APPLICATION-ORIENTED POLYTOPIC SYSTEM MODELLING AND CONTROLLER DESIGN

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Antal Bejczy Center for
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RESEARCH INFRASTRUCTURE

FABLAB

OUTLINE

1. Linear systems overview
2. Linear Parameter Varying modelling overview
3. Polytopic model based control
4. Polytopic Tensor Product model based control
5. Current mathematical challenges
6. Application example

Linear systems overview

STATE SPACE DESCRIPTION

The system dynamics can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_w\mathbf{w}(t),$$

where $\mathbf{u}(t)$ stands for the control input and $\mathbf{w}(t)$ represents the disturbances and noises.

The measured output can be written as

$$\mathbf{y}(t) = \mathbf{C}_y\mathbf{x}(t) + \mathbf{D}_{yu}\mathbf{u}(t) + \mathbf{D}_{yw}\mathbf{w}(t),$$

and the performance (cost) output

$$\mathbf{z}(t) = \mathbf{C}_z\mathbf{x}(t) + \mathbf{D}_{zu}\mathbf{u}(t) + \mathbf{D}_{zw}\mathbf{w}(t).$$

LINEAR CONTROLLER TYPES

State feedback:

$$\mathbf{u}(t) = \mathbf{F}\mathbf{x}(t)$$

(measurable states without sensor noises)

Static output feedback:

$$\mathbf{u}(t) = \mathbf{G}\mathbf{y}(t)$$

(easy to implement)

Dynamic output feedback:

$$\begin{aligned}\dot{\mathbf{x}}_c(t) &= \mathbf{A}_c\mathbf{x}_c(t) + \mathbf{B}_c\mathbf{y}(t),\\ \mathbf{u}(t) &= \mathbf{C}_c\mathbf{x}_c(t) + \mathbf{D}_c\mathbf{y}(t)\end{aligned}$$

STABILITY BASED ON LYAPUNOV DIRECT METHOD

The closed loop autonomous system $\dot{\mathbf{x}}_{cl}(t) = \mathbf{A}_{cl}\mathbf{x}_{cl}(t)$ is stable if and only if

there exists the Lyapunov function $V(t) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$ with positive definite, symmetric matrix ($\mathbf{P} \succ 0$),

such a way, that in its derivative:

$$\dot{V}(t) = \mathbf{x}^T(t)\left(\mathbf{P}\mathbf{A}_{cl} + \mathbf{A}_{cl}^T\mathbf{P}\right)\mathbf{x}(t)$$

the following definite condition holds

$$\mathbf{P}\mathbf{A}_{cl} + \mathbf{A}_{cl}^T\mathbf{P} \prec 0.$$

Special approaches: for time delay systems Lyapunov-Kraszovski, Lyapunov-Razumikhin functionals, etc.

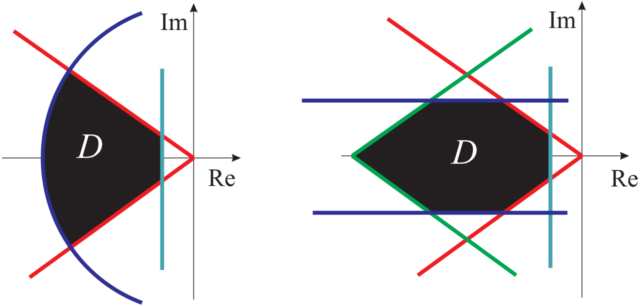
Pole placement

Controller to reallocate the eigenvalus of $\mathbf{A}_{cl}$ rewritting the dynamics

Their role in the shape of transients

Exact placement: Ackermann formula.

Constrain the poles to a region: LMI regions (definite constraints, as the stability condtions) as



H_∞ CRITERIA

$$||\mathfrak{S}_{zw}||_\infty = \sup_{||\mathbf{w}||_2 \neq 0} \frac{||\mathbf{z}||_2}{||\mathbf{w}||_2}$$

Constrain or minimise the H_∞ norm of the system,

- to minimise the gain from the noises, disturbances to the performance,
- to maximise the effect of the command signal to the performance

according to their bandwidth.

