Lineáris feltételes, szeparábilis konkáv minimalizálási feladat alkalmazásai és megoldása

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Linearly constrained, separable concave minimization problem

$$\left.\begin{array}{r} \min\ F(\mathbf{x}) \\ A\,\mathbf{x} \leq \mathbf{b} \\ \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{array}\right\} \quad (P)$$

where $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{l}, \mathbf{u} \in \mathbb{R}^n$ and $\mathbf{l} \geq \mathbf{0}$.

Objective function: $F(\mathbf{x}) := \sum_{j=1}^{n} f_j(x_j)$, where $f_j : \mathbb{R} \to \mathbb{R}$ are concave functions and for the *domain of f_j* $[l_j, u_j] \subseteq \mathcal{D}_{f_j}$ holds. Let us introduce the sets $\mathcal{A} := \{\mathbf{x} \in \mathbb{R}^n : A\,\mathbf{x} \leq \mathbf{b}\}$ and $\mathcal{T} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$.

Feasible solution set: $\mathcal{P} = \mathcal{A} \cap \mathcal{T}$

set of the optimal solutions: $\mathcal{P}^* := \{\bar{\mathbf{x}} \in \mathcal{P} : F(\bar{\mathbf{x}}) \leq F(\mathbf{x}),\ \ \mathbf{x} \in \mathcal{P}\}$

Known results:

1. If $\mathcal{P} \neq \emptyset$ then $\mathcal{P}^* \neq \emptyset$ holds, since F is continuous and $\mathcal{P}$ is bounded and closed.
2. There is optimal solution at a vertex of the polytop $\mathcal{P}$. (Luenberger, 1973)
3. The problem (P) is in the class of NP-complete problems. (Murty and Kabadi, 1987)

Practical problems

Several practical problem can be formulated by problem (P) like

- some control problems (e.g. Apkarian and Tuan, 1999),
- concave knapsack problems (e.g. Moré and Vavasis, 1990/91),
- some production and transportation problems (e.g. Kuno and Utsunomiya, 2000),
- production planning problems (e.g. Liu, Sahinidis and Shectman, 1996),
- process network synthesis problems (e.g. Friedler, Fan and Imreh, 1998),
- some network flow problems (e.g. Yajima and Konno, 1999),
- ...

Solution methods

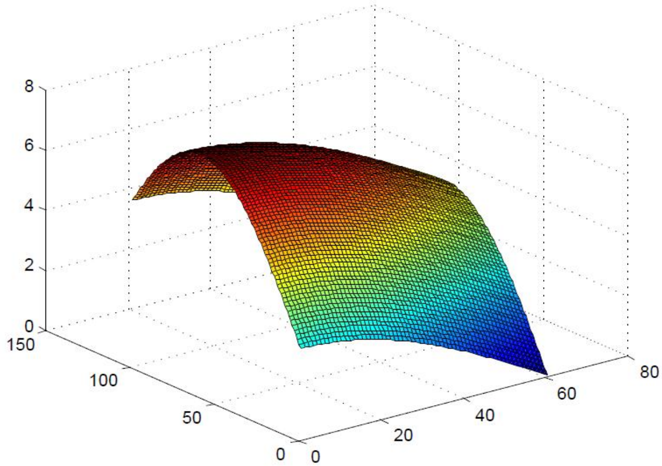
- *listing vertices of the polyhedron* $\mathcal{P}$ (e.g. Dyer, 1983; Dyer and Proll, 1977),
- *cutting plane methods* (e.g. Hoffman, 1981; Tuy, Thieu and Thai, 1985),
- *branch-and-bound algorithms, BB* (e.g. Falk and Soland, 1969; Shectman and Sahinidis, 1998; Phillips and Rosen, 1993; Locatelli and Thoai, 2000) and
- other methods ...

Example

$$\min \quad 5\,\sin\left(\tfrac{\pi}{6}\,x_1\right) + 3\,\cos\left(\tfrac{\pi}{6}\,x_2\right)$$

$$\begin{array}{rrlrrlrrl}
x_1 & -3\,x_2 & \le 2, & x_1 & -\ x_2 & \le 3, & 3\,x_1 & -\ x_2 & \le 12 \\
2\,x_1 & +\ x_2 & \le 11, & -\ x_1 & +5\,x_2 & \le 10, & -3\,x_1 & +2\,x_2 & \le 0 \\
 & & & -3\,x_1 & -\ x_2 & \le -3, & & &
\end{array}$$

$$0 \le x_1 \le 5 \qquad\qquad 0 \le x_2 \le 3$$



Elementary properties of concave functions

Theorem. Let f be one dimensional function on interval $I \subset D_f$.
The following statements are equivalent

(a) Function f is concave on interval I.

(b) Let $x, y \in I,\ x \neq y$ and $m(x,y) = \frac{f(y)-f(x)}{y-x}$. If $a, b, c \in I$, $a < b < c$ then the following holds $m(a,b) \geq m(a,c) \geq m(b,c)$.

(c) For any $t \in I, m_t(x) = m(t,x)$ function is decreasing on $I \setminus \{t\}$.

(a) If $a, b, c \in I,\ a < b < c$ then $m(a,b) \geq m(b,c)$. •

Theorem. Let f be one dimensional concave function on open interval $I \subset D_f$, then

(a) Function f is continuous on interval I.

(b) At any $t \in I$ the function is left and right differentiable and

$$f'_-(t) \geq f'_+(t).$$

(c) If $a, b, \in I,\ a < b$ then $f'_+(a) \geq m(a,b) \geq f'_-(b)$, moreover, if f is strict concave on interval I, then $f'_+(a) > m(a,b) > f'_-(b)$. •

Linear relaxation of concave functions

BB-type linear relaxation of the concave functions $f_j : \mathbb{R} \to \mathbb{R}$ on the closed interval $[l_j,\, u_j]$ is

$$g_j(x_j) = m(l_j, u_j)\,(x_j - l_j) + f_j(l_j) = \frac{f_j(u_j) - f_j(l_j)}{u_j - l_j}\, x_j + \left(f_j(l_j) - \frac{f_j(u_j) - f_j(l_j)}{u_j - l_j}\, l_j \right) = c_j\, x_j + d_j,$$

where $c_j = m(l_j, u_j)$ and $d_j = f_j(l_j) - m(l_j, u_j)\, l_j$. Then the objective function $F(\mathbf{x}) = \sum_{j=1}^{n} f_j(x_j)$ is approximated by the linear function

$$G(\mathbf{x}) = \sum_{j=1}^{n} g_j(x_j) = \sum_{j=1}^{n} (c_j\, x_j + f_j(l_j) - c_j\, l_j) = \mathbf{c}^T\mathbf{x} + (F(\mathbf{l}) - \mathbf{c}^T\mathbf{l}) = \mathbf{c}^T\mathbf{x} + \delta$$

on the set $\mathcal{P} = \mathcal{A} \cap \mathcal{T}$, where $\delta = F(\mathbf{l}) - \mathbf{c}^T\mathbf{l}$. It is easy to show that

$$F(\mathbf{x}) \geq G(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \delta, \qquad \text{holds for all } \mathbf{x} \in \mathcal{P}.$$

Example (continue) $\quad x_1 \in [0, 5]$ and $x_2 \in [0, 3]$

$$f_1(x_1) = 5\,\sin(\frac{\pi}{6}\, x_1): \qquad c_1 = \frac{5\,\sin(\frac{\pi}{6}\,5)}{5} = \frac{1}{2}, \;\; d_1 = 0 \;\Rightarrow\; g_1(x_1) = \frac{1}{2}\, x_1$$

$$f_2(x_2) = 3\,\cos(\frac{\pi}{6}\, x_2): \qquad c_2 = \frac{3\,\cos(\frac{\pi}{6}\,3) - 3\,\cos(0)}{3} = -1, \;\; d_2 = 3 \;\Rightarrow\; g_2(x_2) = -x_2 + 3$$

Linear relaxation of the problem

Lower bound for the objective value of (P) can be computed using the following linear programming problem

$$\min_{\mathbf{x}\in\mathcal{P}} \; \mathbf{c}^T\mathbf{x} + \delta \qquad (P_{LP})$$

