On the limits of human balancing: delay, sensory uncertainties and movement constraints

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Outline

- Delay-differential equations and machine tool vibrations
- Human balancing model (stick balancing & postural sway)
- Reflex delay
- Linear control concepts (PD, PDA, MP, AAW)
- Linear control concepts in case of sensory uncertainties
- Intermittent control concepts (clock-driven and event-driven)
- Conclusions

Classification of linear systems

Time Invariant ODEs

$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x} \in \mathbb{R}^n$

$\det(\lambda\mathbf{I} - \mathbf{A}) = 0,$

λ - characteristic root (exponent)

as. stab. $\Rightarrow \operatorname{Re}(\lambda_j) < 0, \; j = 1,2,\dots,n$

Time Invariant DDEs

$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau), \quad \mathbf{x} \in \mathbb{R}^n$

$\det\left(\lambda\mathbf{I} - \mathbf{A} - \mathbf{B}\mathrm{e}^{-\lambda\tau}\right) = 0,$

λ - characteristic root (exponent)

as. stab. $\Rightarrow \operatorname{Re}(\lambda_j) < 0, \; j = 1,2,\dots,\infty$

Time Periodic ODEs

$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \qquad \mathbf{A}(t+T) = \mathbf{A}(t)$

$\mathbf{x}(T) = \boldsymbol{\Phi}\,\mathbf{x}(0)$ Floquet theory

monodromy matrix
(state transition matrix)

μ - characteristic multiplier

as. stab. $\Rightarrow |\mu_j| < 1, \; j = 1,2,\dots,n$

Time Periodic DDEs

$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{x}(t-\tau)$

$\mathbf{A}(t+T) = \mathbf{A}(t), \quad \mathbf{B}(t+T) = \mathbf{B}(t)$

$\mathbf{x}_T = \mathbf{U}\mathbf{x}_0$ monodromy operator

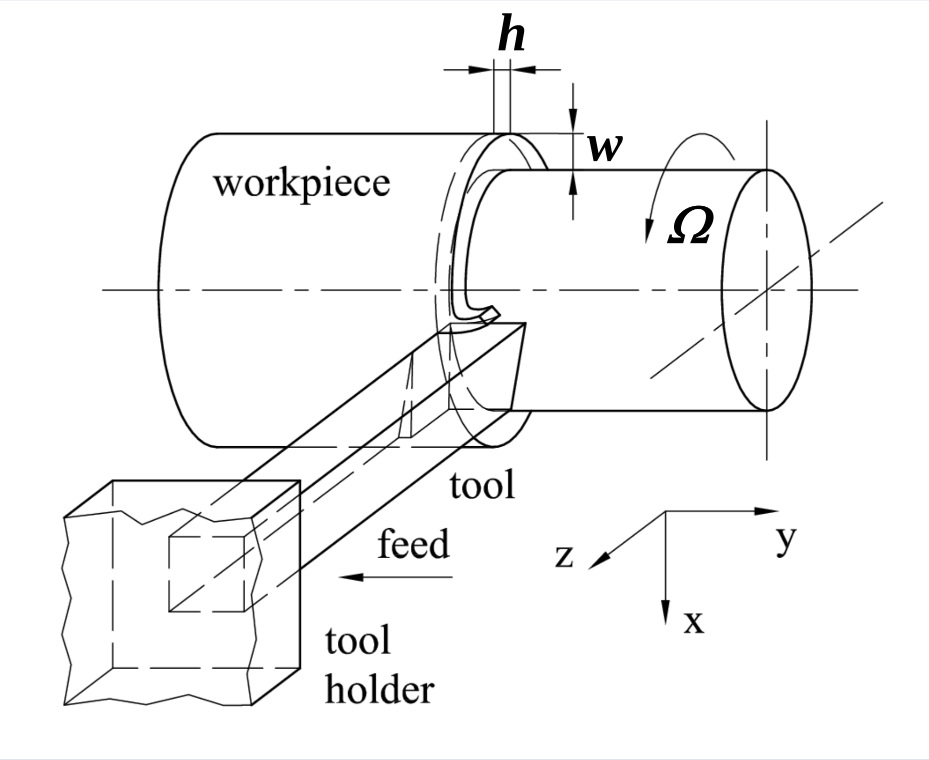
$\mathbf{x}_t(s) = \mathbf{x}(t+s), \qquad s \in [-\tau, 0]$

as. stab. $\Rightarrow |\mu_j| < 1, \; j = 1,2,\dots,\infty$

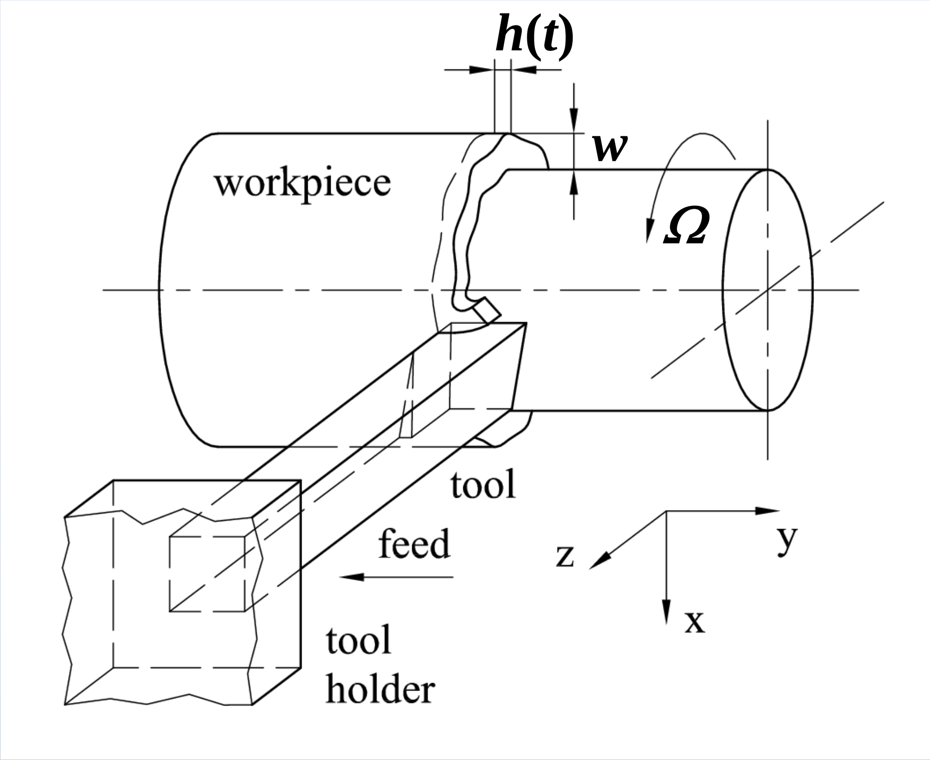
Machine tool vibrations (chatter)

Machine tool vibrations (chatter)

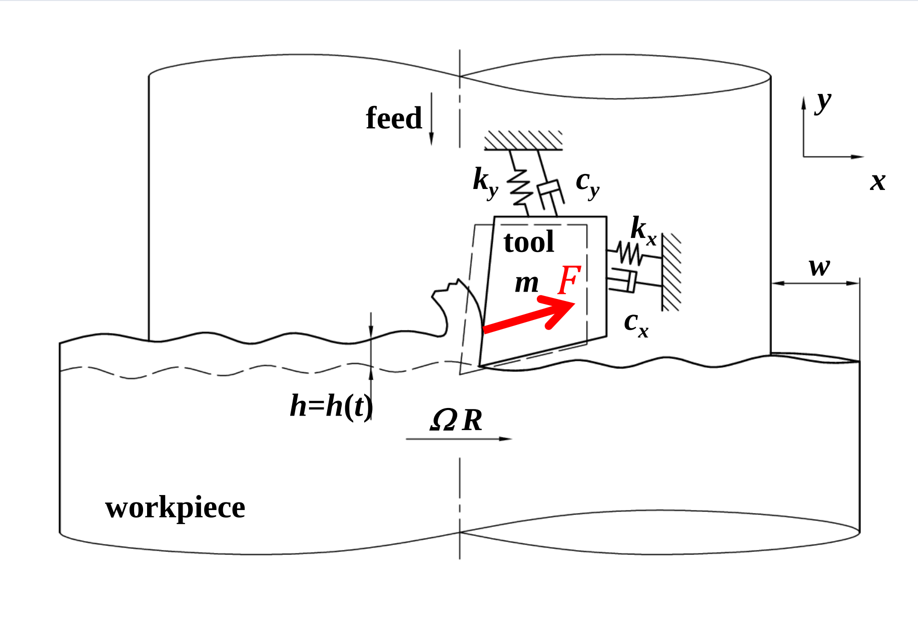
Ideally rigid tool (no vibrations)

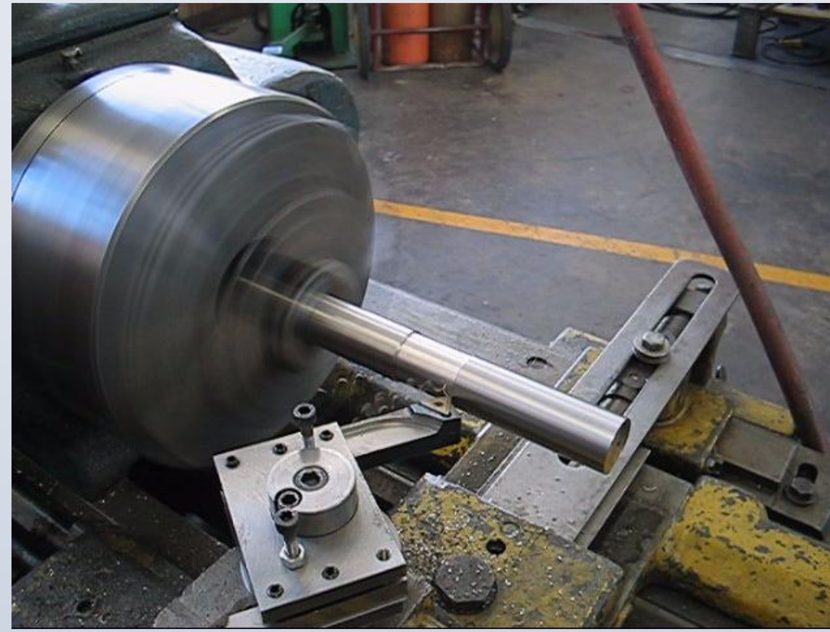


Real compliant tool (vibrations)



Turning process



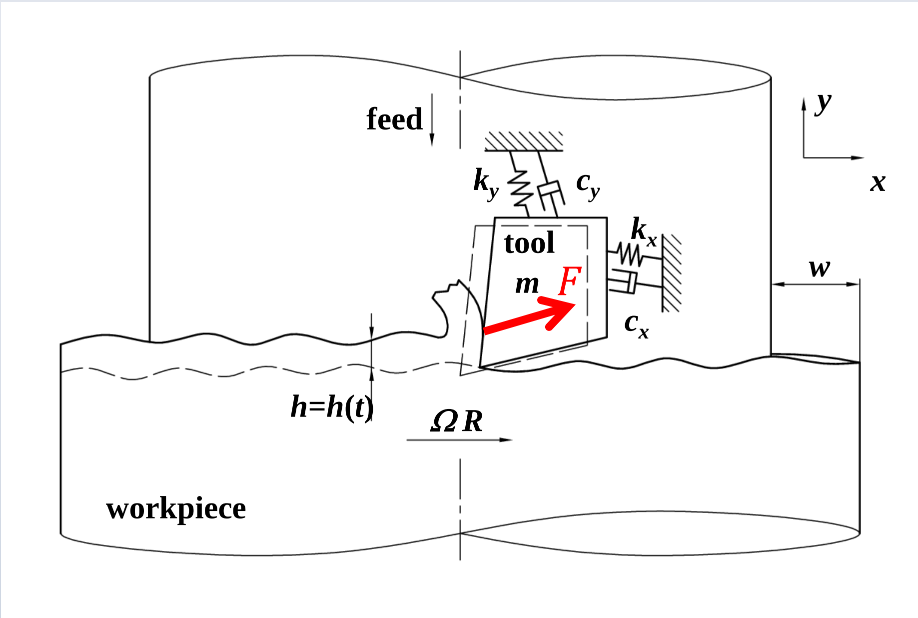


$$m\ddot{y}(t) + c_y\dot{y}(t) + k_y y(t) = \underbrace{K_y w(v_f\tau + y(t-\tau) - y(t))^q}_{\text{Cutting force}}$$