Ricatti equations: the optimal controller, H_∞ value

Bounded Real Lemma: definite condition

LINEAR MATRIX INEQUALITY BASED CONTROLLER DESIGN

Linear Matrix Inequality: definite condition on affine variables (α_1, α_2, ...)

$$\mathbf{M}_0 + \sum_i \alpha_i \mathbf{M}_i \prec 0$$

- convex feasible domain
- convex optimisation on them
- inner point methods
- criteria can easily be combined

Definite conditions are LMIs for

- state feedback
- full order dynamic feedback

Linear Parameter Varying modelling overview

LPV MODELLING

Linear Parameter Varying (LPV) models:

- Linear but parameter dependent physical phenomena (temperature, etc.)
- Uncertain/varying mechanical parameters
- etc.

Notation:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \\ \mathbf{z}(t) \end{bmatrix} = \mathbf{S}(\mathbf{p}(t)) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix},$$

where

$$\mathbf{S}(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B}_u(\mathbf{p}(t)) & \mathbf{C}_w(\mathbf{p}(t)) \\ \mathbf{C}_y(\mathbf{p}(t)) & \mathbf{D}_{yu}(\mathbf{p}(t)) & \mathbf{D}_{yw}(\mathbf{p}(t)) \\ \mathbf{C}_z(\mathbf{p}(t)) & \mathbf{D}_{zu}(\mathbf{p}(t)) & \mathbf{D}_{zw}(\mathbf{p}(t)) \end{bmatrix}$$

qLPV MODELLING

quasi-Linear Parameter Varying (qLPV) models:

- Nonlinearity appears as parameters that depend on the states
- External parameters as well

Apply feedback linearisation – if it is possible – to decrease nonlinearity

CONTROL APPROACHES

- Linear Fractional Transformation
- Grid approach
- Affine LPV model based control
- Polytopic (TP) model based control

Polytopic model based control

POLYTOPIC MODEL DEFINITION

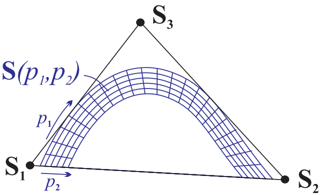
The

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \\ \mathbf{z}(t) \end{bmatrix} = \mathbf{S}(\mathbf{p}(t)) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix}, \qquad \mathbf{p} \in \Omega = [\underline{p}_1, \overline{p}_1] \times [\underline{p}_2, \overline{p}_2] \times \dots,$$

LPV/qLPV model given by convex combinations of so-called vertex systems as

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{j=1}^{J} h_j(\mathbf{p})\mathbf{S}_j, \quad \text{where} \quad \sum_{j=1}^{J} h_j(\mathbf{p}) = 1, \; h_j(\mathbf{p}) \geq 0 \; \forall \mathbf{p} \in \Omega.$$

Geometrically vertices of an enclosing polytope



FORMULATION OF LMI CONTROL CRITERIA

Stability verification:
stable if $\exists \mathbf{P} = \mathbf{P}^T \succ 0$ such that

$$\sum_j h_j \left(\mathbf{P}\mathbf{A}_j + \mathbf{A}_j^T \mathbf{P} \right) \prec 0 \quad \forall possible\ h_1, ..., h_J$$

$\rightarrow$ LMIs: $\mathbf{P}\mathbf{A}_j + \mathbf{A}_j^T \mathbf{P} \prec 0 \ \forall j$.

Static gain, state feedback controller design: $\mathbf{X} = \mathbf{P}^{-1}$, $\mathbf{M} = \mathbf{F}\mathbf{P}^{-1}$
the closed loop system is stable if $\exists \mathbf{X} = \mathbf{X}^T \succ 0$, $\mathbf{M}$ such that

$$\sum_j h_j \left(\mathbf{A}_j\mathbf{X} + \mathbf{B}_j\mathbf{M} + \mathbf{X}\mathbf{A}_j^T + \mathbf{M}^T\mathbf{B}_j^T \right) \prec 0 \quad \forall possible\ h_1, ..., h_J$$

$\rightarrow$ LMIs: $\mathbf{A}_j\mathbf{X} + \mathbf{B}_j\mathbf{M} + \mathbf{X}\mathbf{A}_j^T + \mathbf{M}^T\mathbf{B}_j^T \prec 0 \ \forall j$.

