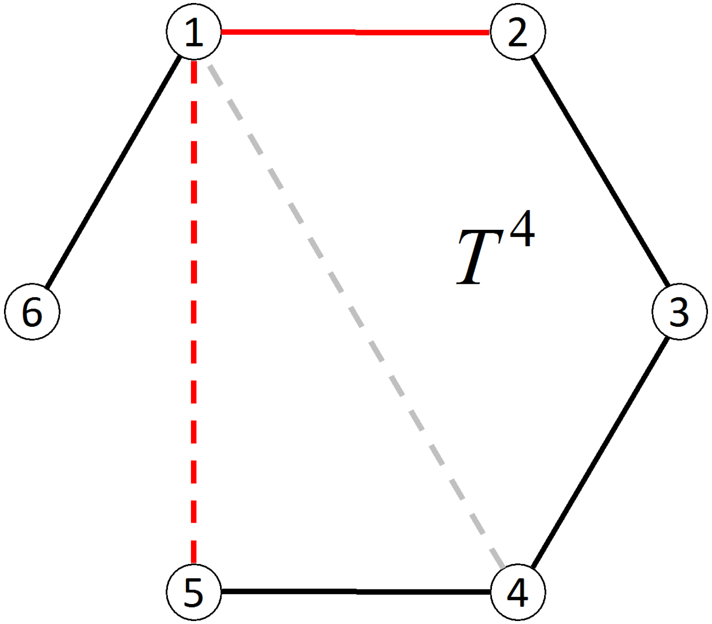
$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$