Proposition. Let $\tilde{\mathbf{x}} \in \mathcal{P}^*_{LP}$ and assume that $F \in \mathcal{C}(int(\mathcal{T}))$ then

$$\beta = \mathbf{c}^T\tilde{\mathbf{x}} + \delta = G(\tilde{\mathbf{x}}) \le F(\mathbf{x}) \le F(\tilde{\mathbf{x}}) + \left(\nabla F(\tilde{\mathbf{x}})\right)^T(\mathbf{x} - \tilde{\mathbf{x}})$$

holds for all $\mathbf{x} \in \mathcal{P}$. •

Example (continue)

$$\min \; \tfrac{1}{2}x_1 - x_2 + 3$$

$$\begin{array}{rrl} x_1 & -3x_2 & \le 2, \\ 2x_1 & + \; x_2 & \le 11, \end{array} \qquad \begin{array}{rrl} x_1 & - \; x_2 & \le 3, \\ - \; x_1 & +5x_2 & \le 10, \\ -3x_1 & - \; x_2 & \le -3, \end{array} \qquad \begin{array}{rrl} 3x_1 & - \; x_2 & \le 12 \\ -3x_1 & +2x_2 & \le 0 \end{array}$$

$$0 \le x_1 \le 5 \qquad\qquad 0 \le x_2 \le 3$$

Optimal solution: $\tilde{x}_1 = 1.53846,\ \tilde{x}_2 = 2.30769,$ and $G(\tilde{\mathbf{x}}) = 1.46154$

$$G(\tilde{\mathbf{x}}) = 1.46154 \le F(\mathbf{x}) = 5\,\sin(\tfrac{\pi}{6}\,x_1) + 3\,\cos(\tfrac{\pi}{6}\,x_2) \le 1.8135\,x_1 - 1.4687\,x_2 + 5.269$$

Linear programming relaxation of the original problem

Let us consider the relaxed LP problem (and it's dual) of (P) in the following form

$$\left.\begin{array}{r} \min\ \mathbf{c}^T\mathbf{x} \\ A\,\mathbf{x} \le \mathbf{b} \\ \mathbf{l} \le \mathbf{x} \le \mathbf{u} \end{array}\right\} \quad (P_{LP}) \qquad\qquad \left.\begin{array}{c} \max\ -\mathbf{b}^T\mathbf{y} + \mathbf{l}^T\mathbf{z} - \mathbf{u}^T\mathbf{s} \\ -A^T\mathbf{y} + \mathbf{z} - \mathbf{s} = \mathbf{c} \\ \mathbf{y} \ge \mathbf{0}, \quad \mathbf{z} \ge \mathbf{0}, \quad \mathbf{s} \ge \mathbf{0} \end{array}\right\} \quad (D_{LP})$$

Set of the dual feasible solutions: $\mathcal{D} = \{(\mathbf{y},\mathbf{z},\mathbf{s}) : -A^T\mathbf{y} + \mathbf{z} - \mathbf{s} = \mathbf{c},\ \mathbf{y} \ge \mathbf{0},\ \mathbf{z} \ge \mathbf{0},\ \mathbf{s} \ge \mathbf{0}\}$

Weak Duality Theorem. Let $\mathbf{x} \in \mathcal{P}$ and $(\mathbf{y},\mathbf{z},\mathbf{s}) \in \mathcal{D}$ vectors then

$$\mathbf{c}^T\mathbf{x} \ge -\mathbf{b}^T\mathbf{y} + \mathbf{l}^T\mathbf{z} - \mathbf{u}^T\mathbf{s}$$

inequality holds. Previous inequality holds with equality if and only if

$$0 = \mathbf{c}^T\mathbf{x} + \mathbf{b}^T\mathbf{y} - \mathbf{l}^T\mathbf{z} + \mathbf{u}^T\mathbf{s} = \mathbf{y}^T(\mathbf{b} - A\,\mathbf{x}) + \mathbf{z}^T(\mathbf{x} - \mathbf{l}) + \mathbf{s}^T(\mathbf{u} - \mathbf{x}). \qquad \bullet$$

Optimality criteria:

$$\begin{array}{c} A\,\mathbf{x} \le \mathbf{b}, \quad \mathbf{l} \le \mathbf{x} \le \mathbf{u} \\ -A^T\mathbf{y} + \mathbf{z} - \mathbf{s} = \mathbf{c}, \quad \mathbf{y} \ge \mathbf{0},\ \mathbf{z} \ge \mathbf{0},\ \mathbf{s} \ge \mathbf{0} \\ \mathbf{y}\,(\mathbf{b} - A\,\mathbf{x}) = \mathbf{0}, \quad \mathbf{z}\,(\mathbf{x} - \mathbf{l}) = \mathbf{0}, \quad \mathbf{s}\,(\mathbf{u} - \mathbf{x}) = \mathbf{0}, \end{array}$$

$\mathcal{P}_c^* = \{\mathbf{x}^* \in \mathcal{P} : \mathbf{c}^T\mathbf{x}^* \le \mathbf{c}^T\mathbf{x},\ \mathbf{x} \in \mathcal{P}\}$ is the *set of the optimal solutions* of the problem (P_{LP}).

Index sets: $\mathcal{J} = \mathcal{J}_B \cup \mathcal{J}_N = \mathcal{J}_B \cup (\mathcal{J}_N^l \cup \mathcal{J}_N^u)$, $\mathcal{J}_B \cap \mathcal{J}_N = \emptyset$.

Basic vectors $\{\mathbf{a}_j : j \in \mathcal{J}_B\}$ are linearly independent. Let $\bar{\mathbf{x}} \in \mathcal{P}$ basic feasible solution, then

$$\bar{\mathbf{x}}_B = B^{-1}\,\mathbf{b} - \sum_{j \in \mathcal{J}_N^l} l_j\,\bar{\mathbf{a}}_j - \sum_{j \in \mathcal{J}_N^u} u_j\,\bar{\mathbf{a}}_j, \qquad \text{where } \bar{\mathbf{a}}_j = B^{-1}\mathbf{a}_j.$$

Let $\mathbf{x}^* \in \mathcal{P}_c^*$ be a basic solution belonging to the basis B and $\mathbf{y}^* = \mathbf{c}_B^T B^{-1} \geq \mathbf{0}$, we get that

- in case of $j \in \mathcal{J}_B$, $l_j < x_j^* < u_j$, $z_j = 0$ and $s_j = 0$ hold and thus $\quad -\mathbf{a}_j^T \mathbf{y} = c_j,$
- in case of $j \in \mathcal{J}_N^l$, $l_j = x_j^*$, $z_j \geq 0$ and $s_j = 0$ hold and thus $\quad z_j = c_j + \mathbf{a}_j^T \mathbf{y} \geq 0,$
- in case of $j \in \mathcal{J}_N^u$, $u_j = x_j^*$, $z_j = 0$ and $s_j \geq 0$ hold and thus $\quad -s_j = c_j + \mathbf{a}_j^T \mathbf{y} \leq 0.$

Finally, we obtain a basic solution $\mathbf{x}^* \in \mathcal{P}$, which is optimal if and only if

$$
\begin{aligned}
\mathbf{y}^* = \mathbf{c}_B^T B^{-1} &\geq \mathbf{0} && && (1)\\
-\mathbf{c}_B^T B^{-1} \mathbf{a}_j &\leq c_j, && \text{any } j \in \mathcal{J}_N^l \text{ and} && (2)\\
-\mathbf{c}_B^T B^{-1} \mathbf{a}_j &\geq c_j, && \text{any } j \in \mathcal{J}_N^u && (3)
\end{aligned}
$$

hold.

Let us consider the set of all objective function coefficients of linear programs for which the current basic solution, $\mathbf{x}^* \in \mathcal{P}$ is an optimal basic solution

$$\mathcal{C}_B = \{\mathbf{c} \in \mathbb{R}^n : \text{constraints } (1) - (3) \text{ are satisfied}\} \neq \emptyset$$

Example (continue) Sensitivity analysis shows that if $c_1 \in [0.2, 1.5]$ and $c_2 \in [-2.5, -0.33]$ then $\tilde{x}_1 = 1.53846,\ \tilde{x}_2 = 2.30769$ remains optimal solution of the relaxed $LP(\mathbf{c})$ problem.