Linearized equation of motion:

$$m\ddot{y}(t) + k\dot{y}(t) + cy(t) = H(y(t-\tau) - y(t))$$

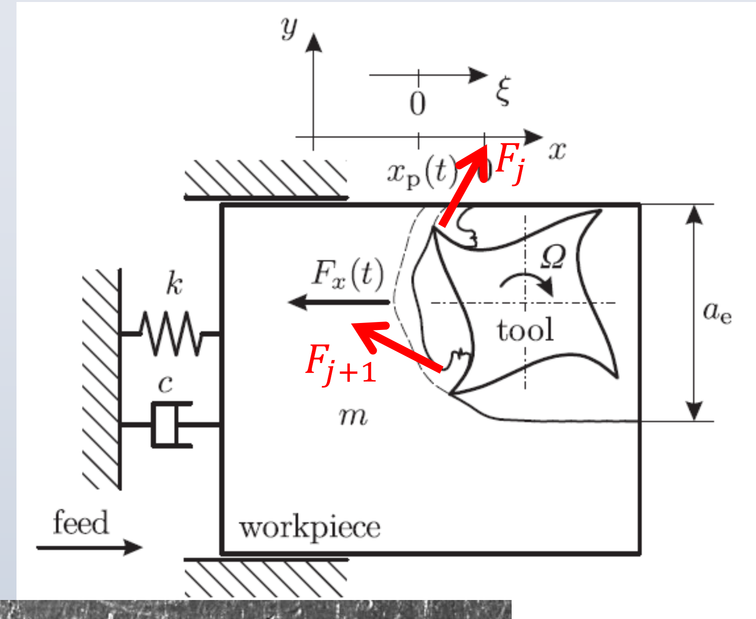
Turning process

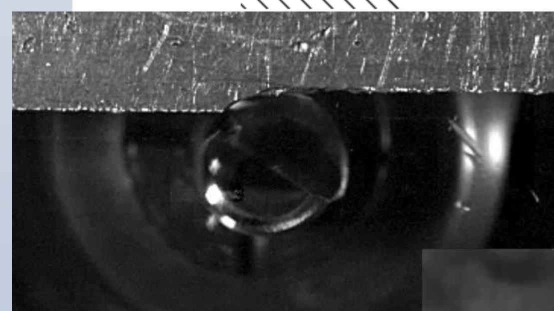


Linearized equation of motion:

$$m\ddot{y}(t) + k\dot{y}(t) + cy(t) = H(y(t-\tau) - y(t))$$

Milling process





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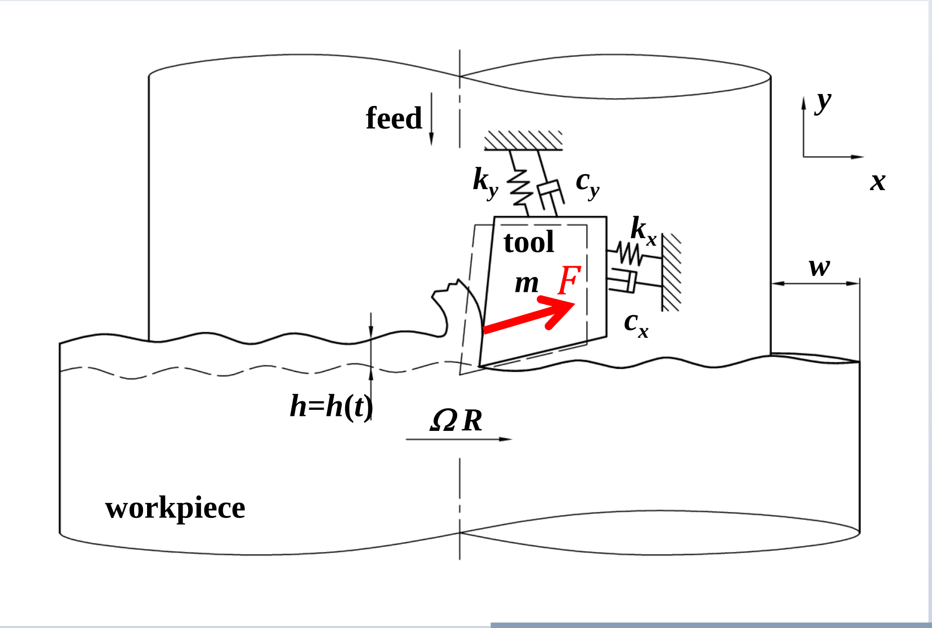
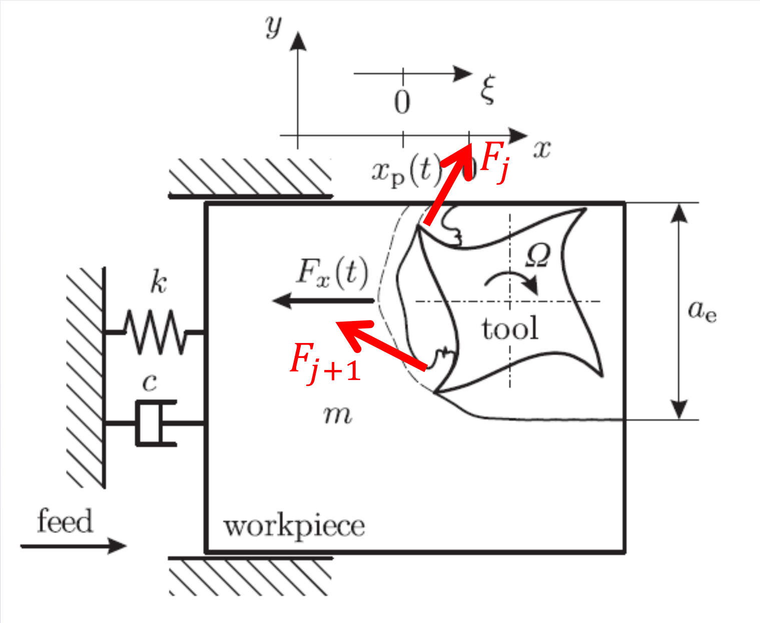
Turning process



Milling process



$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \underbrace{-Q(t)\left(v_f\tau + x(t) - x(t-\tau)\right)^q}_{\text{Cutting force}}$$

$$Q(t) = Q(t+\tau)$$

Linearized equation of motion:

$$m\ddot{y}(t) + k\dot{y}(t) + cy(t) = H(y(t-\tau) - y(t))$$

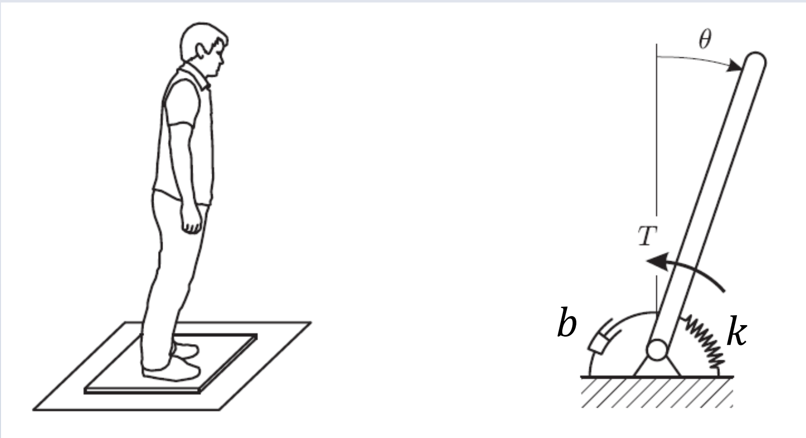
Linearized equation of motion:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -H(t)\big(x(t) - x(t-\tau)\big)$$

Balancing models

Balancing models

Postural sway



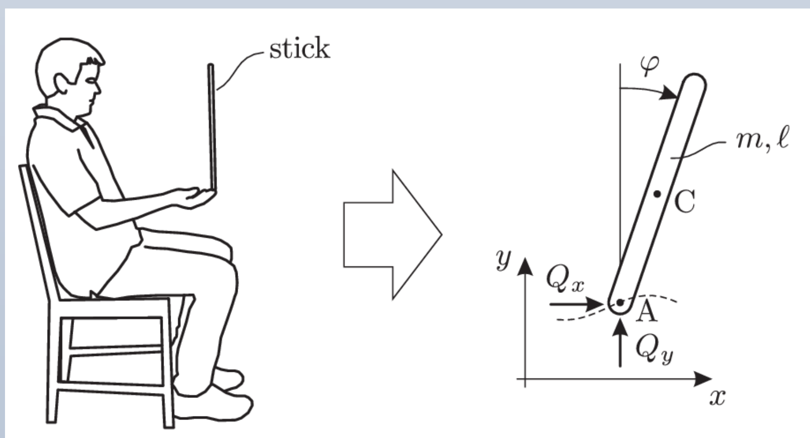
feedback torque

$$\ddot{\varphi}(t) + b\dot{\varphi}(t) + \underbrace{\left(k - \frac{3g}{2l}\right)}_{\approx -0.2\frac{3g}{2l} < 0}\varphi(t) = \frac{12}{ml^2}T(t)$$

(Loram, Lakie, Asai, Nomura)

Upper position: unstable position

Stick balancing



$$\ddot{\varphi}(t) - \frac{6g}{l}\varphi(t) = -\frac{6}{ml}Q_x(t)$$

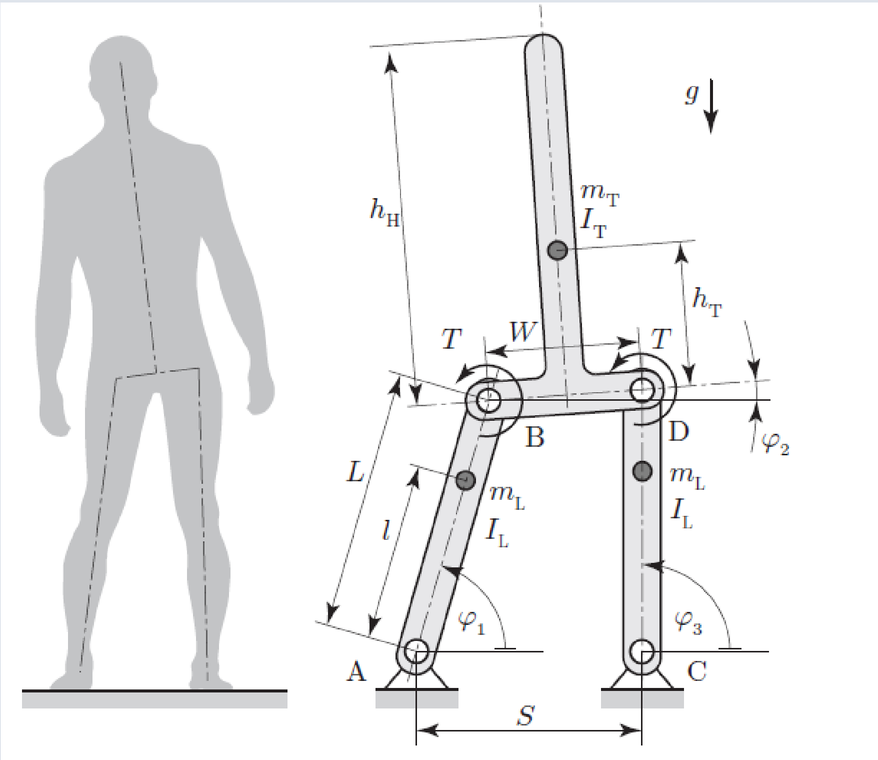
feedback force

$$\boxed{\ddot{\varphi}(t) - a\varphi(t) = -\frac{6}{ml}Q_x(t)}$$

$$a = \frac{6g}{l}$$ system parameter

Balancing models

Frontal plane mediolateral balance



$$I\,\ddot{\varphi}_1(t) - G\,\varphi_1(t) = -C\,T(t)$$

(Henry, Fung, Horak, 2001; Bingham, Ting, 2013)

$$I = 2(m_L L^2 + I_L) + \frac{m_T(h_T\alpha - W\beta)^2 + I_T\alpha^2}{W^2}$$

$$G = -g\left(\frac{m_T(h_T\alpha)^2}{W^2} - \frac{(2lm_L + Lm_T)(\alpha\beta^2 - L^2S)}{LW\beta}\right)$$

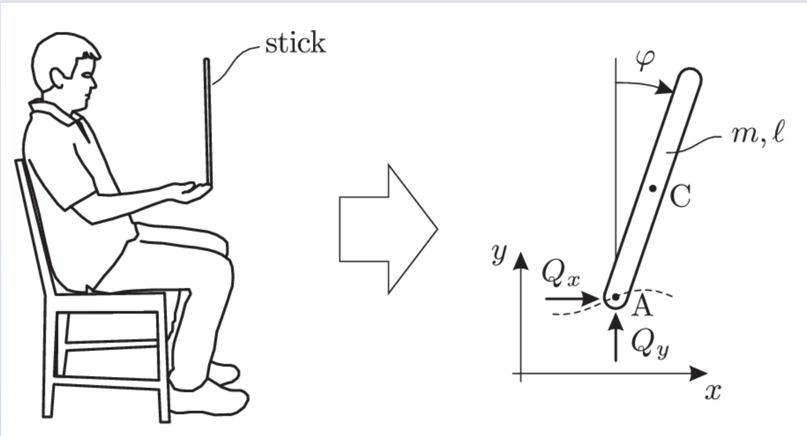
$$C = \frac{S}{W}\left(\frac{\alpha h_H}{W} - \beta - \frac{\alpha}{W}\right)$$

$$\boxed{\ddot{\varphi}(t) - a\varphi(t) = -Q(t)}$$

Upper position: unstable position

Balancing models

Stick balancing



$$\ddot{\varphi}(t) - \frac{6g}{l}\varphi(t) = -\frac{6}{ml}Q_x(t)$$

feedback force

$$\boxed{\ddot{\varphi}(t) - a\varphi(t) = -\frac{6}{ml}Q_x(t)}$$

system parameter

Stick balancing:
- Newtonian dynamics
- unstable system
- one degree-of-freedom model