$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

proof of the lemma

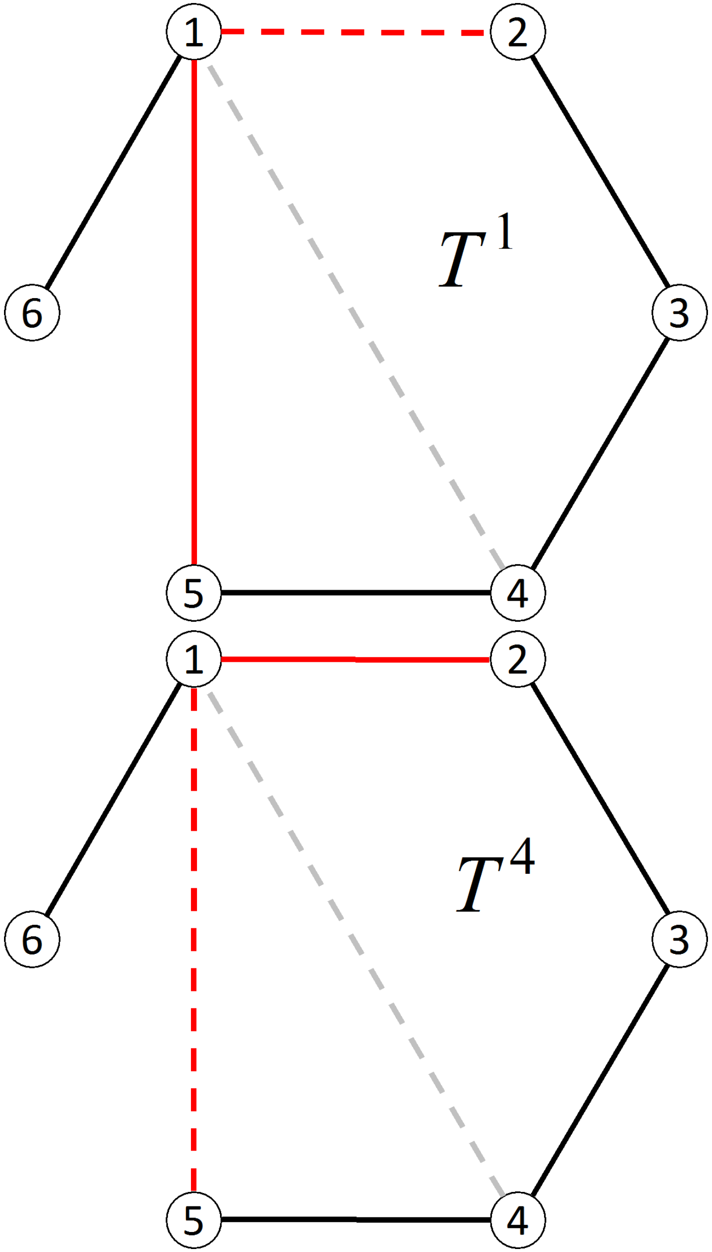




$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

proof of the lemma

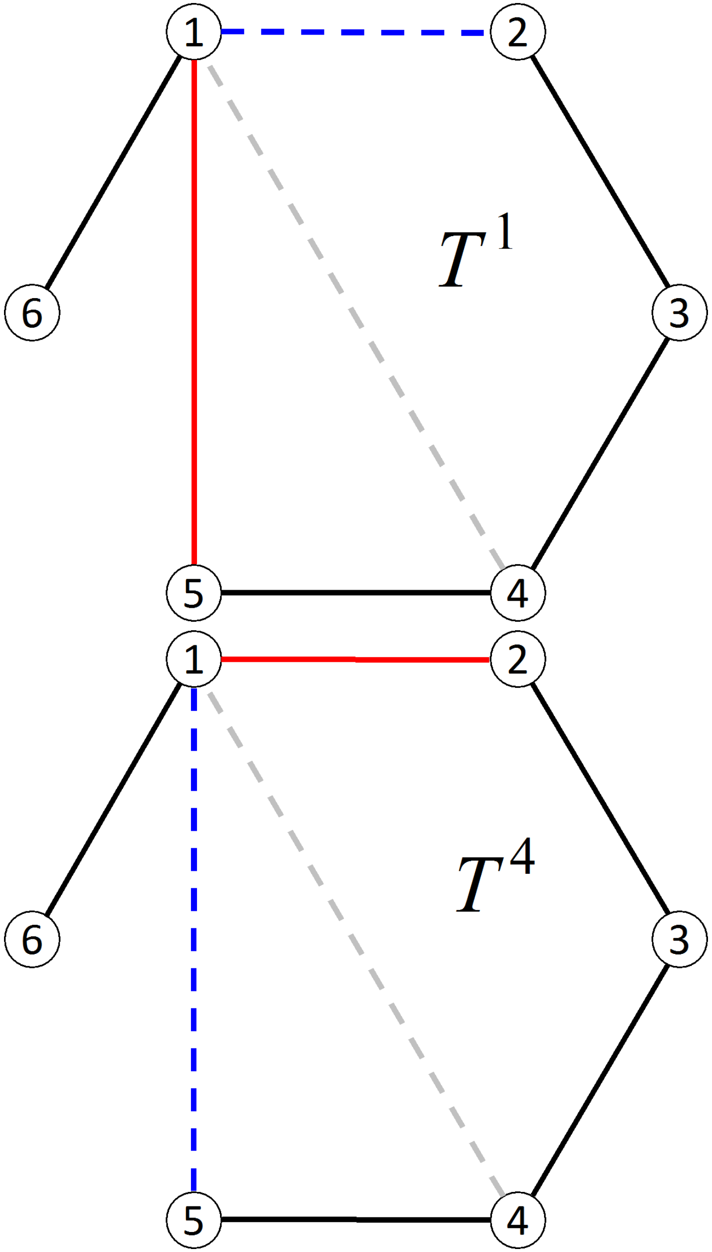


$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

$$b^1_{12} + b^4_{15} = b_{12} + b_{15}$$

proof of the lemma



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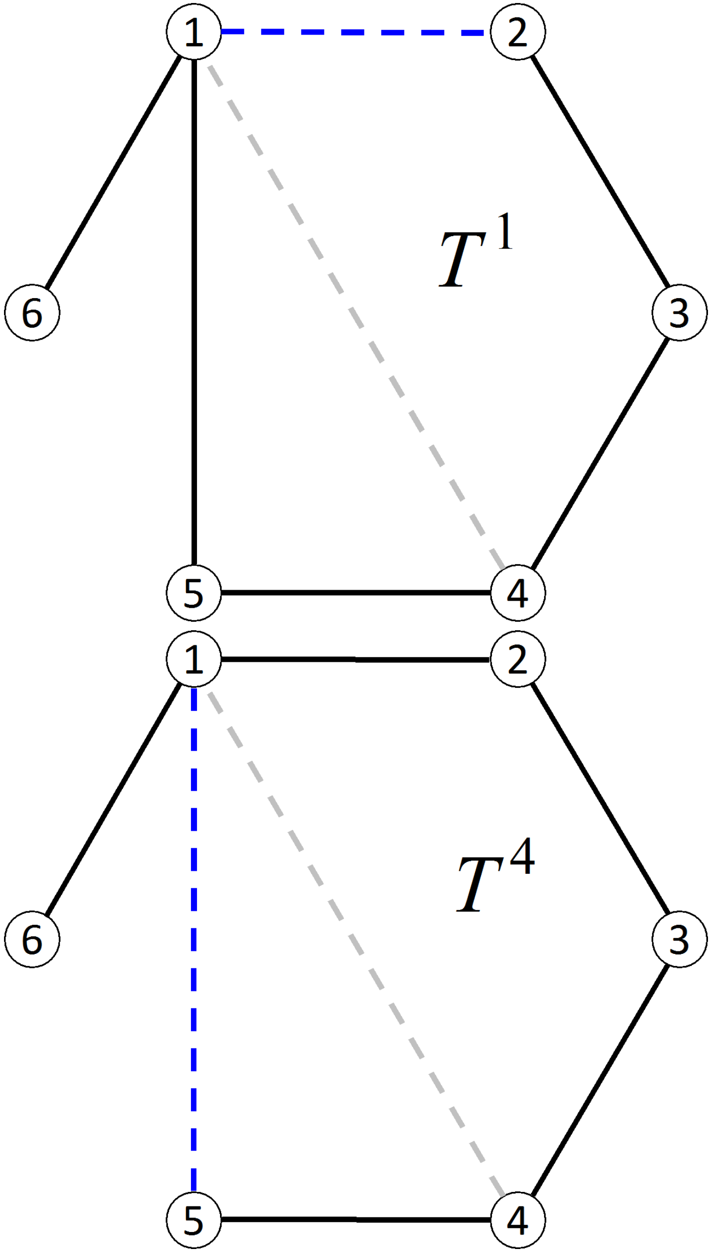