Linear approximation of concave functions

General linear approximation of the concave functions $f_j : \mathbb{R} \to \mathbb{R}$ on the closed interval $[a_j,\, b_j]$ is

$$h_j(x_j) = m(a_j, b_j)\,(x_j - a_j) + f_j(a_j) = h_j\,x_j + r_j$$

where $l_j \leq a_j < b_j \leq u_j,\ h_j = m(a_j, b_j)$ and $r_j = f_j(a_j) - m(a_j, b_j)\,a_j$. Then for the function

$$H(\mathbf{x}) = \sum_{j=1}^{n} h_j(x_j) = \sum_{j=1}^{n} (h_j\,x_j + f_j(a_j) - h_j\,a_j) = \mathbf{h}^T\mathbf{x} + (F(\mathbf{a}) - \mathbf{h}^T\mathbf{a}) = \mathbf{h}^T\mathbf{x} + \varrho,$$

where $\varrho = F(\mathbf{a}) - \mathbf{h}^T\mathbf{a}$, and the following inequalities holds

$$F(\mathbf{x}) \geq H(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathcal{P}(\mathbf{a},\mathbf{b}), \qquad \text{and} \qquad F(\mathbf{x}) \leq H(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathcal{P} \setminus \mathcal{P}(\mathbf{a},\mathbf{b}),$$

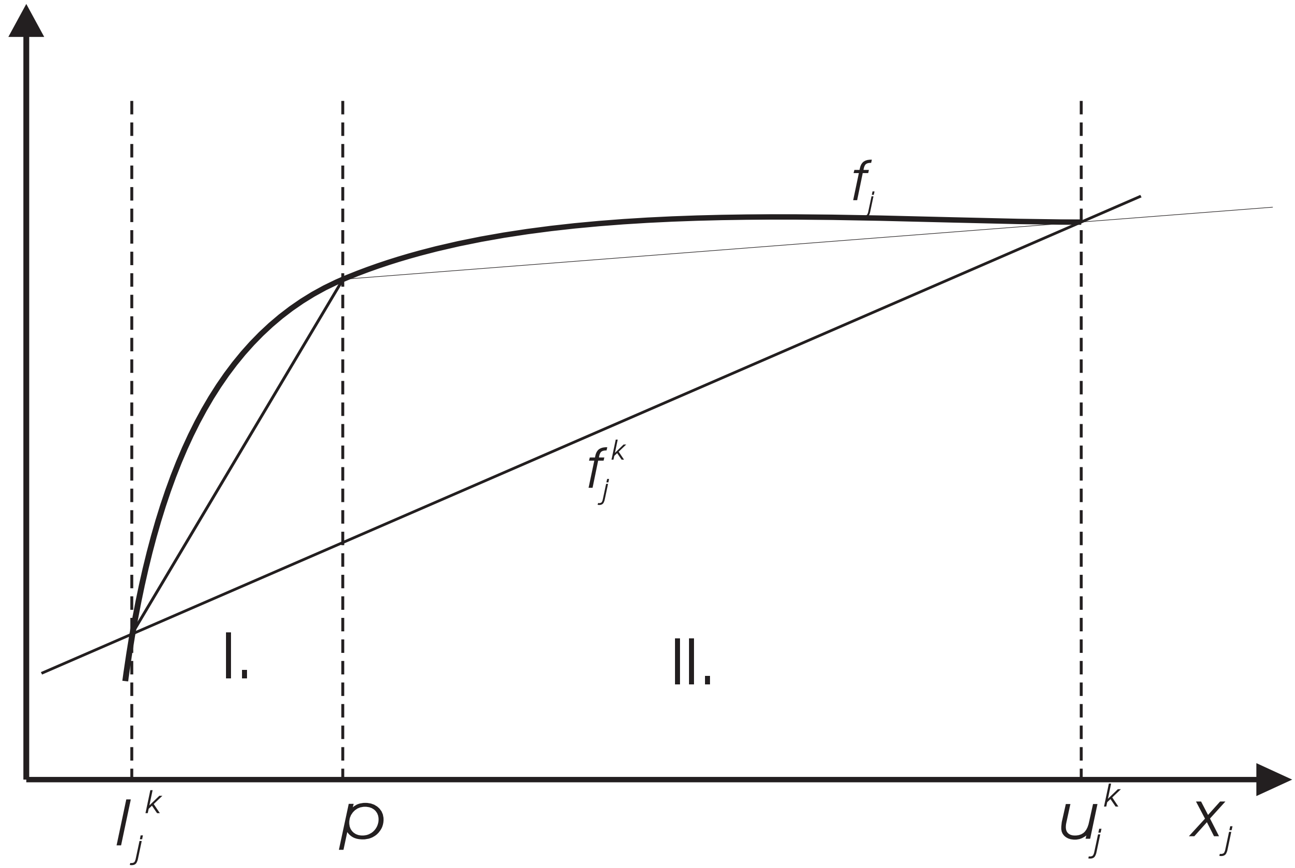
where $\mathbf{a}, \mathbf{b} \in \mathcal{T},\ \mathbf{a} < \mathbf{b}$ and $\mathcal{P}(\mathbf{a},\mathbf{b}) = \mathcal{A} \cap \{\mathbf{x} \in \mathbb{R}^n :\ \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\}$.

Set of the normal vectors of the general linear approximations:

$$\begin{aligned}\mathcal{H}_f \quad := \quad & \{\mathbf{h} \in \mathbb{R}^n :\ h_j = f'_{j,+}(t),\ t \in [l_j, u_j)\} \cup \{\mathbf{h} \in \mathbb{R}^n :\ h_j = f'_{j,-}(t),\ t \in (l_j, u_j]\} \cup \\ & \{\mathbf{h} \in \mathbb{R}^n :\ h_j = m(a_j, b_j),\ l_j \leq a_j < b_j \leq u_j\}\end{aligned}$$

Question. Is there any relations between the sets $\mathcal{C}_B$ and $\mathcal{H}_f$?

Linear approximation of concave function



Example summary: iteration 1

separable concave, objective function:

$$F(\mathbf{x}) = 5\,\sin(\frac{\pi}{6}\,x_1) + 3\,\cos(\frac{\pi}{6}\,x_2) = f_1(x_1) + f_2(x_2)$$

feasible solution set:

$$\mathcal{P} = \mathcal{A} \cap \mathcal{T} = \{\mathbf{x} \in \mathbb{R}^n : A\,\mathbf{x} \le \mathbf{b}\} \cap \{\mathbf{x} \in \mathbb{R}^n :\ 0 \le x_1 \le 5,\ 0 \le x_2 \le 3\}$$

linear approximation of the objective function: $G(\mathbf{x}) = g_1(x_1) + g_2(x_2) = \frac{1}{2}\,x_1 - x_2 + 3$

optimal solution of the linear approximation: $\tilde{x}_1 = 1.53846,\ \tilde{x}_2 = 2.30769,$ and $G(\tilde{\mathbf{x}}) = 1.46154$

$$G(\tilde{\mathbf{x}}) = 1.46154 \le F(\mathbf{x}) = 5\,\sin(\tfrac{\pi}{6}\,x_1) + 3\,\cos(\tfrac{\pi}{6}\,x_2) \le 1.8135\,x_1 - 1.4687\,x_2 + 5.269 = F_{\tilde{\mathbf{x}}}(\mathbf{x})$$

sensitivity analysis: if $c_1 \in [0.2, 1.5]$ and $c_2 \in [-2.5, -0.33]$ then $\tilde{x}_1 = 1.53846,\ \tilde{x}_2 = 2.30769$ remains optimal solution of the relaxed $LP(\mathbf{c})$ problem.