By exploiting that all convex combinations of matrices are positive definite iff they are positive definite.

FORMULATION OF LMI CONTROL CRITERIA

Scheduling state feedback controller design:
$\mathbf{X} = \mathbf{P}^{-1}$, $\mathbf{M}(\mathbf{p}) = \mathbf{F}(\mathbf{p})\mathbf{P}^{-1} = \sum_j h_j(\mathbf{p})\mathbf{M}_j$
the closed loop system is stable if $\exists \mathbf{X} = \mathbf{X}^T \succ 0$, $\mathbf{M}_1, \ldots, \mathbf{M}_J$ such that

$$\sum_j \sum_k h_j h_k \left(\mathbf{A}_j\mathbf{X} + \mathbf{B}_j\mathbf{M}_k + \mathbf{X}\mathbf{A}_j^T + \mathbf{M}_k^T\mathbf{B}_j^T \right) \prec 0 \quad \forall possible\ h_1, \ldots, h_J$$

Different methods to obtain sufficient LMIs conditions, e.g.,
$\rightarrow \mathbf{A}_j\mathbf{X} + \mathbf{B}_j\mathbf{M}_j + \mathbf{X}\mathbf{A}_j^T + \mathbf{M}_j^T\mathbf{B}_j^T \prec 0\ \forall j$,
$\rightarrow (\mathbf{A}_j + \mathbf{A}_k)\mathbf{X} + \mathbf{B}_j\mathbf{M}_k + \mathbf{B}_k\mathbf{M}_j + (*)^T \prec 0\ \forall j, k < j$.

CONVEXITY

The scheduling controller design is a convex problem in general for:

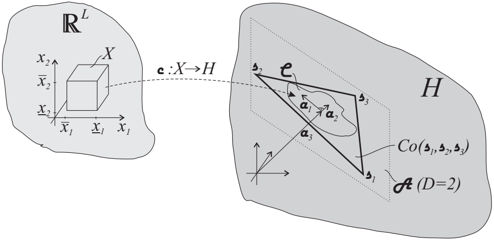
- state feedback design
- full order dynamic output feedback

The robust controller design problem is a convex problem in general for:

- state feedback design

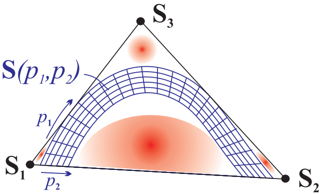
DERIVATION BASED ON AFFINE GEOMETRY

To obtain polytopic form for a function (e.g., parameter varying system matrices)



1. The affine hull ($\mathfrak{A}$) of its image set can be identified
2. The image can be described on its basis with coordinates of appropriate dimension
3. An enclosing polytope ($\mathfrak{C}$) can be constructed

ENCLOSING POLYTOPE ON THE AFFINE HULL



The role of enclosing polytopes is essential in control applications:

- The design methods see the whole polytopic model without taking into account what system matrices belong to the investigated model and which are irrelevant (red regions)
- If there is any non-stabilizable system within the polytopic model, the design is infeasible.

Problem: to derive vertices considering the control aspects.

Two phase approach:

1. Generation based on clear geometric criteria.
2. Manipulation of the enclosing polytopes by identifying the roles of the vertices in the control problem.

Generation methods:

- (near) Minimal Volume Simplex

and methods to manipulate them taking into account the control properties of the considered LPV/qLPV models:

- manipulation of vertices of the MVS,
- cutting off regions with hyperplanes obtaining non-simplex polytopes,
- local volume minimalization of Non-Simplex polytopes

Polytopic TP model based control

POLYTOPIC TP MODEL DEFINITON

Consider the function $\mathbf{S}(\mathbf{p}) : \Omega \to \mathbb{S}$.