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proof

Finally, to complete the proof, take the sum of equations

$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b^s_{ik} \quad \text{for all } i = 1,\ldots,n$$

for all $s = 1, 2, \ldots, S$ and apply the lemma

$$\sum_{s=1}^{S} \left(\sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b^s_{ik} \right) = S \sum_{k:e(i,k)\in E(G)} b_{ik}$$

to conclude that $\mathbf{y}^{LLS} = \frac{1}{S} \sum_{s=1}^{S} \mathbf{y}^s$. $\square$

Applications of incomplete pairwise comparison matrices

- MCDM problems
- ranking chess teams in a tournament (Csató, 2013)
- ranking universities/faculties (Csató, 2016)
- ranking top tennis players (Bozóki, Csató, Temesi, 2016)
- ranking scientific papers submitted to a conference
- ranking Go players
- ...

Efficiency (Pareto optimality)

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix},$$

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$$\left[\frac{w_i^{EM}}{w_j^{EM}}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}$$

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$$\left[\frac{w_i'}{w_j'}\right] = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{3.9546} & \mathbf{8.8989} \\ \mathbf{1} & 1 & 3.9546 & 8.8989 \\ \mathbf{0.2529} & 0.2529 & 1 & 2.2503 \\ \mathbf{0.1124} & 0.1124 & 0.4444 & 1 \end{pmatrix}.$$

The multi-objective optimization problem is as follows:

$$\min_{x_i > 0 \; \forall i} \left(\left| a_{ij} - \frac{x_i}{x_j} \right| \right)_{i \neq j}$$

Efficiency (Pareto optimality)

Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n}$ be an $n \times n$ pairwise comparison matrix and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector.

Definition: weight vector $\mathbf{w}$ is called *efficient,* if there exists no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i, j \leq n,$$

$$\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \leq k, \ell \leq n.$$

An efficient weight vector cannot be improved such that every element of the pairwise comparison matrix is approximated at least as good, and at least one element is approximated strictly better.

Test of efficiency

Given pairwise comparison matrix $\mathbf{A}$ and weight vector $\mathbf{w}$, our goal is check whether $\mathbf{w}$ is efficient.

Let $v_i = \log w_i$, $1 \leq i \leq n$, and $b_{ij} = \log a_{ij}$, $1 \leq i, j \leq n$,

$$I = \left\{ (i,j) \middle| a_{ij} < \frac{w_i}{w_j} \right\}$$

$$J = \left\{ (i,j) \middle| a_{ij} = \frac{w_i}{w_j}, i < j \right\}$$

$$\min \sum_{(i,j) \in I} -s_{ij}$$

$$
\begin{aligned}
y_j - y_i &\leq -b_{ij} && \text{for all } (i,j) \in I, \\
y_i - y_j + s_{ij} &\leq v_i - v_j && \text{for all } (i,j) \in I, \\
y_i - y_j &= b_{ij} && \text{for all } (i,j) \in J, \\
s_{ij} &\geq 0 && \text{for all } (i,j) \in I, \\
y_1 &= 0
\end{aligned}
$$

Variables are y_i, $1 \leq i \leq n$ and $s_{ij} \geq 0$, $(i,j) \in I$.