finding branching point p:

$$(Eq_1)\ \ y_1 = 0.2\,x_1 \text{ and } y_1 = 5\,\sin(\frac{\pi}{6}\,x_1) \qquad \text{or} \qquad (Eq_2)\ \ y_1 = 1.5\,x_1 \text{ and } y_1 = 5\,\sin(\frac{\pi}{6}\,x_1)$$

p: $1.5\,p - 5\,\sin(\frac{\pi}{6}\,p) = 0 \quad p \in (3.29393, 3.29394)$ then $\mathcal{P}_1 = \mathcal{A} \cap \mathcal{T}_1$ and $\mathcal{P}_2 = \mathcal{A} \cap \mathcal{T}_2$

$$\mathcal{T}_1 = \{\mathbf{x} \in \mathbb{R}^n :\ 0 \le x_1 \le 3.29393,\ 0 \le x_2 \le 3\} \text{ and } \mathcal{T}_2 = \{\mathbf{x} \in \mathbb{R}^n :\ 3.29394 \le x_1 \le 5,\ 0 \le x_2 \le 3\}$$

$$G_1(x) = \frac{3}{2}\,x_1 - x_2 + 3,\ \ \tilde{\mathbf{x}} \in \mathcal{P}_1 \text{ is an optimal solution and } G_1(\tilde{\mathbf{x}}) = 3 \le F(\mathbf{x}) \le F(\tilde{\mathbf{x}}) = 4.6698.$$

Example: iteration 2

$$(LP): \min_{\mathbf{x}\in\mathcal{P}} G(\mathbf{x}) \qquad \text{optimal solution } \tilde{x}_1 = 1.53846,\ \tilde{x}_2 = 2.30769, \text{ and } G(\tilde{\mathbf{x}}) = 1.46154$$

branching point $p \in (3.29393, 3.29394)$*:* (LP_1) and (LP_2)

$$(LP_1): \min_{\mathbf{x}\in\mathcal{P}_1} G_1(\mathbf{x}) \qquad \tilde{\mathbf{x}} = (1.53846, 2.30769) \in \mathcal{P}_1, \quad G_1(\tilde{\mathbf{x}}) = 3 < F(\tilde{\mathbf{x}}) = 4.6698;$$

sensitivity analysis: $c_1 \in [1.5, +\infty)$ and $c_2 \in [-1, 0.5]$

$$(LP_2): \min_{\mathbf{x}\in\mathcal{P}_2} G_2(\mathbf{x})$$

$$G_2(\mathbf{x}) = g_1^2(x_1) + g_2^2(x_2) = -1.4307\,x_1 + 9.6535 - x_2 + 3 = -1.4307\,x_1 - x_2 + 12.6535$$

optimal solution $\hat{x}_1 = 4.09091,\ \hat{x}_2 = 2.818182,$ and $G_2(\hat{\mathbf{x}}) = 3.982454$

$$G_2(\hat{\mathbf{x}}) = 3.982454 < F(\hat{\mathbf{x}}) = 4.4914 \quad \text{and} \quad G_2(\hat{\mathbf{x}}) \le F(\mathbf{x}) \le -1.4154\,x_1 - 1.5637\,x_2 + 14.6885$$

Problem	$\mathcal{T}$	$\tilde{\mathbf{x}}$	$G(\tilde{\mathbf{x}})$	$F(\tilde{\mathbf{x}})$	Status
(LP)	$0 \le x_1 \le 5,\quad 0 \le x_2 \le 3$	$(1.53846, 2.30769)$	1.46154	4.6698	*1*
(LP_1)	$0 \le x_1 \le 3.29393,\quad 0 \le x_2 \le 3$	$(1.53846, 2.30769)$	3	4.6698	1
(LP_2)	$3.29394 \le x_1 \le 5,\quad 0 \le x_2 \le 3$	$(4.09091, 2.818182)$	3.982454	4.4914	1

Branch & bound algorithm: based on sensitivity analysis

Step 0

Solve the relaxed LP problem. (Solution: $\tilde{\mathbf{x}}$, $G(\tilde{\mathbf{x}})$, $F(\tilde{\mathbf{x}})$, sensitivity analysis data. Current best solution: $\mathbf{x}^* := \tilde{\mathbf{x}}$ and $F^* := F(\tilde{\mathbf{x}})$.)
Choose a decision variable for branching and compute the branching point, p.
Put the LP problem into the list of problems.

Step 1

Define $\mathcal{T}'$ and $\mathcal{T}''$ using the branching point of the previous problem.
Produce the corresponding LP' and LP'' problems and put them into the list of problems.

Step 2

Select an LP problem from the list of problems that has not bee analyzed or solved, yet.
If the list of problems is empty then stop.

Step 3

Solve the selected LP problem: $\tilde{\mathbf{x}}$, $G(\tilde{\mathbf{x}})$, $F(\tilde{\mathbf{x}})$, sensitivity analysis data.
If $G(\tilde{\mathbf{x}}) \geq F^*$ then delete this problem from the list of problems and go to Step 2.
Choose a decision variable for branching and compute the branching point, p.
Put the LP problem into the list of problems.

If $F(\tilde{\mathbf{x}}) < F^*$ then $F^* := F(\tilde{\mathbf{x}})$ and delete all problems from the list for which $G(\tilde{\mathbf{x}}) \geq F^*$.
Go to Step 1.

Example: result

Vertex	Function value	Vertex	Function value
$P_{12} = (\frac{7}{2}, \frac{1}{2})$	7.7274	$P_{56} = (\frac{20}{13}, \frac{30}{13})$	4.6698
$P_{23} = (\frac{9}{2}, \frac{3}{2})$	5.6569	$P_{67} = (\frac{2}{3}, 1)$	4.3082
$P_{34} = (\frac{23}{5}, \frac{9}{5})$	5.109	$P_{70} = (1, 0)$	5.5
$P_{45} = (\frac{45}{11}, \frac{31}{11})$	4.4914	$P_{01} = (2, 0)$	7.3301

Necessary optimality condition

Lemma. Consider problem (P). Let $\hat{\mathbf{x}} \in \mathcal{P}^*$ then $\bar{G}(\hat{\mathbf{x}}) = \min_{\mathbf{x}\in\mathcal{P}} \bar{G}(\mathbf{x})$, where $\bar{G}(\mathbf{x}) = (\nabla F(\hat{\mathbf{x}}))^T(\mathbf{x} - \hat{\mathbf{x}}) + F(\hat{\mathbf{x}})$. Thus $\hat{\mathbf{x}} \in \mathcal{P}^*_{\mathbf{h}}$, where $\mathbf{h} = \nabla F(\hat{\mathbf{x}})$. Furthermore, if $\hat{\mathbf{x}}$ is a basic solution belonging to basis B then $\nabla F(\hat{\mathbf{x}}) \in \mathcal{C}_B$.

Proof. Because of the concavity of function F

$$F(\mathbf{x}) \le \bar{G}(\mathbf{x}) = (\nabla F(\hat{\mathbf{x}}))^T(\mathbf{x} - \hat{\mathbf{x}}) + F(\hat{\mathbf{x}}),$$

with equality at $\hat{\mathbf{x}}$, namely $F(\hat{\mathbf{x}}) = \bar{G}(\hat{\mathbf{x}})$. Then $F(\hat{\mathbf{x}}) = \min_{\mathbf{x}\in\mathcal{P}} F(\mathbf{x}) \le \min_{\mathbf{x}\in\mathcal{P}} \bar{G}(\mathbf{x}) \le \bar{G}(\hat{\mathbf{x}}) = F(\hat{\mathbf{x}})$, from which

$$\min_{\mathbf{x}\in\mathcal{P}} \bar{G}(\mathbf{x}) = \bar{G}(\hat{\mathbf{x}})$$

is obtained. Furthermore

$$\mathbf{c} \in \mathcal{C}_B \iff \mathbf{c}^T\hat{\mathbf{x}} \le \mathbf{c}^T\mathbf{x}, \ \text{for all}\, \mathbf{x} \in \mathcal{P} \iff \mathbf{c} \in cone(P \setminus \{\hat{\mathbf{x}}\})^+.$$

Since $\hat{\mathbf{x}} \in \mathcal{P}^*$, then there is no $\mathbf{x} \in \mathcal{P}$, such that the function $F(\mathbf{x})$ is decreasing in the direction $\mathbf{x} - \hat{\mathbf{x}}$, namely $(\nabla F(\hat{\mathbf{x}})) \in cone(P \setminus \{\hat{\mathbf{x}}\})^+ = \mathcal{C}_B$. •

Remark. 1. If F is not differentiable at $\hat{\mathbf{x}}$ then any inner point of the set of subgradients is also suitable for function $\bar{G}$.