Stick balancing tests

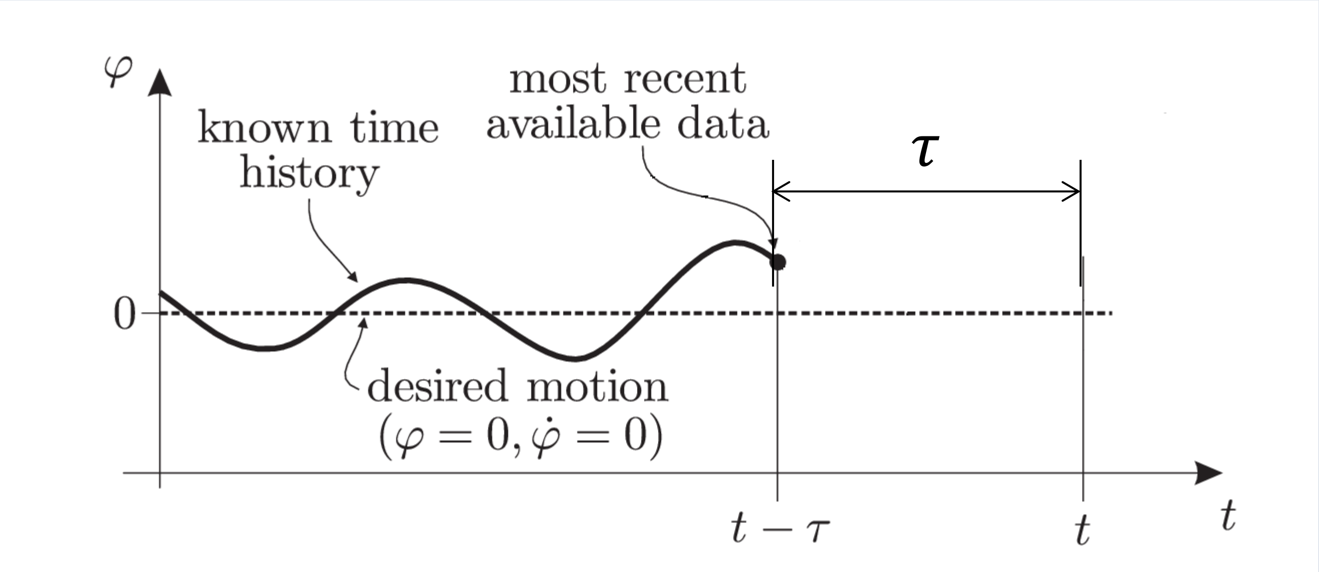
(John Milton, Claremont, 2000-)

Reflex delay

$$\boxed{\ddot{\varphi}(t) - a\varphi(t) = -\frac{6}{ml}Q(t)}$$

$$Q(t) = f(\varphi(\vartheta), \dot{\varphi}(\vartheta), \ddot{\varphi}(\vartheta)) \quad \vartheta \in [0, t-\tau]$$

For example $Q(t) = f(\varphi(t-\tau), \dot{\varphi}(t-\tau))$

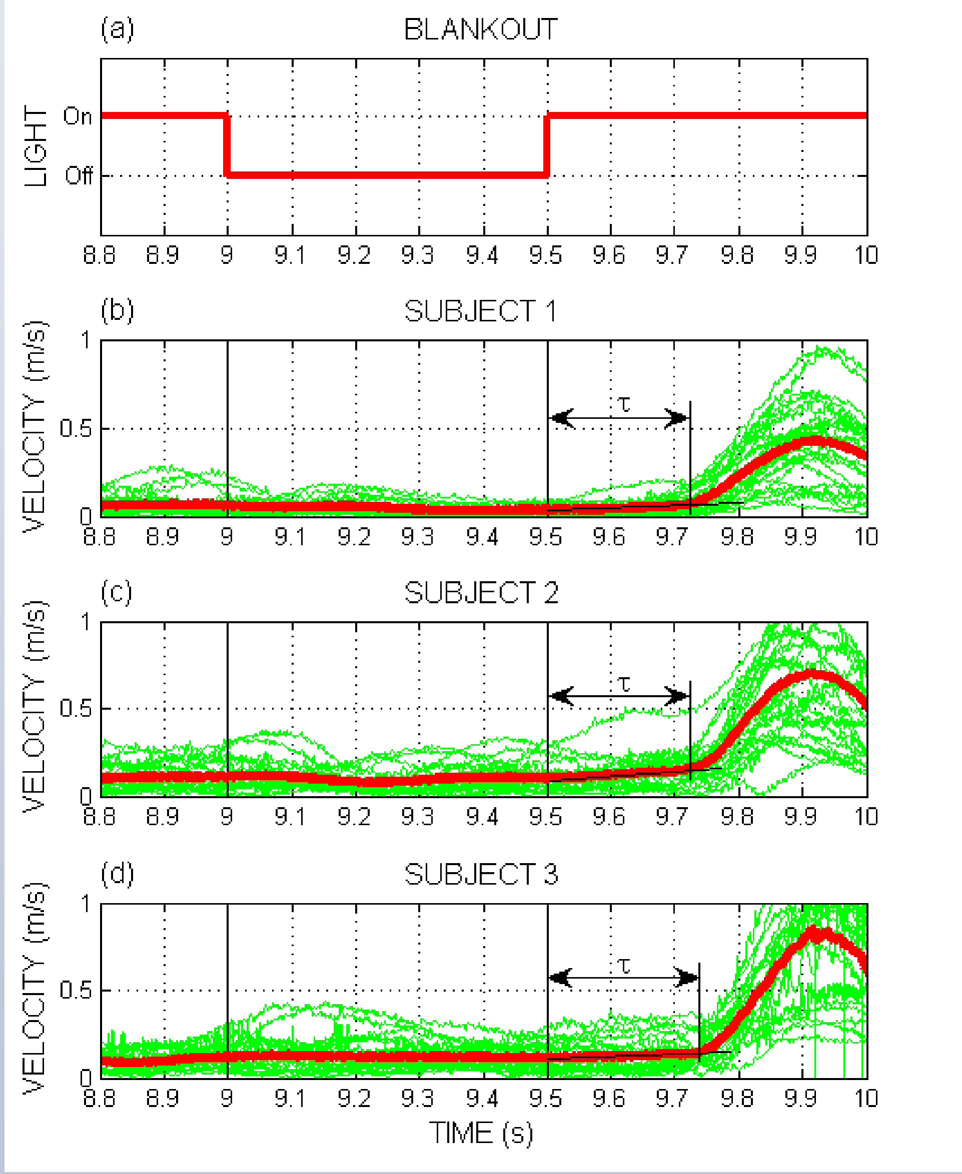


delay for visual tracking
Nasher (1976): 150~250ms
Miall (1993): 200~250ms
Jordan (1996): 100~200ms
Kawato (1999): 150~250ms

delay for stick balancing using cross-correlation:
Cabrera, Milton (2004): 80~200ms

Reflex delay

blankout tests:
Milton (2011):
$\tau \approx 230\text{ms}$



Delayed PD feedback

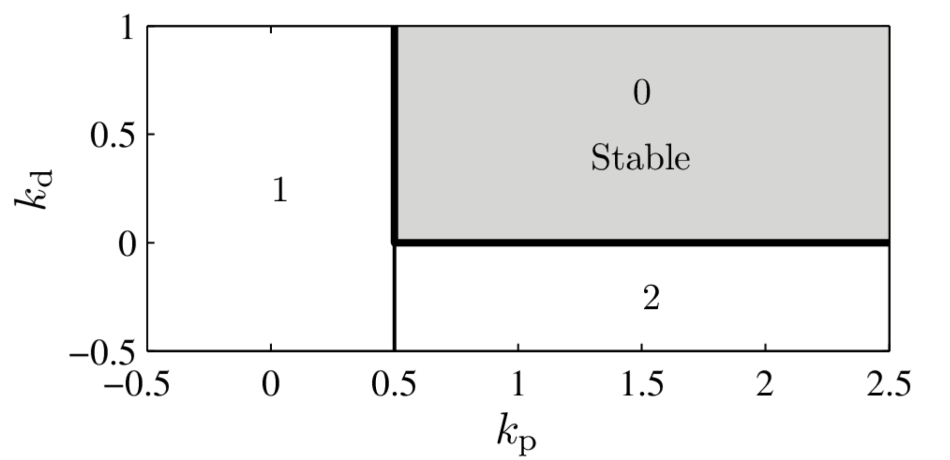
$$Q(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau)$$

feedback delay

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau)$$

$\boxed{\tau = 0}$ $$\ddot{\varphi}(t) + k_{\mathrm{d}}\dot{\varphi}(t) + \left(k_{\mathrm{p}} - a\right)\varphi(t) = 0$$

$a = 0.5$



Stability:

$$k_{\mathrm{p}} > a$$

$$k_{\mathrm{d}} > 0$$

Delayed PD feedback

$$\ddot{\varphi}(t) - a\varphi(t) = -k_\mathrm{p}\varphi(t-\tau) - k_\mathrm{d}\dot{\varphi}(t-\tau)$$

$\boxed{\tau \neq 0}$

D-subdivision:

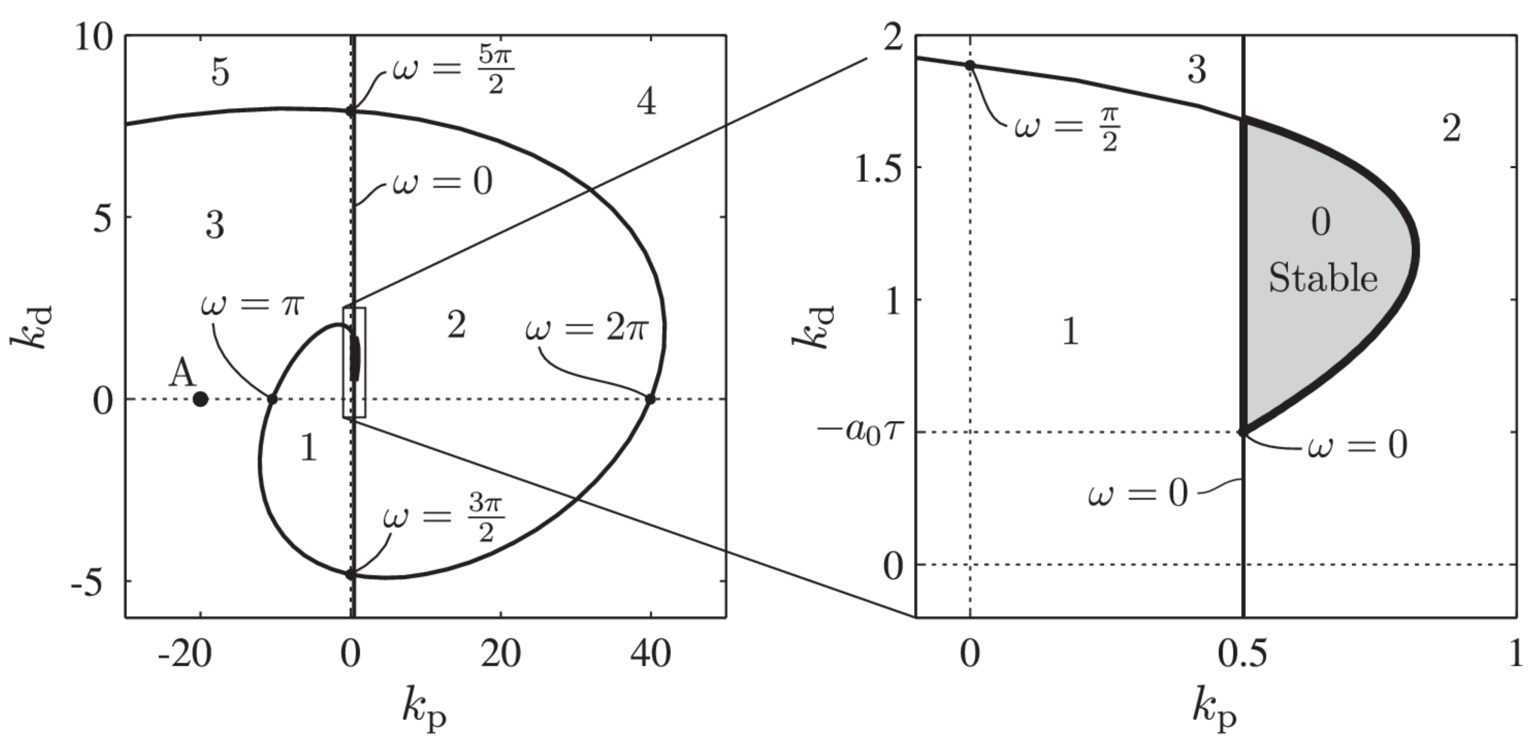
$(\varphi(t) = A\mathrm{e}^{\lambda t}, \lambda = \mathrm{i}\omega)$