According to the chosen parameter sets (denoted as $\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \dots \subseteq \mathbf{p}$ and their domains accordingly as $\Omega_1, \Omega_2, \dots$), the TP form

$$\mathbf{S}(\mathbf{p}) = \mathcal{S} \times_1 \mathbf{w}^{(1)}(\mathbf{p}^{(1)}) \times_2 \mathbf{w}^{(2)}(\mathbf{p}^{(2)}) \dots$$

$$\text{where} \quad \mathbf{w}^{(k)} : \Omega_k \to \mathbb{R}^{J_k}, \quad \text{and} \quad \mathcal{S} \in \mathbb{S}^{J_1 \times J_2 \times \dots}$$

is called a Polytopic TP form if

$$\sum_j w_j^{(k)}(\mathbf{p}^{(k)}) = 1, \quad \mathbf{w}^{(k)}(\mathbf{p}^{(k)}) \geq \mathbf{0} \qquad \forall \mathbf{p}^{(k)} \in \Omega_k, \; k.$$

MULTIPLE PARAMETER DEPENDENCIES

TP forms can depend on multivariate parameter sets (optionally with two times or higher multiplicities)

$$\mathbf{X}(\mathbf{p}) = \mathcal{X} \underbrace{\times_1 \mathbf{w}^{(1)}(\mathbf{p}^{(1)}) \times_2 \mathbf{w}^{(1)}(\mathbf{p}^{(1)}) \cdots \times_{M_1} \mathbf{w}^{(1)}(\mathbf{p}^{(1)})}_{M_1}$$

$$\underbrace{\times_{M_1+1} \mathbf{w}^{(2)}(\mathbf{p}^{(2)}) \times_{M_1+2} \cdots \times_{M_1+M_2} \mathbf{w}^{(2)}(\mathbf{p}^{(2)})}_{M_2} \ldots,$$

can be denoted as

$$\mathbf{X}(\mathbf{p}) = \mathcal{X} \mathop{\boxtimes}_{k=1}^{K(\mathbf{M})} \mathbf{w}^{(I(k,\mathbf{M}))}(\mathbf{p}^{(I(k,\mathbf{M}))}),$$

where

$$K(\mathbf{M}) = \sum_i M_i, \qquad I(k,\mathbf{M}) = i \ \text{ if } \sum_{a=1}^{i-1} M_a < k < \sum_{a=1}^{i} M_a,$$

and the **M** multiplicity vector describes the structure.

POLYTOPIC TP MODEL BASED CONTROL ANALYSIS AND SYNTHESIS

The variables of controller and Lyapunov function candidates are in TP forms with appropriately chosen multiplicities:

If a system has measured and non-measured parameters as well

- robustness against the non-measured ones
- use measurable scheduling parameters to reach better performance

If a system has non-varying and varying parameters as well

- the Lyapunov-function can depend on the non-varying ones (to decrease conservatism)
- the varying parameters does not appear in the Lyapunov-function (or they are appropriately handled)

Definite condition on TP variables can be extracted to (Linear) Matrix Inequalities

DERIVATION OF POLYTOPIC TP MODELS

The polytopic TP model can be derived through the Affine TP model

$$\mathbf{S}(\mathbf{p}) = \mathcal{S}^{aff} \times_1 \mathbf{v}^{(1)}(\mathbf{p}^{(1)}) \times_2 \mathbf{v}^{(2)}(\mathbf{p}^{(2)}) \ldots,$$

The derivation requires enclosing polytopes for the weighting functions $\mathbf{v}^{(k)}(\mathbf{p}^{(k)})$ in the euclidean space with the given dimension.

Generation then manipulation until satisfactory results or too complex polytope is obtained.