A property of linear relaxation

Consider problem (P). Let us define the set $\mathcal{H} \subseteq \mathbb{R}^n$ such that the elements of this set are

- coefficients of the the objective functions of (general) linear programming relaxations of the problem (P);
- if the optimal solutions of linear programming problem related to all elements of set $\mathcal{H}$ were known, then the optimal solution of problem (P) could be generated, too.

Remark. One possibility to approximate $\mathcal{H}$ is $\mathcal{H}_f$, which uses the information given in the problem (P) about the function F and about the box constraints $\mathcal{T}$. However, no information about the set $\mathcal{A}$ is taken into consideration.

Proposition. Consider the basic solution $\bar{\mathbf{x}} \in \mathcal{P}$, with basis B and let $\bar{\mathbf{h}} \in \mathcal{H}$ be a given vector. If $\bar{\mathbf{h}} \in \mathcal{C}_B = \{\mathbf{c} \in \mathbb{R}^n :$ vector $\mathbf{c}$ satisfies equation (1) – (3) $\}$ then the $\bar{\mathbf{x}}$ is an optimal solution of the following linear programming problem.

$$\left.\min_{\mathbf{x}\in\mathcal{P}} \bar{\mathbf{h}}^T\mathbf{x}\right\} \quad (P_{\bar{h}}),$$

namely $\bar{\mathbf{x}} \in \mathcal{P}^*_{\bar{h}}$, where $\mathcal{P}^*_{\bar{h}}$ denotes the set of optimal solutions of problem $(P_{\bar{h}})$. •

From this result follows that

$$\text{if} \quad \mathcal{H} \subseteq \mathcal{C}_B \quad \text{then} \quad \bar{\mathbf{x}} \in \mathcal{P}^*_h \tag{4}$$

holds for any $\mathbf{h} \in \mathcal{H}$.

Sufficient optimality condition

Theorem. Consider the linearly constrained, separable concave minimization problem (P), and suppose the functions f_j are strictly concave. Let $\bar{\mathbf{x}} \in \mathcal{P}$ be a basic solution with basis B that $\mathcal{H} \subseteq \mathcal{C}_B$ holds, then $\mathcal{P}^* = \{\bar{\mathbf{x}}\}$.

Proof. Since $\mathcal{H} \subseteq \mathcal{C}_B$ thus $\bar{\mathbf{x}} \in \mathcal{P}_h^*$ holds for any $\mathbf{h} \in \mathcal{H}$.

There exist global minimal solution $\hat{\mathbf{x}}$ of (P) which is an extremal point of the set $\mathcal{P}$. Suppose that $\hat{\mathbf{x}} \neq \bar{\mathbf{x}}$.

Let $\hat{\mathbf{h}} = \nabla f(\hat{\mathbf{x}})$. Since previous lemma asserts $\hat{\mathbf{x}} \in \mathcal{P}_{\hat{\mathbf{h}}}^*$, otherwise $\bar{\mathbf{x}} \in \mathcal{P}_{\hat{\mathbf{h}}}^*$. The following relations hold,

$$F(\hat{\mathbf{x}}) = \bar{G}(\hat{\mathbf{x}}) = \bar{G}(\bar{\mathbf{x}}) > F(\bar{\mathbf{x}}), \tag{5}$$

which is a contradiction, thus $\hat{\mathbf{x}} = \bar{\mathbf{x}}$, then $\mathcal{P}^* = \{\bar{\mathbf{x}}\}$. •

Remark. The strict inequality comes from the strict concavity. If the condition of strict concavity is removed from the previous Theorem then the inequality (5) will be modified as

$$F(\bar{\mathbf{x}}) \geq F(\hat{\mathbf{x}}) = \bar{G}(\hat{\mathbf{x}}) = \bar{G}(\bar{\mathbf{x}}) \geq F(\bar{\mathbf{x}})$$

so $F(\bar{\mathbf{x}}) = F(\hat{\mathbf{x}})$, thus $\bar{\mathbf{x}} \in \mathcal{P}^*$, but the equality $|\mathcal{P}^*| = 1$ cannot be guaranteed.

It has been proved that the *sufficient optimality condition* for a basic solution $\bar{\mathbf{x}} \in \mathcal{P}$ of problem (P) with basis B is

$$\mathcal{H} \subseteq \mathcal{C}_B.$$

Approximating set $\mathcal{H}$

If the set approximating $\mathcal{H}$ is only based on the properties of problem (P), we can get

$$\mathcal{H}_f = \{\mathbf{h} \in \mathbb{R}^n : h_j \in [f'_{j-}(u_j), f'_{j+}(l_j)]\}$$

and $\mathcal{H} \subseteq \mathcal{H}_f$ holds. Based on the previous result for a basic solution $\bar{\mathbf{x}} \in \mathcal{P}$ of problem (P) with basis B, if $\mathcal{H} \subseteq \mathcal{H}_f \subseteq \mathcal{C}_B$ then $\mathcal{P}^* = \{\bar{\mathbf{x}}\}$.

Let us determine set approximating $\mathcal{H}$ for a given basic solution $\bar{\mathbf{x}} \in \mathcal{P}$ then

$$\mathcal{H}_{f,\bar{\mathbf{x}}} = \{\mathbf{h} \in \mathbb{R}^n : h_j \in [c_j^l, c_j^u]\} \qquad \text{(Phillips and Rosen, 1993)}$$

this set (hyper rectangle) will contain the coefficients of all possible relaxed linear functions, where

$$c_j^u = \begin{cases} m(l_j, \bar{x}_j), & \bar{x}_j \neq l_j \\ f'_{j+}(l_j), & \text{otherwise} \end{cases} \quad \text{and} \quad c_j^l = \begin{cases} m(\bar{x}_j, u_j), & \bar{x}_j \neq u_j \\ f'_{j-}(u_j), & \text{otherwise} \end{cases}$$

From the concavity of the function F, we can get the inequalities

$$f'_{j-}(u_j) \leq c_j^l = m(\bar{x}_j, u_j) \leq m(l_j, \bar{x}_j) = c_j^u \leq f'_{j+}(l_j) \tag{6}$$

and therefore $\mathcal{H}_{f,\bar{\mathbf{x}}} \subseteq \mathcal{H}_f$ holds.

2^n relaxed LP problems have a common optimal solution $\iff$ $\mathcal{H}_{f,\bar{\mathbf{x}}} \subseteq \mathcal{C}_B$

extremal points of $\mathcal{H}_{f,\bar{\mathbf{x}}}$ are elements of $\mathcal{C}_B$ $\iff$ $\mathcal{H}_{f,\bar{\mathbf{x}}} \subseteq \mathcal{C}_B$

Test points

Theorem. (Phillips and Rosen, 1993) Consider the linearly constrained, separable concave minimization problem (P). Let $\bar{\mathbf{x}} \in \mathcal{P}$ be a basic solution with basis B such that $\mathcal{H}_{f,\bar{\mathbf{x}}} \subseteq \mathcal{C}_B$ holds, then $\bar{\mathbf{x}} \in \mathcal{P}^*$. •

Let us define a test point, which belongs to $\mathcal{H}_{f,\bar{\mathbf{x}}}$, and violate the constraint indexed by $j \in \mathcal{J}_N^l$

$$-\mathbf{c}_B^T B^{-1}\mathbf{a}_j = -\mathbf{c}_B^T \bar{\mathbf{a}}_j \leq c_j.$$

It means, we choose such vertex of $\mathcal{H}_{f,\bar{\mathbf{x}}}$, which increase the left side of inequality and decrease the right side as much as possible. Therefore the test point $\bar{\mathbf{h}}_j$ can be defined as follows

$$\bar{h}_{ij} = \begin{cases} c_j^l, & \text{if } i = j \\ c_j^l, & \text{if } \bar{a}_{ij} > 0, i \in \mathcal{J}_B \\ c_j^u, & \text{if } \bar{a}_{ij} < 0, i \in \mathcal{J}_B \\ h_{ij}, & \text{if } , i \notin (\mathcal{J}_B \setminus \{i \,:\, \bar{a}_{ij} = 0\}) \cup \{j\}, \text{ where } h_{ij} \in [c_i^l, c_i^u]. \end{cases}$$

It is obvious that $\bar{\mathbf{h}}_j \in \mathcal{H}_{f,\bar{\mathbf{x}}}$ holds. From the construction of the test point it is clear that

$$\bar{\mathbf{h}}_B^T \bar{\mathbf{a}}_j + \bar{h}_{jj} \leq \mathbf{h}_B^T \bar{\mathbf{a}}_j + h_{jj} \quad \text{holds for any } \mathbf{h} \in \mathcal{H}_{f,\bar{\mathbf{x}}}, \text{ which is}$$

$$0 \geq -\bar{\mathbf{h}}_B^T \bar{\mathbf{a}}_j - \bar{h}_{jj} \geq -\mathbf{h}_B^T \bar{\mathbf{a}}_j - h_{jj}. \tag{7}$$

Now, if the test point does not violate the inequality, that is the red inequality holds, then there is no element of set $\mathcal{H}_{f,\bar{\mathbf{x}}}$ which can violate the inequality $j \in \mathcal{J}_N^l$.