$\omega = 0$: $k_\mathrm{p} = a$

$\omega \neq 0$: $k_\mathrm{p} = (\omega^2 + a)\cos(\omega\tau)$

$$k_\mathrm{d} = \frac{\omega^2 + a}{\omega}\sin(\omega\tau)$$

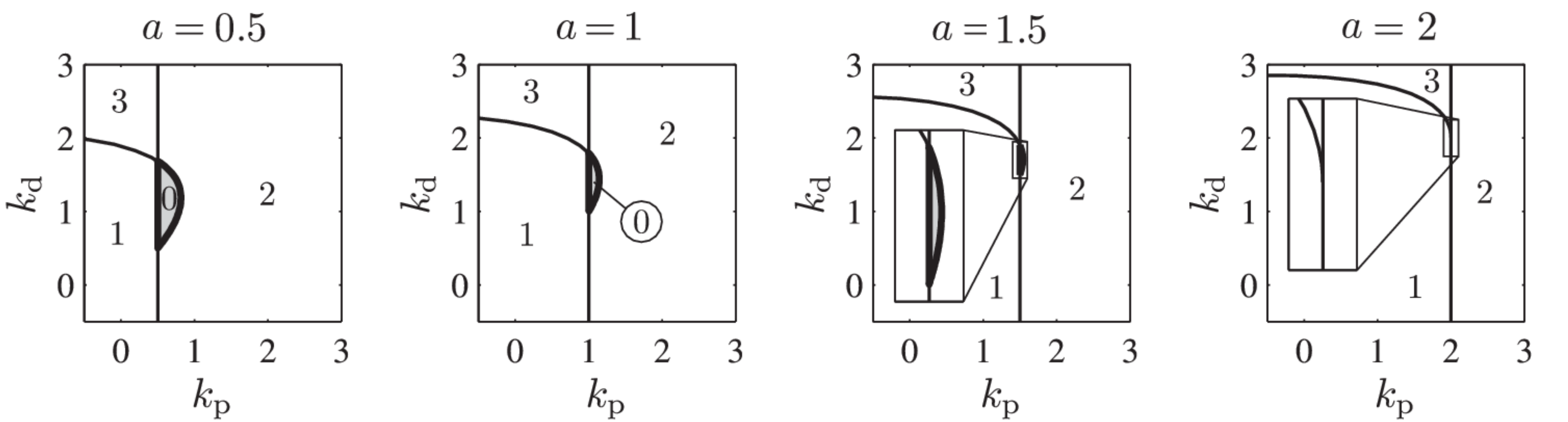
$\tau = 1, a = 0.5$



Delayed PD feedback

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau)$$

$\tau = 1$



$$a_{\mathrm{crit,PD}} = \frac{2}{\tau^2}$$ (Schürer, 1948) or $$\tau_{\mathrm{crit,PD}} = \sqrt{2/a}$$

Stick balancing: $a = \frac{6g}{l}, \quad \tau = 230\mathrm{ms}$

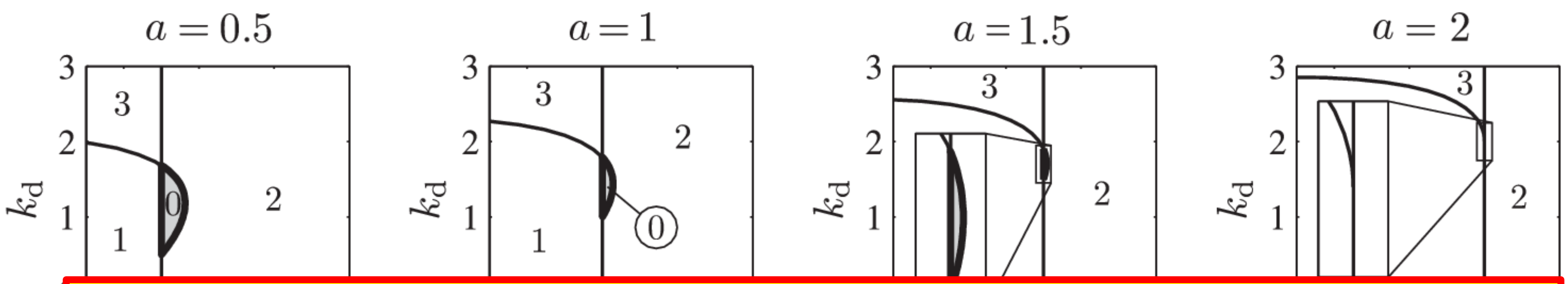
$$l_{\mathrm{crit,PD}} = \frac{6g}{a_{\mathrm{crit,PD}}} = 3g\tau^2 = 156\mathrm{cm}$$

Experiments: $l_{\mathrm{crit,PD}} = 30 \sim 50\mathrm{cm}$ (Milton et al., 1990–)

Delayed PD feedback

$$\ddot{\varphi}(t) - a\varphi(t) = -k_\mathrm{p}\varphi(t-\tau) - k_\mathrm{d}\dot{\varphi}(t-\tau)$$

$\tau = 1$



What control law does the human brain use?

Stick balancing: $a = \frac{6g}{l}$, $\tau = 230\mathrm{ms}$

$$l_\mathrm{crit,PD} = \frac{6g}{a_\mathrm{crit,PD}} = 3g\tau^2 = 156\mathrm{cm}$$

Experiments: $l_\mathrm{crit,PD} = 30\sim50\mathrm{cm}$ (Milton et al., 1990–)

Different linear control concepts

Time-invariant controllers:

- proportional-derivative (PD)

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau)$$

- proportional-derivative-acceleration (PDA)

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau) - k_{\mathrm{a}}\ddot{\varphi}(t-\tau)$$

- model predictive (MP) controllers

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi_{\mathrm{p}}(t) - k_{\mathrm{d}}\dot{\varphi}_{\mathrm{p}}(t)$$

Time-varying controller:

- act-and-wait (AAW)

$$\ddot{\varphi}(t) - a\varphi(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < t_{\mathrm{w}} \ \text{(wait)} \\ -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau) & \text{if} \quad \mathrm{t_w} \le t < t_{\mathrm{w}} + t_{\mathrm{a}} = T \ \text{(act)} \end{cases}$$

Different linear control concepts

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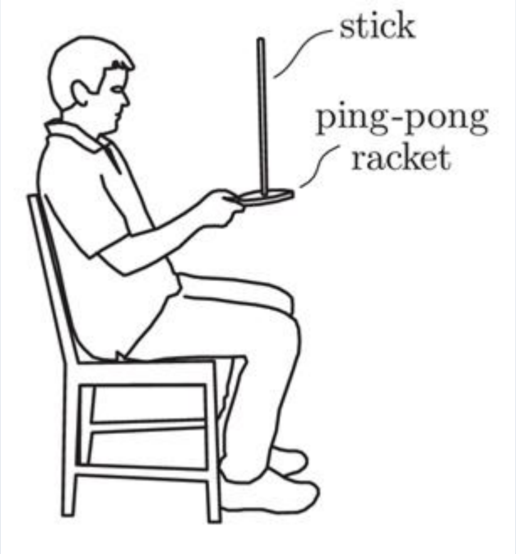
Proportional-derivative-acceleration (PDA) controller

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau) - k_{\mathrm{a}}\ddot{\varphi}(t-\tau)$$



No sensory feedback from fingertip ⟹ PD

Sensory feedback from fingertip ⟹ PDA (?)

Proportional-derivative-acceleration (PDA) controller

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau) - k_{\mathrm{a}}\ddot{\varphi}(t-\tau)$$

Neutral Functional Differential Equations (NFDE):

Necessary condition for stability: the difference part must be stable.

$$\ddot{\varphi}(t) = -k_{\mathrm{a}}\ddot{\varphi}(t-\tau)$$

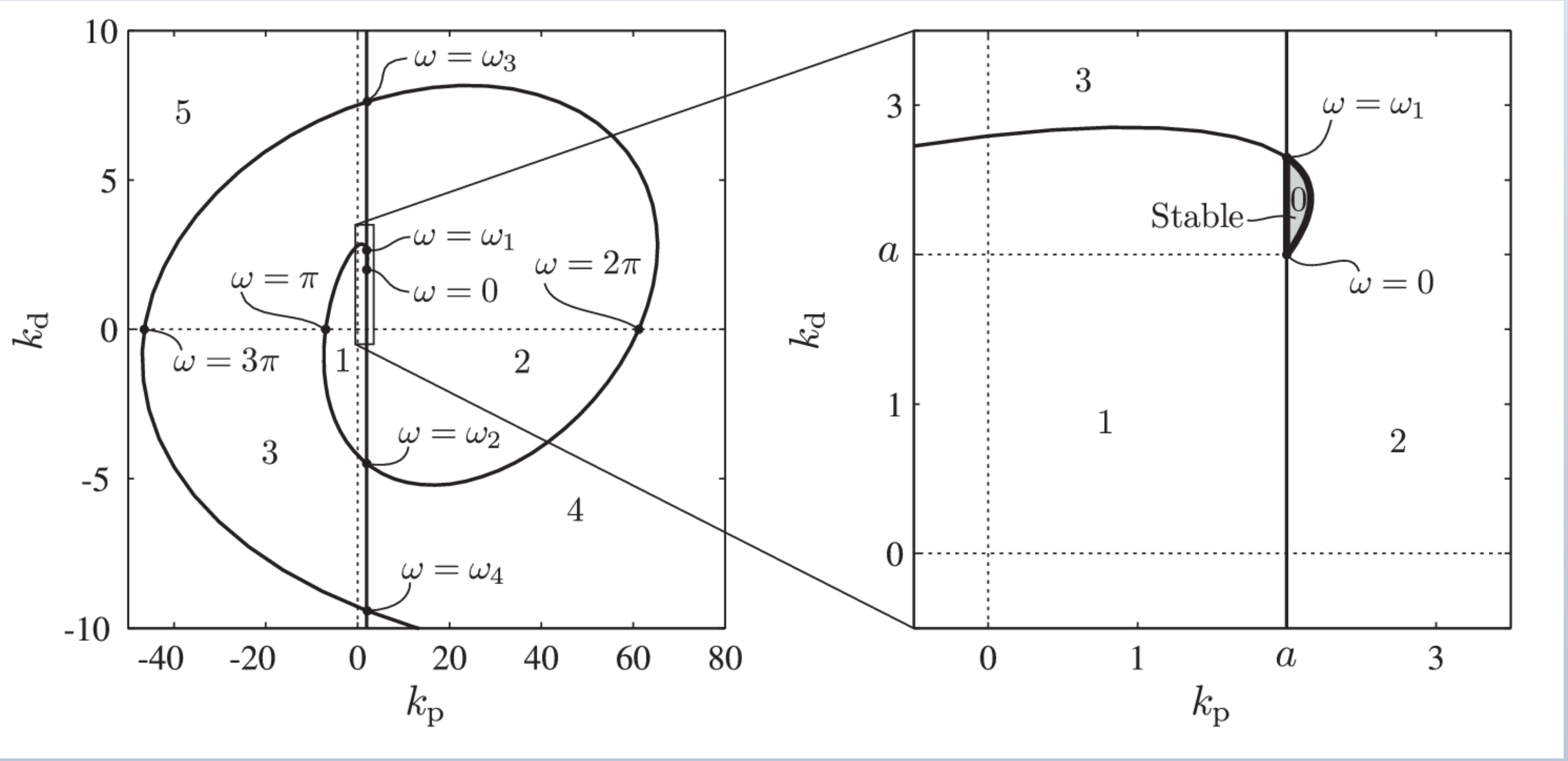
$$\Downarrow$$

$$|k_{\mathrm{a}}| \leq 1$$

Proportional-derivative-acceleration (PDA) controller

$$\ddot{\varphi}(t) - a\varphi(t) = -k_\mathrm{p}\varphi(t-\tau) - k_\mathrm{d}\dot{\varphi}(t-\tau) - k_\mathrm{a}\ddot{\varphi}(t-\tau)$$