Current mathematical challenges

Non-convex optimisation problems:

- Minimal Volume Enclosing Polytopes with given number of verteces/facets
- Controller design:
  - static output feedback design
  - robust dynamic output feedback design

Application of affine structure of the non-simplex polytopic model during controller design

- To allow parametrisation of the parameter dependent matrices with less scalar variables
- To provide less complex structures in the controller candidate (less online computation)
- Does it decrease the achievable performances?

Application example

MECHANICAL MODEL

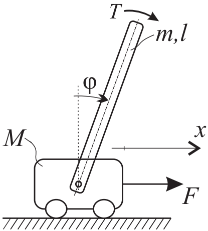


FIGURE: Notations

Equation of motions:

$$\ddot{\varphi} = \frac{6\frac{g}{l}(m+M) - 3m\dot{\varphi}^2\cos\varphi}{D}\sin\varphi - \frac{6\cos\varphi}{l\cdot D}F(t) + \frac{12(m+M)}{ml^2D}T(t)$$

$$\ddot{x} = \frac{m(2\dot{\varphi}^2 l - 3g\cos\varphi)}{D}\sin\varphi + \frac{4}{D}F(t) - \frac{6\cos\varphi}{l\cdot D}T(t),$$

where $D = D(\varphi, m, M) = 4M + m(1+3\sin^2\varphi)$ and the parameters: $m = 0.1[kg]$, $l = 0.3[m]$, $M \in [0.9, 1.3]\,[kg]$, $|\varphi| \in [0, 50]\,[deg]$.

qLPV MODELLING

By applying feedback linearisation, the following qLPV model can be derived

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 6g\dfrac{M-M_0}{l\cdot D}\dfrac{\sin\varphi}{\varphi} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & g\dfrac{D_0}{D}\dfrac{\tan\varphi}{\varphi} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}(\varphi,M)} \mathbf{x}(t) + \underbrace{\begin{bmatrix} \dfrac{D_0}{D} \\ 0 \\ \dfrac{-2}{3\cos\varphi}\dfrac{D_0}{D} \\ 0 \end{bmatrix}}_{\mathbf{B}_u(\varphi,M)} u(t) +$$

$$+ \underbrace{\begin{bmatrix} \dfrac{2}{ml\cos\varphi} \\ 0 \\ \dfrac{-1}{M+m} \\ 0 \end{bmatrix}}_{\mathbf{B}_w(\varphi,M)} w(t),$$

where the state variables are $\mathbf{x}(t) = \begin{bmatrix} \dot{\varphi} & \varphi & \dot{x} & x \end{bmatrix}^T$ and the parameters are the φ angle and the M mass.

CONTROL DESIGN GOALS

- State feedback controller
- To minimise the effect of $T(t)$ disturbance torque to the pendulum and car
- By minimising the H_∞ norm from $w(t) = T(t)$ disturbance to $\mathbf{z}(t) = \begin{bmatrix} x(t) & \dot{\varphi}(t) \end{bmatrix}^T$

Chosen parameter sets:

- $p^{(1)} = |\varphi|$, $p^{(2)} = M$
- $\mathbf{p}^{(3)} = \begin{bmatrix} |\varphi| & M \end{bmatrix}$: for the not exactly separable part

LMI CONTROL CRITERIA

METHOD (BOUNDED REAL LEMMA)

Consider the following SDP:

$$\min_{\mathbf{X}(\mathbf{p}),\mathbf{M}(\mathbf{p})} \gamma_\infty$$

$$\text{s.t.} \quad \mathbf{X}(\mathbf{p}) \succ 0, \quad \dot{\mathbf{X}}(\mathbf{p}) = 0,$$

$$\begin{bmatrix} -\mathrm{Sym}(\mathbf{A}(\mathbf{p})\mathbf{X}(\mathbf{p}) + \mathbf{B}_u(\mathbf{p})\mathbf{M}(\mathbf{p})) & \mathbf{B}_w(\mathbf{p}) & (\mathbf{C}\mathbf{X}(\mathbf{p}))^T \\ * & \gamma_\infty & \mathbf{0} \\ * & * & \gamma_\infty \mathbf{E} \end{bmatrix} \succ 0,$$

where $\mathbf{X}(\mathbf{p})$, $\mathbf{M}(\mathbf{p})$ *are TP functions with the same polytopic structure and with given multiplicities.*