Test points (continue)

In general, the test point $\bar{\mathbf{h}}_k$ for any index $k \in \mathcal{J}_N^l \cup \mathcal{J}_N^u$ will be defined, using sets $\mathcal{J}_i^+$, $\mathcal{J}_i^-$, and $i \in \mathcal{J}_B$, as follows

$$\bar{h}_{ik} = \begin{cases} c_i^l, & \text{if } k \in \mathcal{J}_i^-,\ i \in \mathcal{J}_B \\ c_i^u, & \text{if } k \in \mathcal{J}_i^+,\ i \in \mathcal{J}_B \\ c_k^l, & \text{if } i = k, \text{ and } k \in \mathcal{J}_N^l \\ c_k^u, & \text{if } i = k, \text{ and } k \in \mathcal{J}_N^u \\ h_i, & i \notin (\mathcal{J}_B \setminus \{i : \bar{a}_{ik} = 0\}) \cup \{k\}, \text{ where } h_i \in [c_i^l, c_i^u] \end{cases}$$

where

$$\begin{aligned} \mathcal{J}_i^+ &= \{k \in \mathcal{J}_N^l : \bar{a}_{ik} < 0\} \cup \{k \in \mathcal{J}_N^u : \bar{a}_{ik} > 0\}, \text{ and} \\ \mathcal{J}_i^- &= \{k \in \mathcal{J}_N^l : \bar{a}_{ik} > 0\} \cup \{k \in \mathcal{J}_N^u : \bar{a}_{ik} < 0\}. \end{aligned}$$

Based on these observations, we can get the following proposition.

Proposition. If test point $\bar{\mathbf{h}}_k$ does not violate the inequality $k \in \mathcal{J}_N^l \cup \mathcal{J}_N^u$ then no point $\mathbf{h} \in \mathcal{H}_{f,\bar{\mathbf{x}}}$ violates either. •

Moreover, in case of $j \in \mathcal{J}_N^l$ $(j \in \mathcal{J}_N^u)$

$$-\bar{\mathbf{h}}_{B,j}^T\, \bar{\mathbf{a}}_j > c_j^l \qquad (-\bar{\mathbf{h}}^T{}_{B,j}\, \bar{\mathbf{a}}_j < c_j^u)$$

the test point $\bar{\mathbf{h}}_j$ violates the optimality criteria which belongs to the variable j.

Procedure for checking the optimality of a basic solution

We can determine a test point for testing inequality system $\bar{\mathbf{h}}_B^T B^{-1} \geq \mathbf{0}$.

Let matrix $\bar{B} = B^{-1}$ and let $\bar{\mathbf{b}}_i$ denote the i^{th} column of matrix $\bar{B}$, then

$$\bar{h}_{ji} = \begin{cases} c_i^l, & \text{if } b_{ji} > 0,\ j \in \mathcal{J}_B \\ c_i^u, & \text{if } b_{ji} < 0,\ j \in \mathcal{J}_B \\ h_i, & \text{if } j \in \mathcal{J}_N^l \cup \mathcal{J}_N^u \cup \{j \in \mathcal{J}_B \,:\, \bar{b}_{ji} = 0\} \text{ where, } h_i \in [c_j^l, c_j^u] \end{cases}$$

In this case, if $\bar{\mathbf{h}}_{j,B}^T\, \bar{\mathbf{b}}_i \geq 0$ holds, then for any vector $\mathbf{h} \in \mathcal{H}_{f,\bar{\mathbf{x}}}$ the i^{th} nonnegativity condition is satisfied.

Remark. Instead of testing 2^n vertices of hyper rectangle $\mathcal{H}_{f,\bar{\mathbf{x}}}$, it is enough to determine n test points in order to check whether the inclusion $\mathcal{H}_{f,\bar{\mathbf{x}}} \subseteq \mathcal{C}_B$ holds or not.

Let us introduce the index set $\mathcal{K} = \{i \,:\, \bar{\mathbf{h}}_i$ test point violates i^{th} inequality $\}$.

It is obvious that, the equality $\mathcal{K} = \emptyset$ leads to $\mathcal{H}_{f,\bar{\mathbf{x}}} \subseteq \mathcal{C}_B$, thus $\bar{\mathbf{x}} \in \mathcal{P}^*$ holds. The decision, whether basic solution $\bar{\mathbf{x}} \in \mathcal{P}$ is optimal for the problem (P), can be performed as follows

1. generate set $\mathcal{H}_{f,\bar{\mathbf{x}}}$,
2. using matrices B^{-1} and $B^{-1}A_N$ generate test point $\bar{\mathbf{h}}_j$,
3. check the test points, if there is no index j for which test point $\bar{\mathbf{h}}_j$ violates j^{th} condition then $\bar{\mathbf{x}}$ is optimal solution for problem (P).

Question: if any test point $\bar{\mathbf{h}}_j$ can be founded which violates j^{th} condition, can we conclude that $\bar{\mathbf{x}} \in \mathcal{P}$ is not an optimal solution of the problem (P) ?