$\tau = 1$ $a = 2$ $k_\mathrm{a} = 0.5$ D-subdivision



Proportional-derivative-acceleration (PDA) controller

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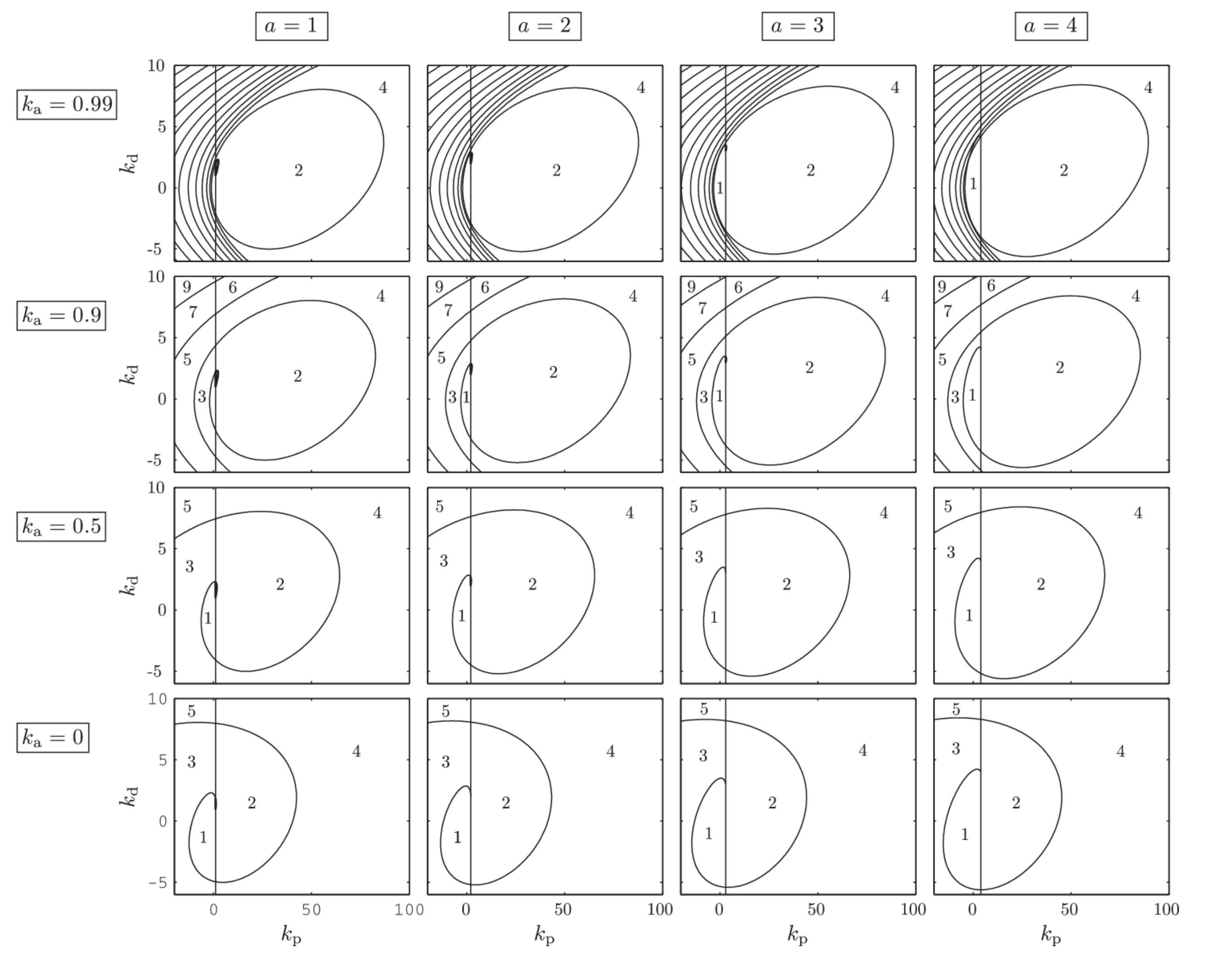

$\tau = 1$

$|k_\mathrm{a}| \leq 1$

Proportional-derivative-acceleration (PDA) controller

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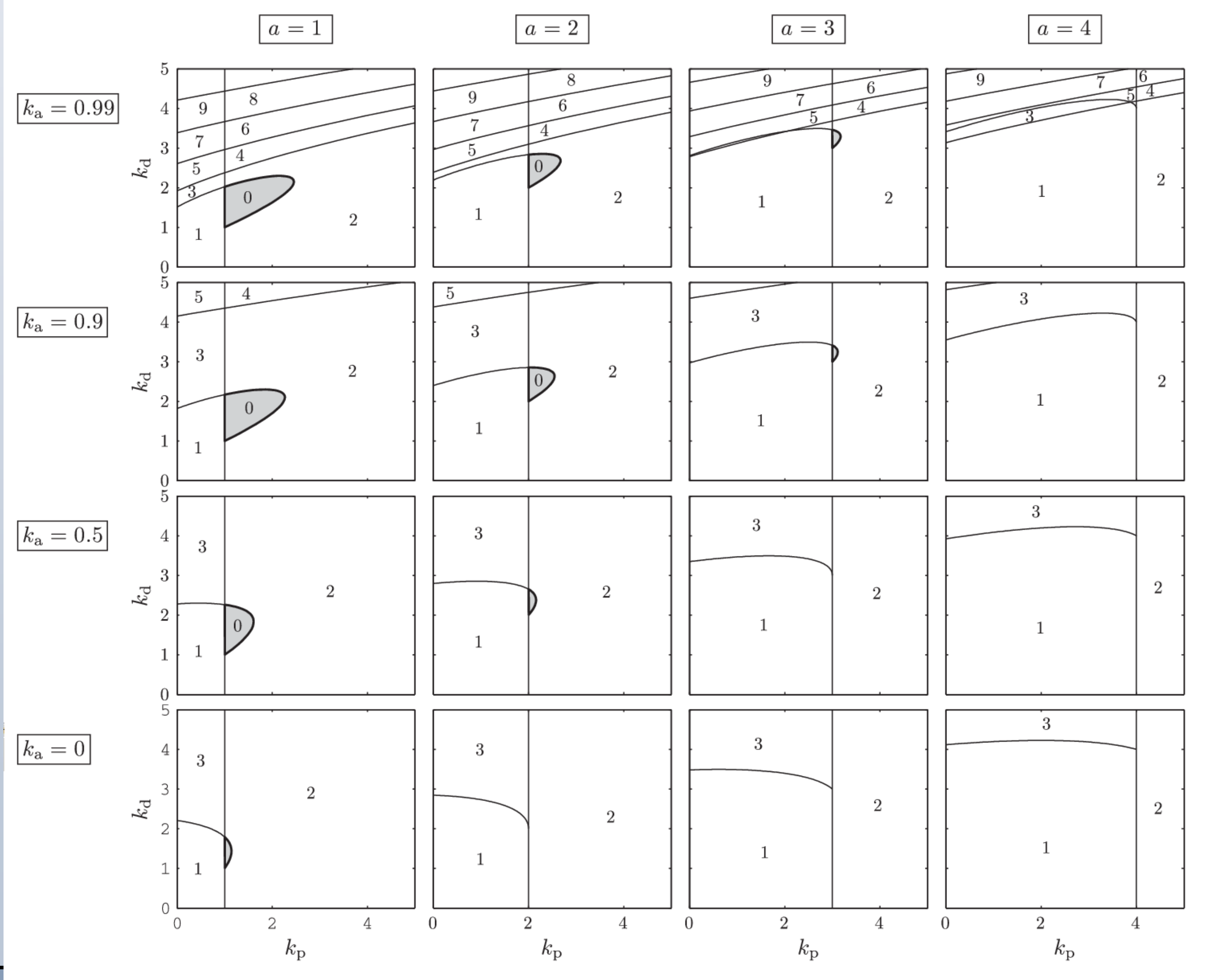

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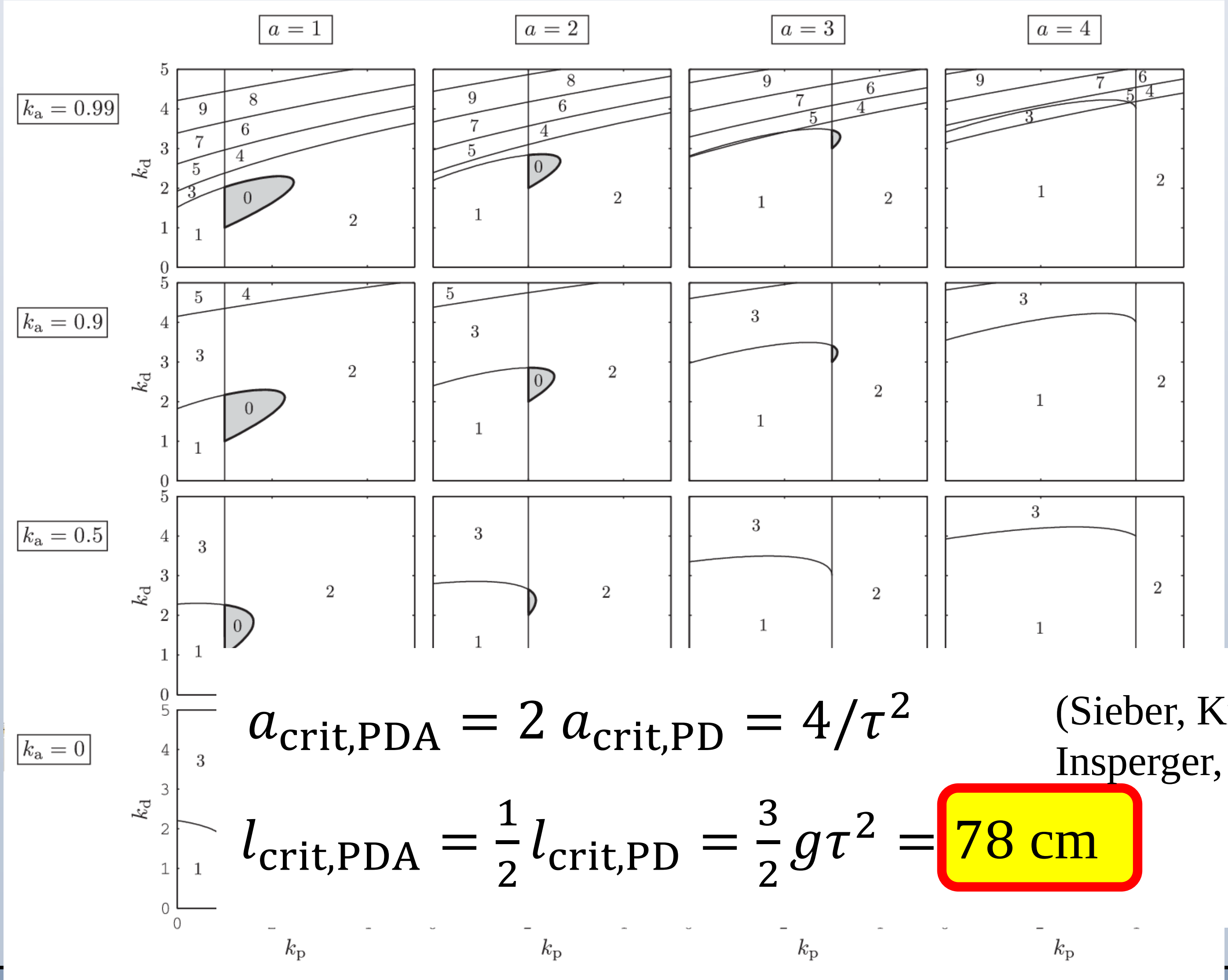

$\tau = 1$

$|k_{\mathrm{a}}| \leq 1$

$$a_{\mathrm{crit,PDA}} = 2\, a_{\mathrm{crit,PD}} = 4/\tau^2$$

$$l_{\mathrm{crit,PDA}} = \frac{1}{2} l_{\mathrm{crit,PD}} = \frac{3}{2} g\tau^2 = \boxed{78\ \mathrm{cm}}$$

(Sieber, Krauskopf 2005; Insperger, Milton, Stépán 2013)

Different linear control concepts

Time-invariant controllers:

- proportional-derivative (PD)

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau)$$

- proportional-derivative-acceleration (PDA)

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau) - k_{\mathrm{a}}\ddot{\varphi}(t-\tau)$$

- model predictive (MP) controllers

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi_{\mathrm{p}}(t) - k_{\mathrm{d}}\dot{\varphi}_{\mathrm{p}}(t)$$

Time-varying controller:

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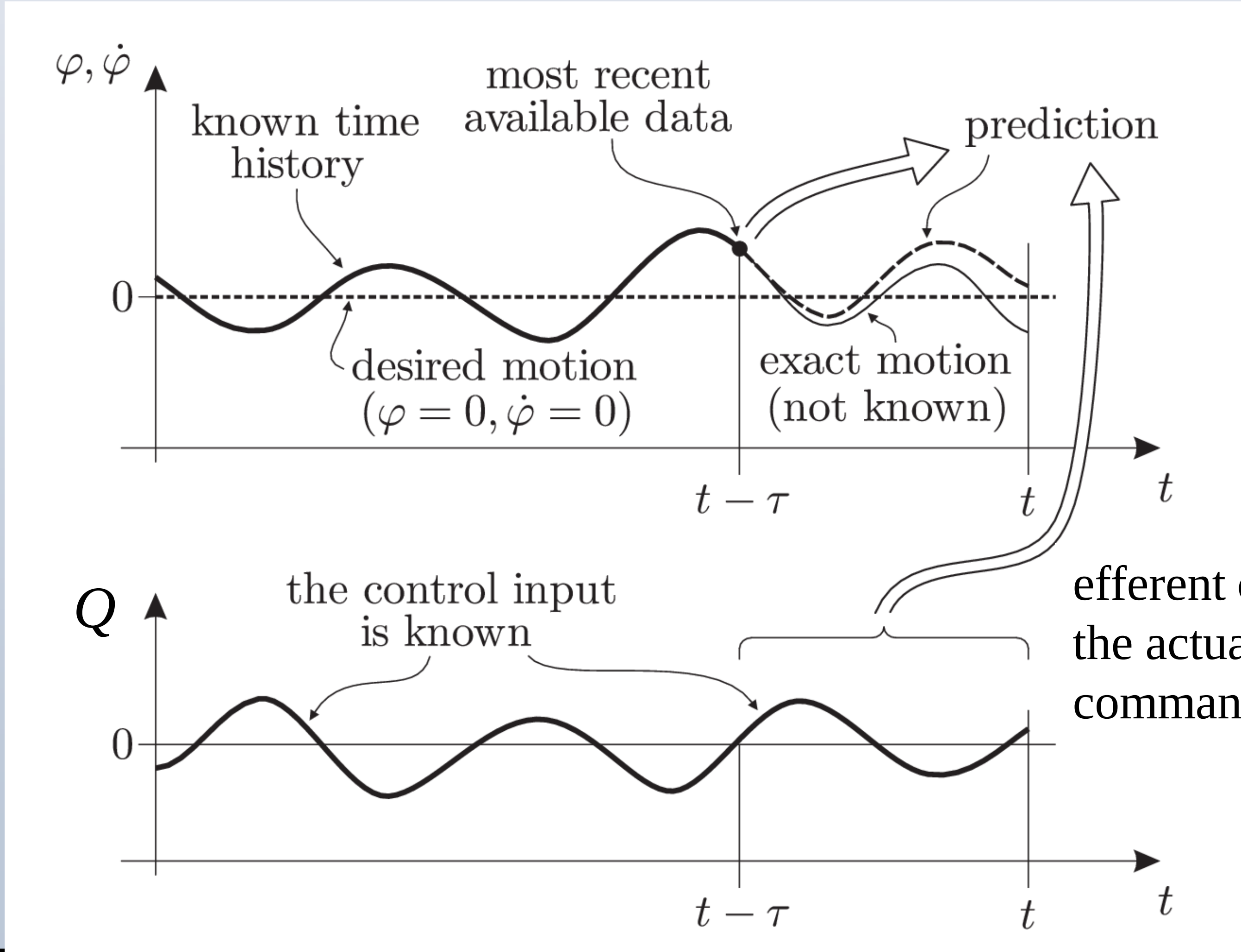
$$\ddot{\varphi}(t) - a\varphi(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < t_{\mathrm{w}} \ \text{(wait)} \\ -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau) & \text{if} \quad \mathrm{t}_{\mathrm{w}} \le t < t_{\mathrm{w}} + t_{\mathrm{a}} = T \ \text{(act)} \end{cases}$$

Model predictive (MP) controller

$$\ddot{\varphi}(t) - a\varphi(t) = -Q(t) \qquad Q(t) = f(\varphi(\vartheta), \dot{\varphi}(\vartheta), \ddot{\varphi}(\vartheta), Q(\xi))$$

$$\vartheta \in [0, t-\tau], \quad \xi \in [0, t]$$



efferent copies of the actual control commands

Model predictive (MP) controller

Predictor-based feedback
Finite Spectrum Assignment
Modified Smith predictor

Mayne (1968), Kleinman (1969)
Manitius and Olbrot (1978)
Michiels, Niculescu, Mondie, Krstic, Jankovic, Wang, Karafyllis, Mirkin, Zhong, …

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t-\tau)$$

$$\mathbf{x}(t) = \begin{pmatrix} \varphi(t) \\ \dot{\varphi}(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}(t-\tau) = Q(t-\tau)$$

Model predictive (MP) controller

Predictor-based feedback
Finite Spectrum Assignment
Modified Smith predictor

Mayne (1968), Kleinman (1969)
Manitius and Olbrot (1978)
Michiels, Niculescu, Mondie, Krstic, Jankovic,
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$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t-\tau)$$

Prediction of $\mathbf{x}(t+\tau)$ from $\mathbf{x}(t)$:

$$\dot{\mathbf{x}}_{\mathrm{p}}(\vartheta) = \widetilde{\mathbf{A}}\mathbf{x}_{\mathrm{p}}(\vartheta) + \widetilde{\mathbf{B}}\mathbf{u}(\vartheta - \tilde{\tau}), \qquad \vartheta \in [t, t+\tilde{\tau}), \qquad \mathbf{x}_{\mathrm{p}}(t) = \mathbf{x}(t)$$

$$\mathbf{x}_{\mathrm{p}}(t+\tilde{\tau}) = \mathrm{e}^{\widetilde{\mathbf{A}}\tilde{\tau}}\mathbf{x}(t) + \int_t^{t+\tilde{\tau}} \mathrm{e}^{\widetilde{\mathbf{A}}(t+\tilde{\tau}-\vartheta)}\widetilde{\mathbf{B}}\mathbf{u}(\vartheta - \tilde{\tau})\mathrm{d}\vartheta$$

Controller:

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}_{\mathrm{p}}(t+\tilde{\tau}) = \mathbf{K}\mathrm{e}^{\widetilde{\mathbf{A}}\tilde{\tau}}\mathbf{x}(t) + \mathbf{K}\int_t^{t+\tilde{\tau}} \mathrm{e}^{\widetilde{\mathbf{A}}(t+\tilde{\tau}-\vartheta)}\widetilde{\mathbf{B}}\mathbf{u}(\vartheta - \tilde{\tau})\mathrm{d}\vartheta$$

If $\widetilde{\mathbf{A}} = \mathbf{A}, \widetilde{\mathbf{B}} = \mathbf{B}$ and $\tilde{\tau} = \tau$ then $\mathbf{x}_{\mathrm{p}}(t+\tilde{\tau}) = \mathbf{x}(t+\tau)$

$$\Longrightarrow \mathbf{u}(t-\tau) = \mathbf{K}\mathbf{x}(t) \quad \Longrightarrow \quad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{x}(t) \Longrightarrow \boxed{l_{\mathrm{crit,MP}} = 0}$$

Different linear control concepts

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$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau) - k_{\mathrm{a}}\ddot{\varphi}(t-\tau)$$

- model predictive (MP) controllers

$$\ddot{\varphi}(t) - a\varphi(t) = -k_{\mathrm{p}}\varphi_{\mathrm{p}}(t) - k_{\mathrm{d}}\dot{\varphi}_{\mathrm{p}}(t)$$

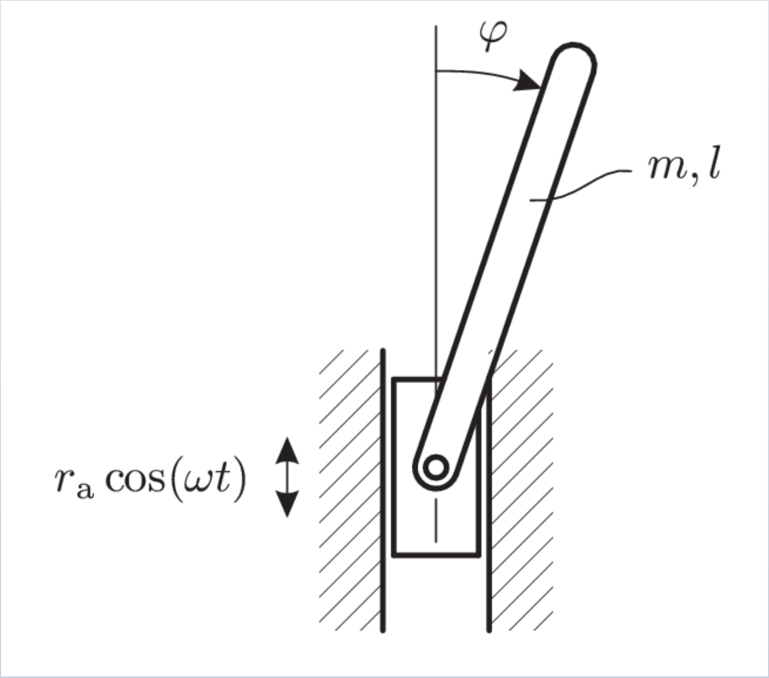
Time-varying controller:

- act-and-wait (AAW)

$$\ddot{\varphi}(t) - a\varphi(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < t_{\mathrm{w}} \ \text{(wait)} \\ -k_{\mathrm{p}}\varphi(t-\tau) - k_{\mathrm{d}}\dot{\varphi}(t-\tau) & \text{if} \quad \mathrm{t}_{\mathrm{w}} \le t < t_{\mathrm{w}} + t_{\mathrm{a}} = T \ \text{(act)} \end{cases}$$

Act-and-wait (AAW) controller

Motivation: parametric forcing of the inverted pendulum

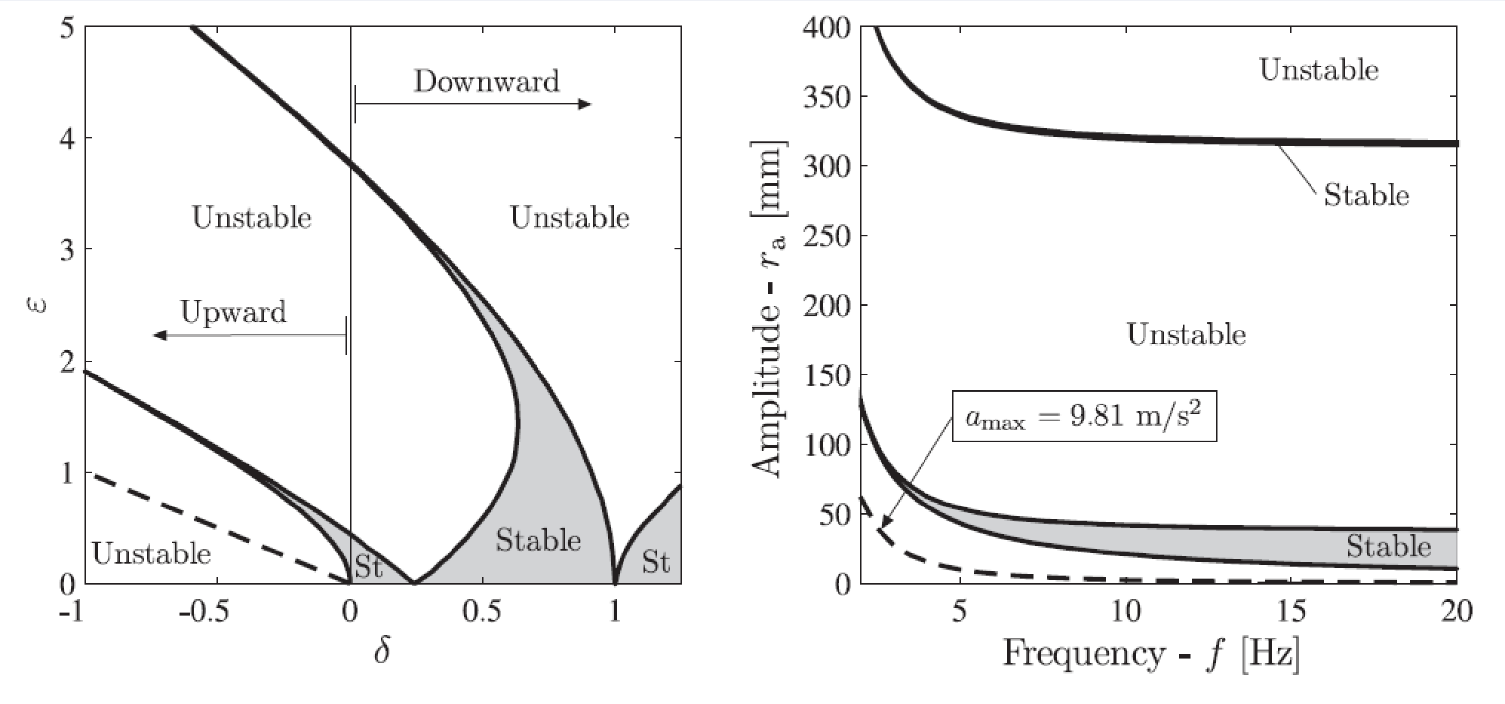


$$\ddot{\varphi}(t) + \left(-\frac{3g}{2l} + \frac{3r_a\omega^2}{2l}\cos(\omega t)\right)\varphi(t) = 0$$