Then with the resulted $u(t) = \mathbf{M}(\mathbf{p})\mathbf{X}(\mathbf{p})^{-1}\mathbf{x}(t)$ *controller, the* $\left\|\frac{\mathbf{C}\mathbf{x}(t)}{w(t)}\right\|_\infty < \gamma_\infty$ *condition holds for all* $\mathbf{p}(t) \in \Omega$ *trajectories.*

AFFINE TP FORM

The Affine TP form:

$$\mathbf{S}(\mathbf{p}) = \mathcal{S}^{aff} \times_1 \mathbf{v}^{(1)}(p^{(1)}) \times_2 \mathbf{v}^{(2)}(p^{(2)}) \times_3 \mathbf{v}^{(3)}(\mathbf{p}^{(3)})$$

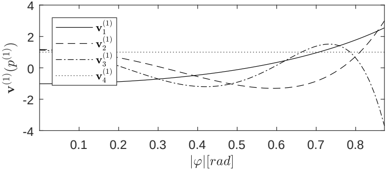


FIGURE: The $\mathbf{v}^{(1)}(p^{(1)})$ weighting functions

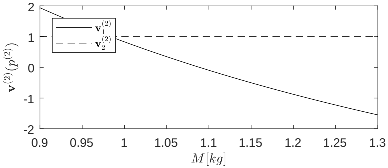


FIGURE: The $\mathbf{v}^{(2)}(p^{(2)})$ weighting functions

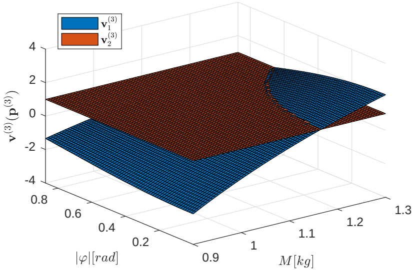


FIGURE: The $\mathbf{v}^{(3)}(\mathbf{p}^{(3)})$ weighting functions

MVS POLYTOPIC TP FORM

The enclosing polytope for the 3 dimensional image from $p^{(1)}$

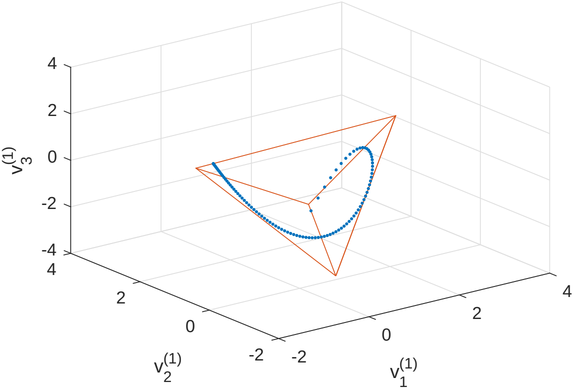


FIGURE: Minimal volume enclosing simplex for $\mathbf{v}^{(1)}(p_1)$ function

The Minimal Volume Simplex Polytopic TP model

$$\mathbf{S}(\mathbf{p}) = \mathcal{S} \times_1 \mathbf{w}^{(MVS,1)}(p^{(1)}) \times_2 \mathbf{w}^{(MVS,2)}(p^{(2)}) \times_3 \mathbf{w}^{(MVS,3)}(\mathbf{p}^{(3)})$$