Illustration: set $\mathcal{C}_B$ for different bases

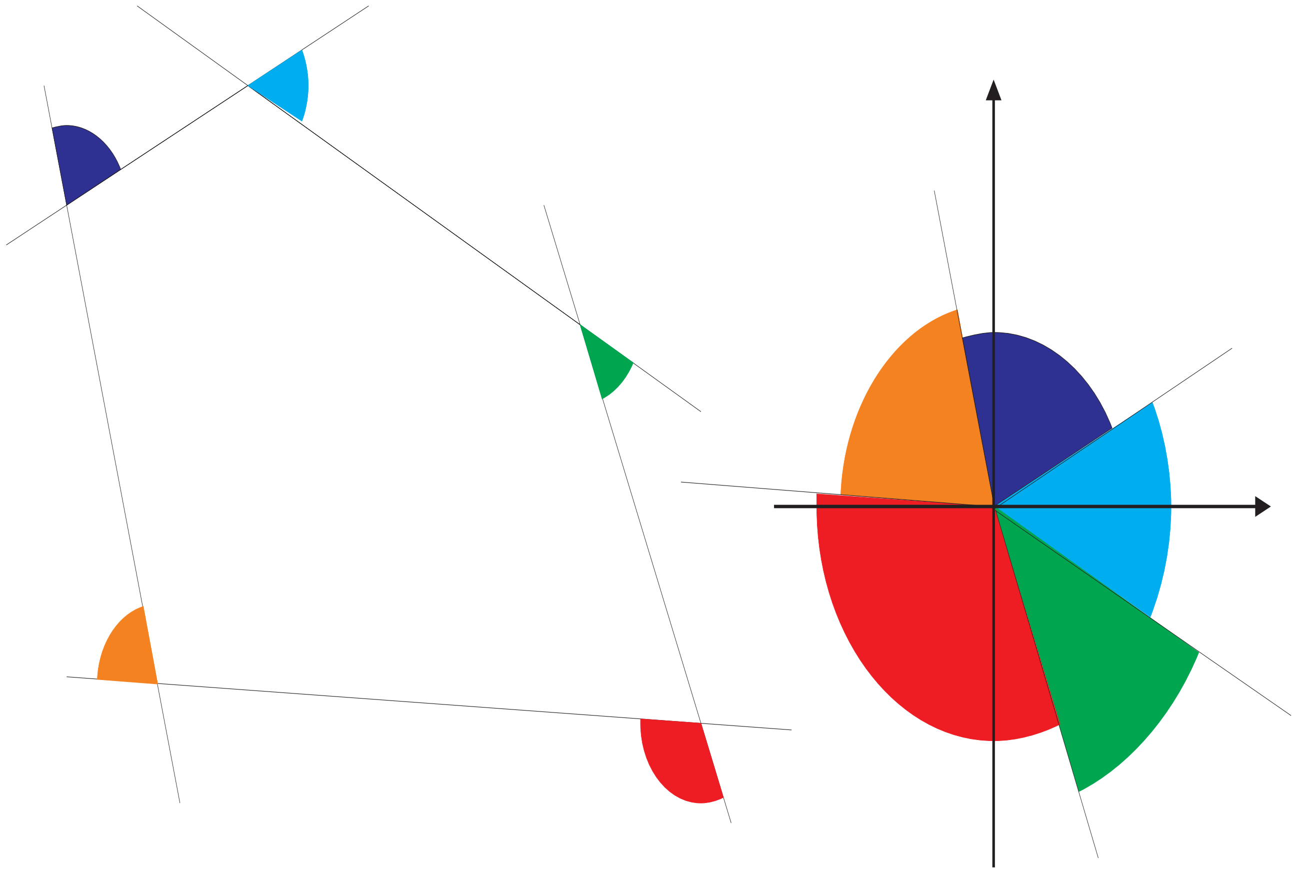


Illustration: a test point violating a constraint

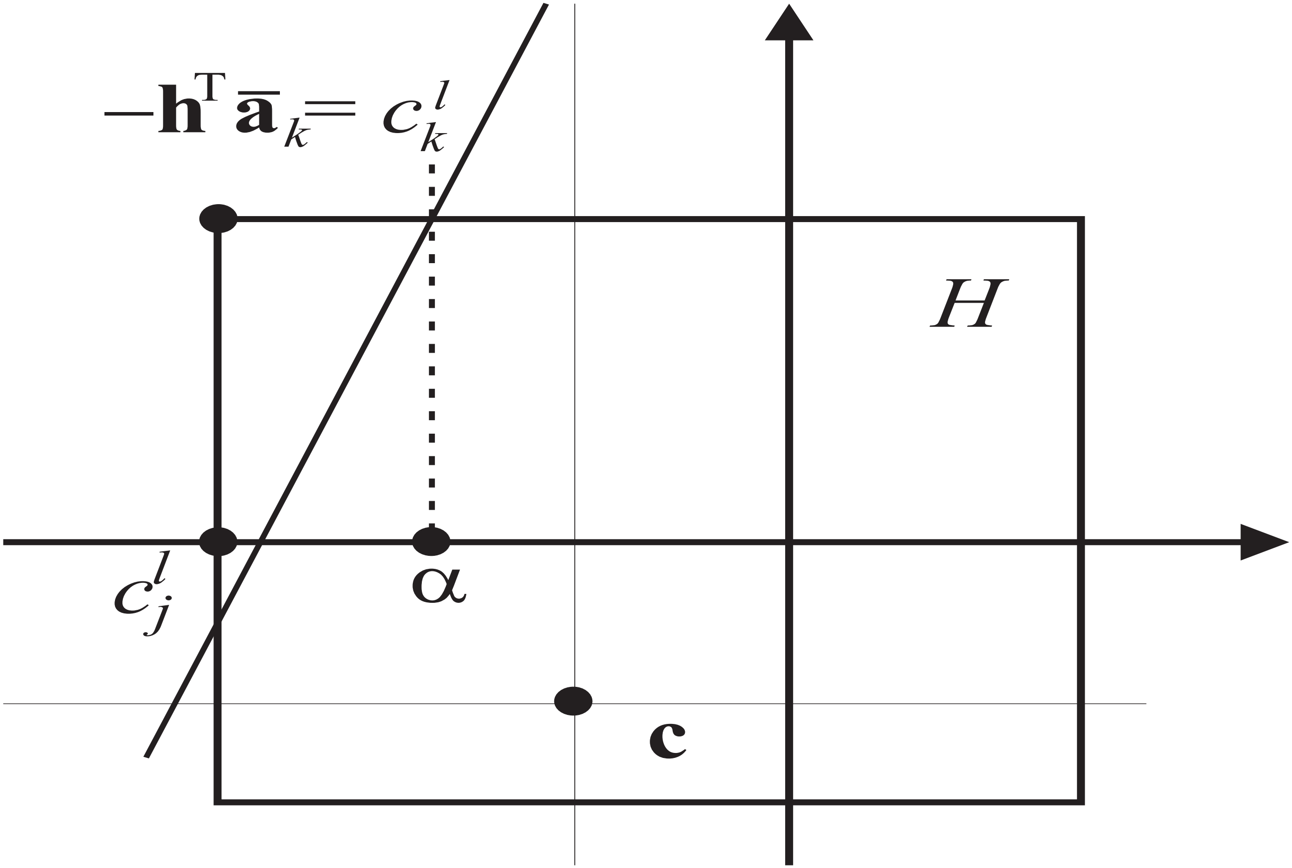


Illustration: a test point violating a constraint

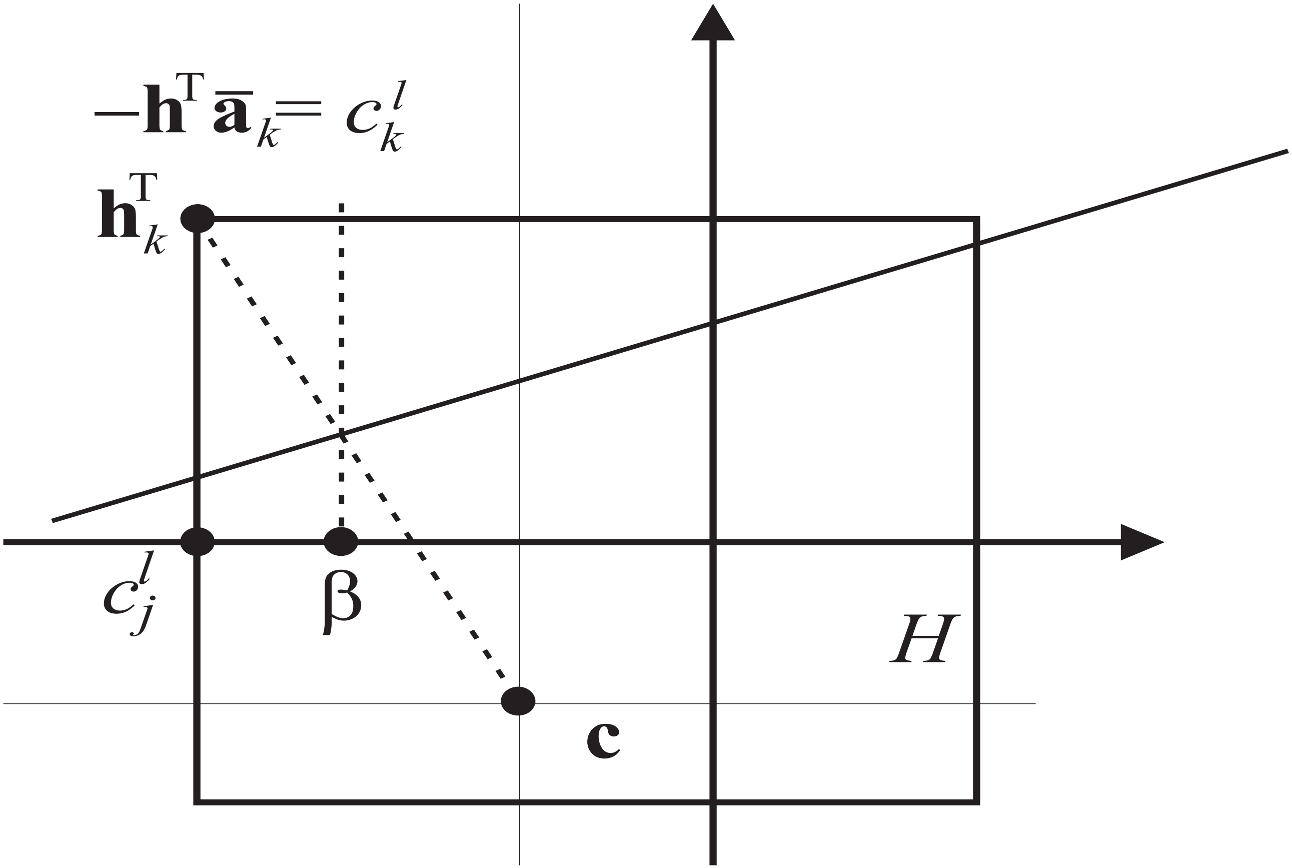


Illustration: projections of the $\mathcal{C}_B$ sets