Mathieu equation: $\ddot{\varphi}(t) + (\delta + \varepsilon\cos(\omega t))\varphi(t) = 0$

Act-and-wait (AAW) controller

Motivation: parametric forcing of the inverted pendulum

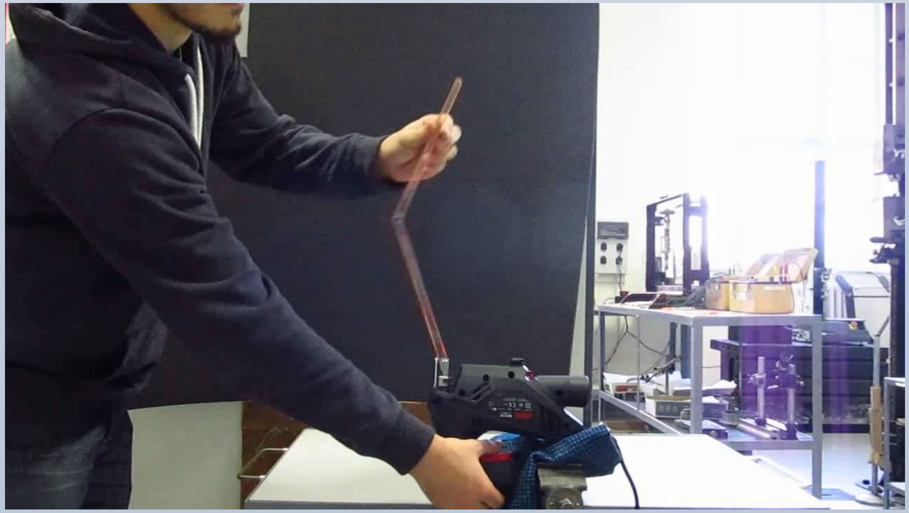


$$\ddot{\varphi}(t) + \left(-\frac{3g}{2l} + \frac{3r_a\omega^2}{2l}\cos(\omega t)\right)\varphi(t) = 0$$

Mathieu equation: $\ddot{\varphi}(t) + (\delta + \varepsilon\cos(\omega t))\varphi(t) = 0$

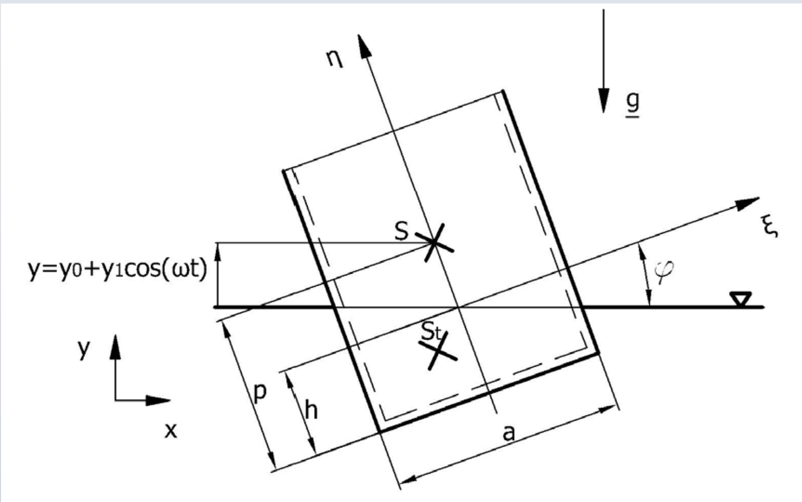
Act-and-wait (AAW) controller

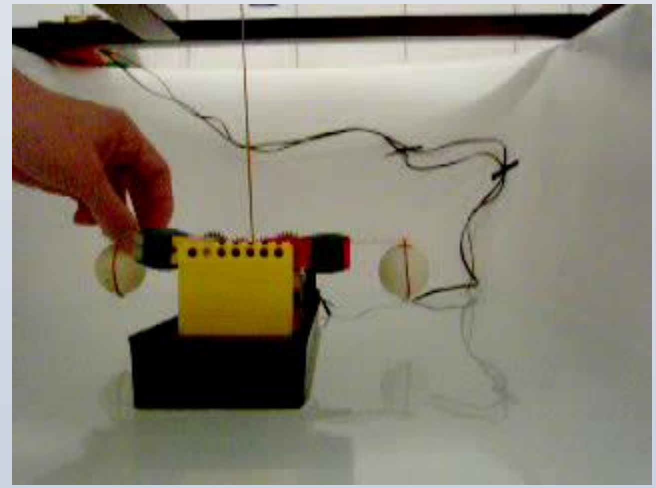
Motivation: parametric forcing of the inverted pendulum



Act-and-wait (AAW) controller

Motivation: parametric forcing of the inverted pendulum



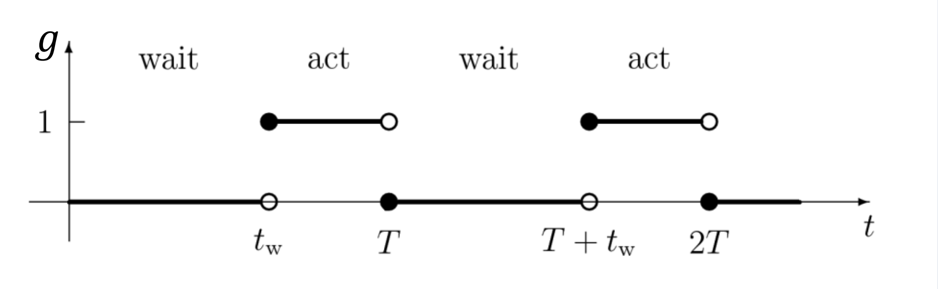


(Ambrus Zelei, BME, 2006)

Kayaking and canoeing...

Act-and-wait (AAW) controller

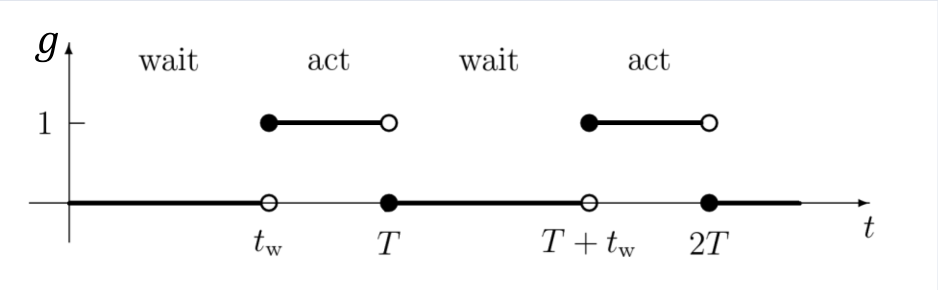
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{u}(t) = g(t)\mathbf{K}\,\mathbf{x}(t-\tau)$$



$$\mathrm{g}(t) = \begin{cases} 0 & \text{if} \quad 0 \le (t \bmod T) < t_\mathrm{w} \quad \text{(wait)} \\ 1 & \text{if} \quad t_\mathrm{w} \le (t \bmod T) < t_\mathrm{w} + t_\mathrm{a} = T \text{ (act)} \end{cases}$$

(Insperger, Stépán 2006)

Act-and-wait (AAW) controller

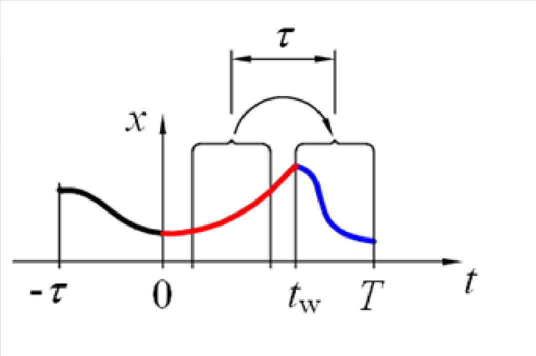


$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{u}(t) = g(t)\mathbf{K}\,\mathbf{x}(t-\tau)$$

$$\mathrm{g}(t) = \begin{cases} 0 & \text{if} \quad 0 \le (t \bmod T) < t_\mathrm{w} \quad \text{(wait)} \\ 1 & \text{if} \quad t_\mathrm{w} \le (t \bmod T) < t_\mathrm{w} + t_\mathrm{a} = T \ \text{(act)} \end{cases}$$

(Insperger, Stépán 2006)

Step-by-step solution ($t_\mathrm{w} \ge \tau$ and $t_\mathrm{a} \le \tau$):



$$t \in [0, t_\mathrm{w}):\ \ \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \ \rightarrow\ \mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0)$$

$$t \in [t_\mathrm{w}, T):\ \ \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{K}\,\mathbf{x}(t-\tau) = \mathbf{A}\mathbf{x}(t) + \mathbf{K}\mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{x}(0)$$

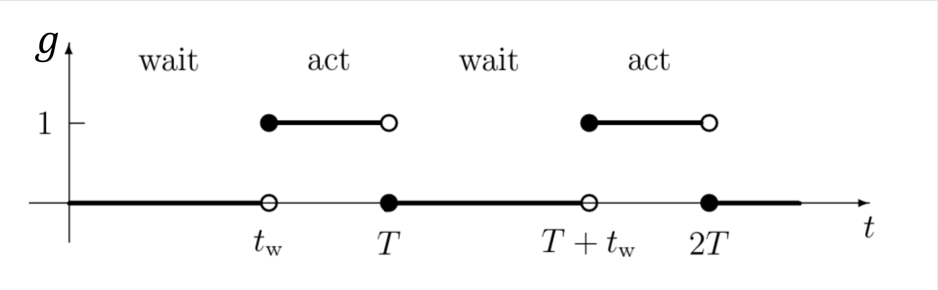
$$\rightarrow \quad \underset{(\mathbf{x} \in \mathbb{R}^n)}{\mathbf{x}(T)} = \underbrace{\left(\mathrm{e}^{\mathbf{A}T} + \int_{t_\mathrm{w}}^{T} \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{B}\mathbf{K}\mathrm{e}^{\mathbf{A}(s-\tau)}\right)}_{\boldsymbol{\Phi} \in \mathbb{R}^{n \times n}}\mathbf{x}(0)$$

$$\boxed{l_{\mathrm{crit,AAW}} = 0}$$

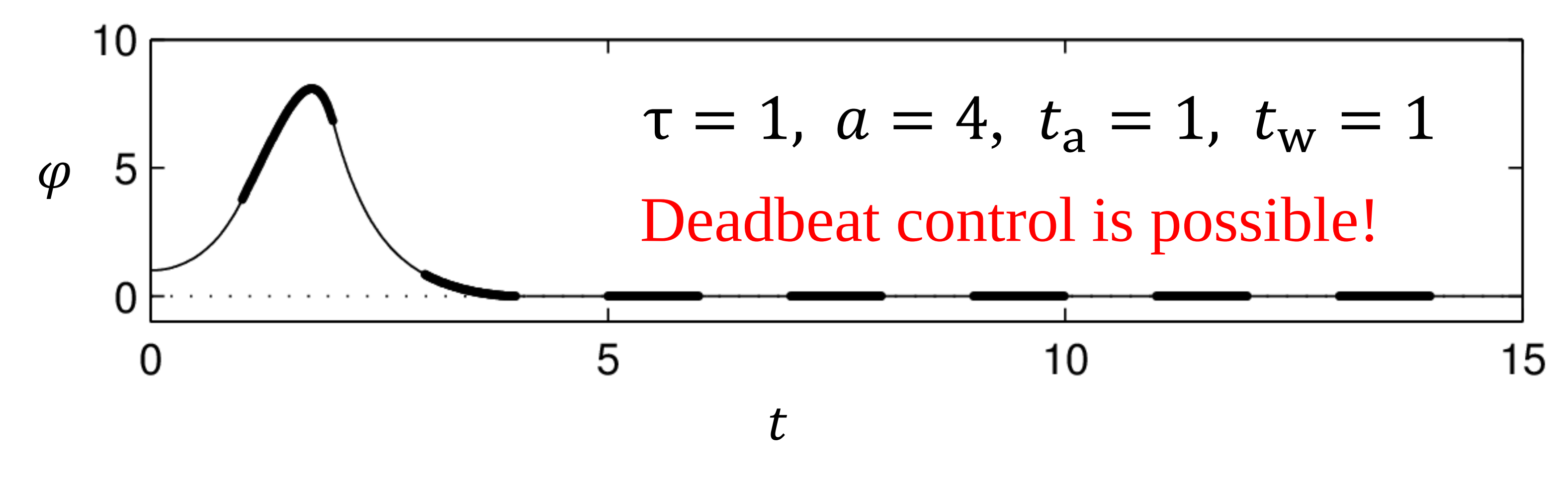
Finite dimensional map

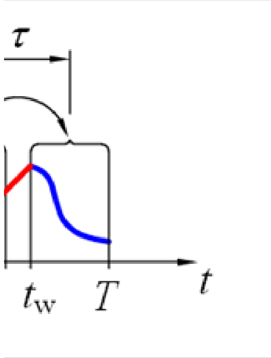
Act-and-wait (AAW) controller

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{u}(t) = g(t)\mathbf{K}\,\mathbf{x}(t-\tau)$$



$$g(t) = \begin{cases} 0 & \text{if} \quad 0 \le (t \bmod T) < t_{\mathrm{w}} \quad \text{(wait)} \\ 1 & \text{if} \quad t_{\mathrm{w}} \le (t \bmod T) < t_{\mathrm{w}} + t_{\mathrm{a}} = T \ \text{(act)} \end{cases}$$