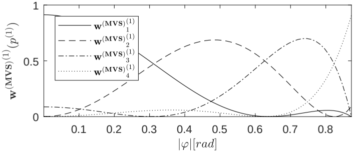


FIGURE: The $\mathbf{w}^{(MVS,1)}(p^{(1)})$ weighting functions

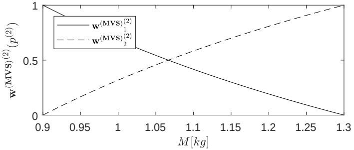


FIGURE: The $\mathbf{w}^{(MVS,2)}(p^{(2)})$ weighting functions

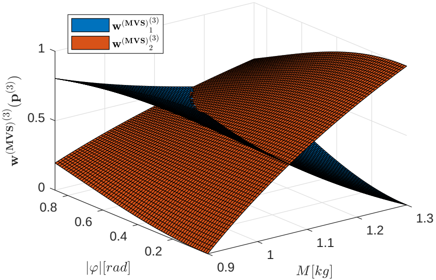


FIGURE: The $\mathbf{w}^{(MVS,3)}(\mathbf{p}^{(3)})$ weighting functions

CONTROLLER DESIGN

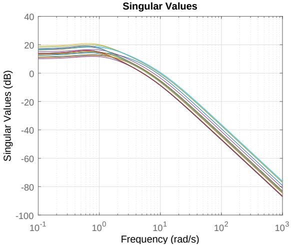


FIGURE: $W_{w\varphi}(j\omega)$

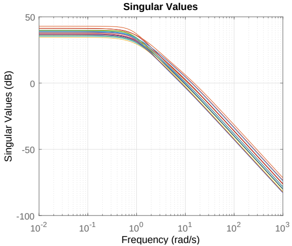


FIGURE: $W_{wx}(j\omega)$

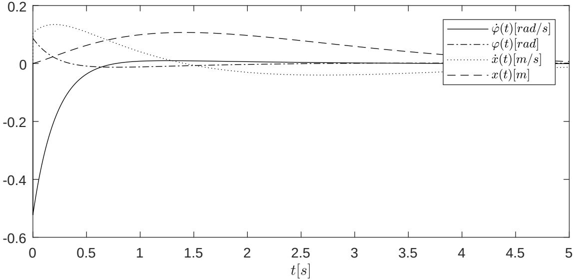


FIGURE: The transient of the controlled system

TP CONTROLLER AND LYAPUNOV FUNCTION

Improve the performance by applying more complex Lyapunov function and controller-candidates:

- m_φ: multiplicity of $\mathbf{w}^{(1)}(|\varphi|)$ in the controller candidate
- m_x: multiplicity of $\mathbf{w}^{(2)}(M)$ in the controller candidate and in the Lyapunov-function

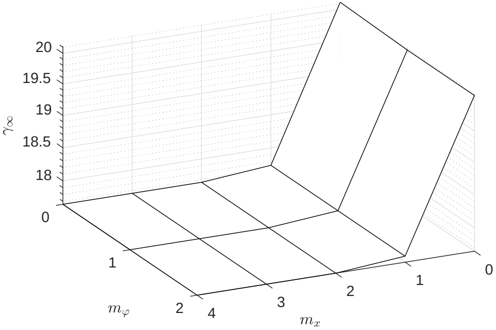


FIGURE: The achievable γ_∞ disturbance rejection

OPTIMISED THE ENCLOSED POLYTOPES

The best result with MVS manipulation: $\gamma^*_\infty = 18.794$

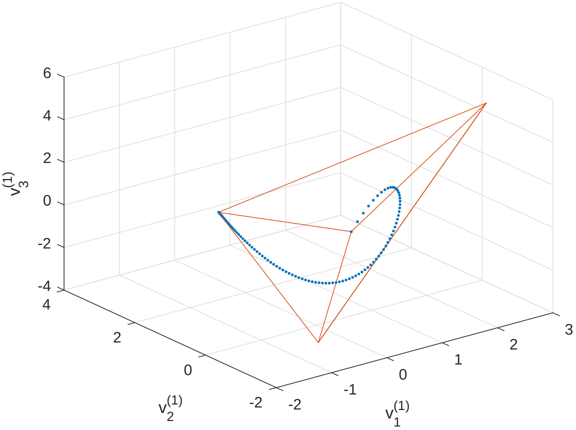


FIGURE: Minimal volume enclosing simplex for $\mathbf{v}^{(1)}(p_1)$ function

Thank you for attention!

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