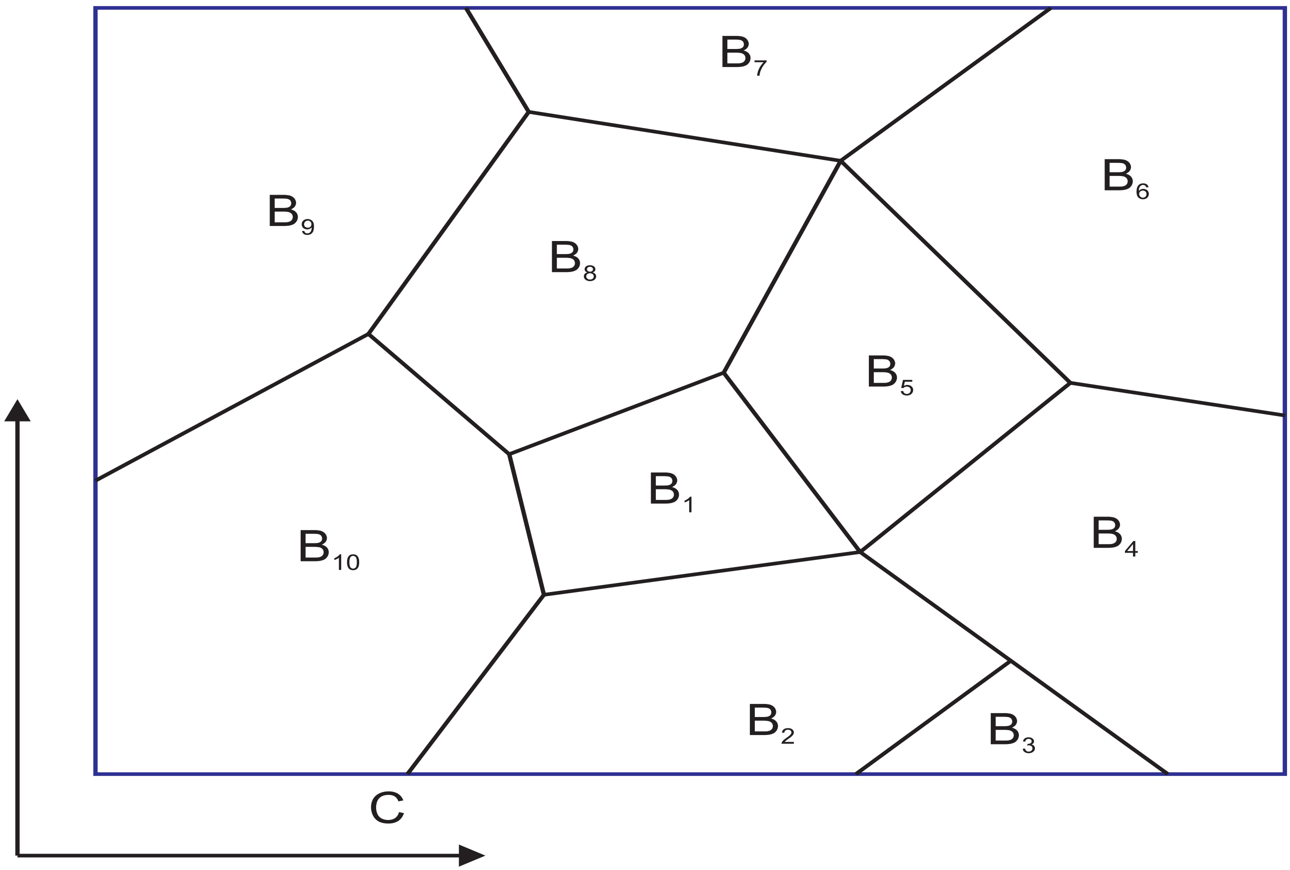
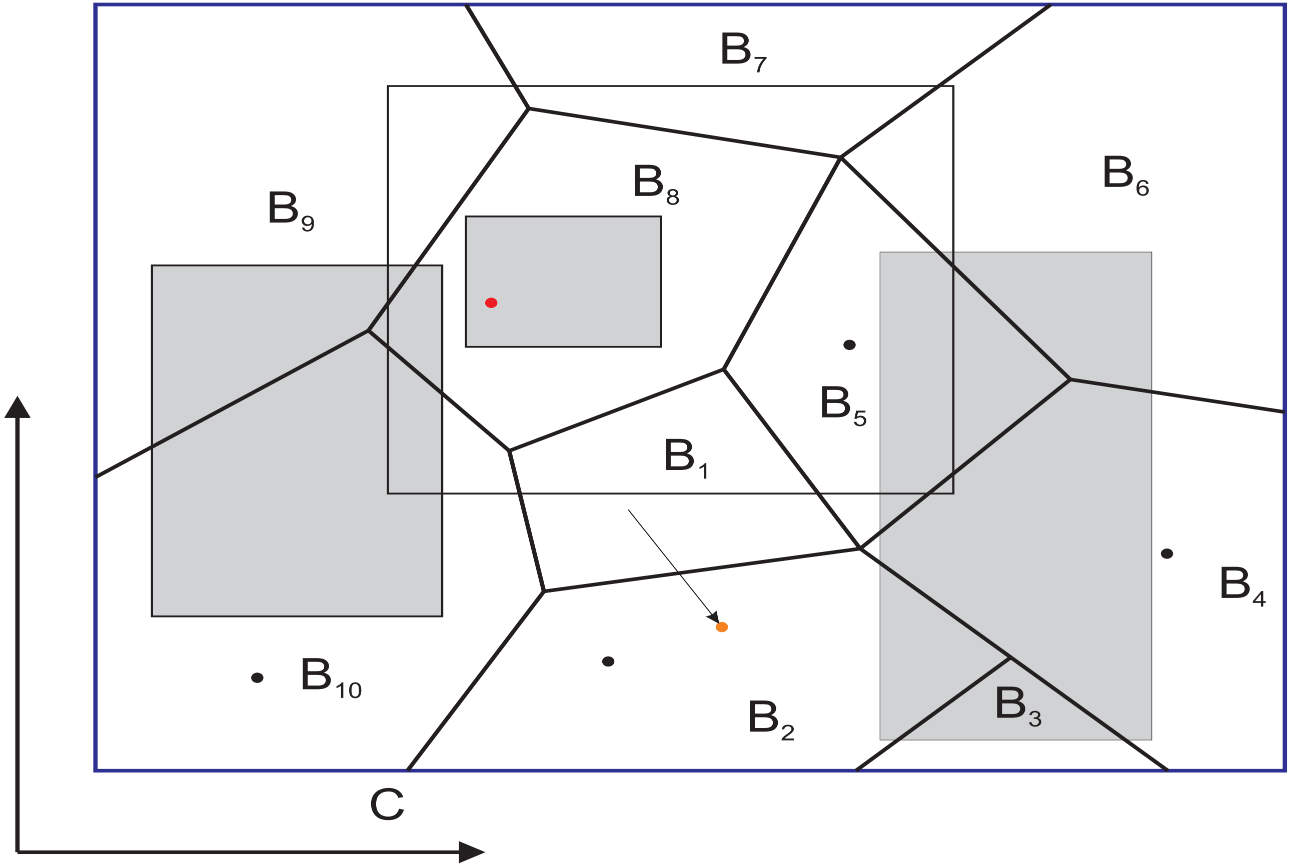


Illustration: projections of the $\mathcal{C}_B$ and the corresponding $\mathcal{H}_{f,\bar{\mathbf{x}}}$ sets



Reference

Illés, T., Nagy, Á. B., Sufficient optimality criterion for linearly constrained, separable concave minimization problems. *J. Optim. Theory Appl.* (2005), 125:3, 559575.