$$\min \sum_{(i,j)\in I} -s_{ij}$$

$$y_j - y_i \le -b_{ij} \qquad \text{for all } (i,j) \in I,$$

$$y_i - y_j + s_{ij} \le v_i - v_j \qquad \text{for all } (i,j) \in I,$$

$$y_i - y_j = b_{ij} \qquad \text{for all } (i,j) \in J,$$

$$s_{ij} \ge 0 \qquad \text{for all } (i,j) \in I,$$

$$y_1 = 0$$

Theorem (Bozóki, Fülöp, 2018):
The optimum value of the linear program above is at most 0 and it is equal to 0 if and only if weight vector $\mathbf{w}$ is efficient. Denote the optimal solution to the LP above by $(\mathbf{y}^*, \mathbf{s}^*) \in \mathbb{R}^{n+|I|}$. If weight vector $\mathbf{w}$ is inefficient, then weight vector $\exp(\mathbf{y}^*)$ is efficient and dominates $\mathbf{w}$ internally.

Pairwise Comparison Matrix Calculator

The efficiency of a weight vector can be tested at

pcmc.online

If the weight vector is found to be inefficient, then a dominating efficient weight vector is found.

Characterization of efficiency

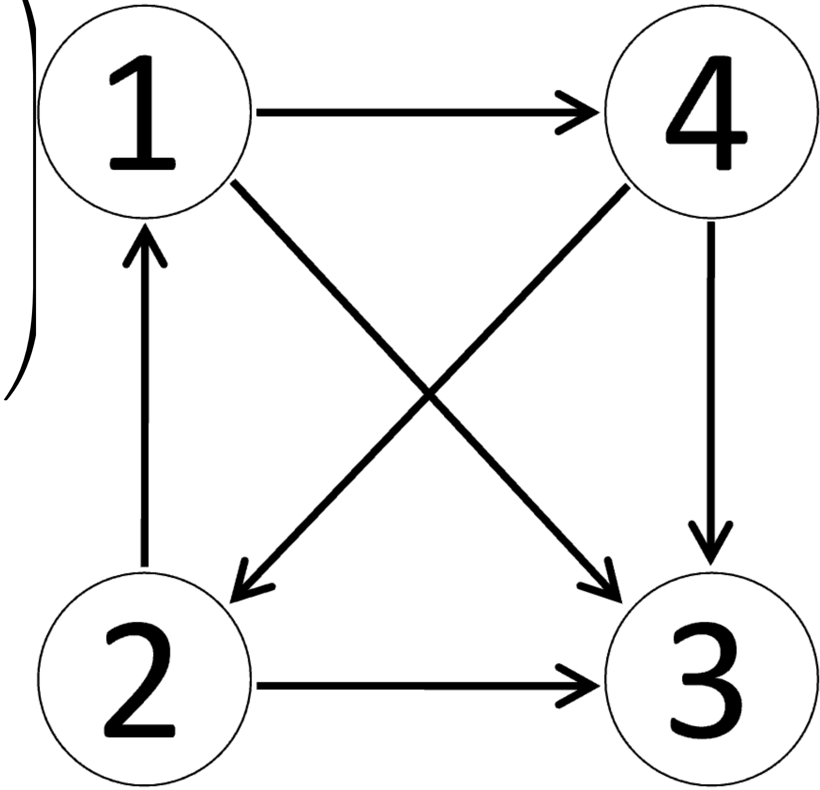
Definition: Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in \mathcal{PCM}_n$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector. Directed graph $(V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ is defined as follows: $V = \{1, 2, \ldots, n\}$ and

$$\overrightarrow{E} = \left\{ \text{arc}(i \rightarrow j) \,\middle|\, \frac{w_i}{w_j} \geq a_{ij}, i \neq j \right\}.$$

Theorem (Blanquero, Carrizosa and Conde, 2006): Weight vector $\mathbf{w}$ is efficient if and only if $(V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ is strongly connected, that is, there exist directed paths from i to j and from j to i for all pairs of $i \neq j$ nodes.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 & 2 \\ 1/2 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/2 & 1/3 & 2 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1 \\ 2.0712416 \end{pmatrix}$$

$$\mathbf{X}^{EM} = \begin{pmatrix} 1 & 1.41 & 6.01 & 2.90 \\ 0.71 & 1 & 4.26 & 2.06 \\ 0.1663 & 0.23 & 1 & 0.48 \\ 0.34 & 0.49 & 2.07 & 1 \end{pmatrix}$$



An open problem

What is a necessary and sufficient condition of the efficiency of the principal eigenvector?

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Thank you for attention.

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