$\mathbf{x}(0)$

$$\rightarrow \quad \mathbf{x}(T) = \underbrace{\left(\mathrm{e}^{\mathbf{A}T} + \int_{t_{\mathrm{w}}}^{T} \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{B}\mathbf{K}\mathrm{e}^{\mathbf{A}(s-\tau)} \right)}_{\boldsymbol{\Phi} \in \mathbb{R}^{n \times n}} \mathbf{x}(0)$$

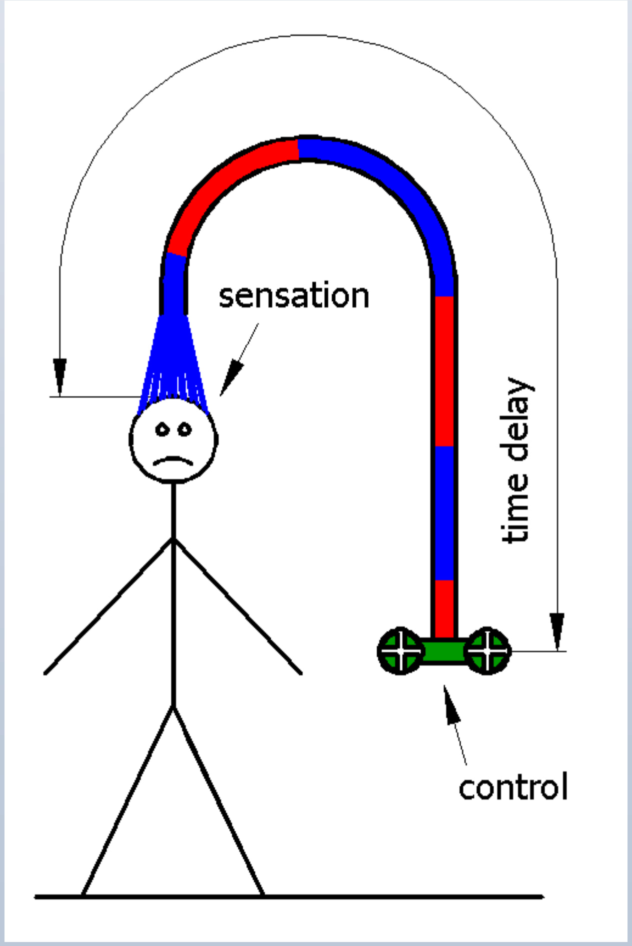
$(\mathbf{x} \in \mathbb{R}^n)$

$\boxed{l_{\mathrm{crit,AAW}} = 0}$

Finite dimensional map

Why wait?

It might seem unnatural not to actuate at all during the wait period in a control process, still... consider the way you take a shower...



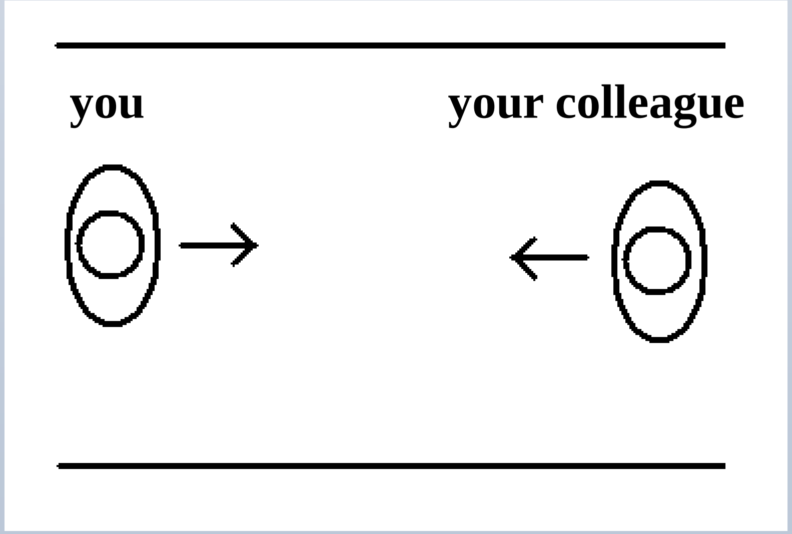
Constant gain control: slow, continuous turning

Act-and-wait: turn and stop, turn and stop

Why wait?

It might seem unnatural not to actuate at all during the wait period in a control process, still...

Bypassing in a narrow corridor...

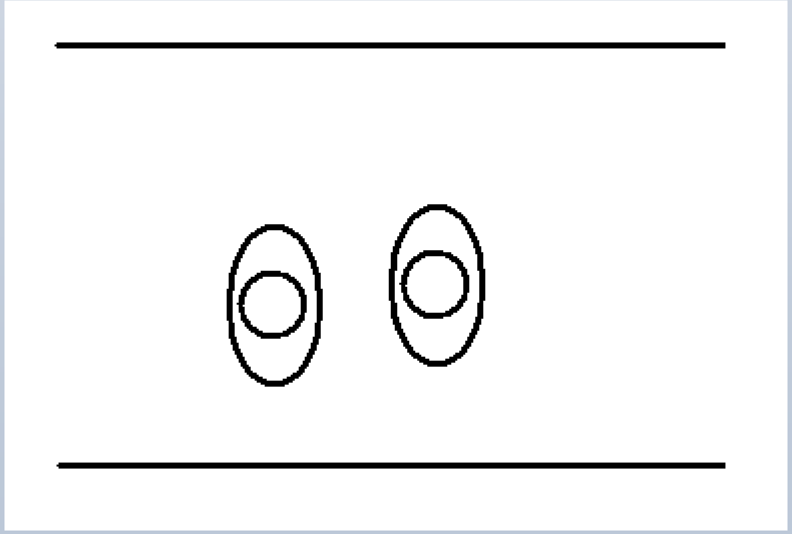


Why wait?

It might seem unnatural not to actuate at all during the wait period in a control process, still...

Bypassing in a narrow corridor...

Stop and wait!

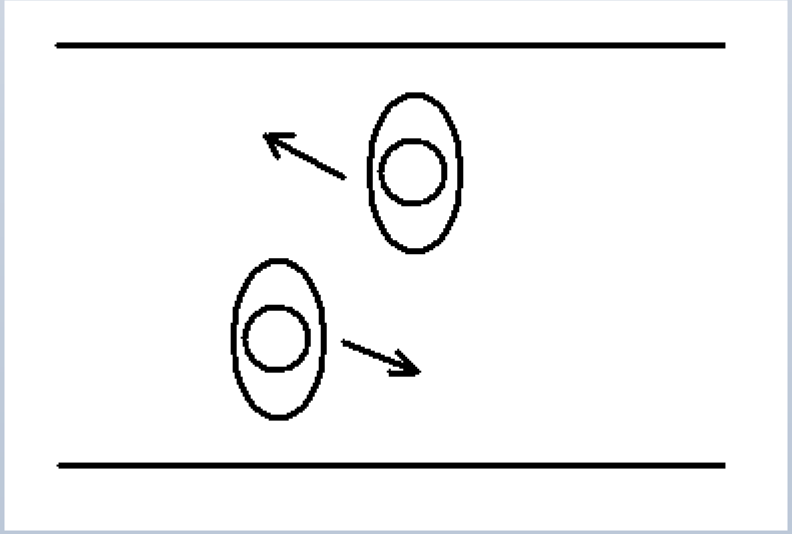


Why wait?

It might seem unnatural not to actuate at all during the wait period in a control process, still...

Bypassing in a narrow corridor...

And then go!



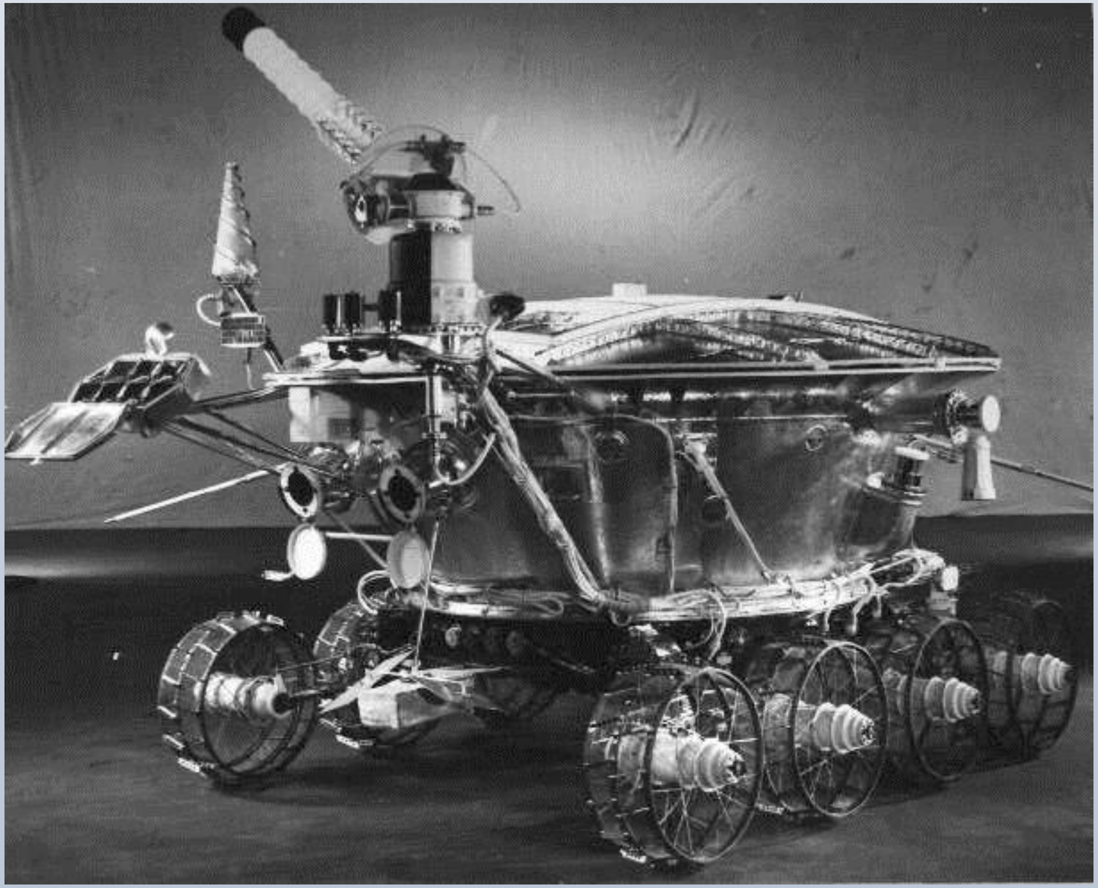
Why wait?

...or the Lunokhod 2...

Lunokhod 2
January-June, 1973
36 km in 137 days

Earth-Moon-Earth:
$2\times1.3s = 2.6s$

Earth-Mars-Earth:
32min



Comparison of different control concepts

- proportional-derivative (PD) $\boxed{l_{\mathrm{crit,PD}} = 156\ \mathrm{cm}}$
- proportional-derivative-acceleration (PDA) $\boxed{l_{\mathrm{crit,PDA}} = 78\ \mathrm{cm}}$
- model predictive (MP) controllers $\boxed{l_{\mathrm{crit,MP}} = 0\ \mathrm{cm}}$
- act-and-wait (AAW) $\boxed{l_{\mathrm{crit,AAW}} = 0\ \mathrm{cm}}$

Modelling sensory uncertainties

(Insperger, Milton, Biol Cybern, 2014)

Perceived sensory inputs:

- $\varphi_{\mathrm{s}}(t) = (1+\delta_{\mathrm{p}})\varphi(t), \quad |\delta_{\mathrm{p}}| \leq \varepsilon_{\mathrm{p}}$

  ε_{p}: sensory uncertainty radius for the angular position

- $\dot{\varphi}_{\mathrm{s}}(t) = (1+\delta_{\mathrm{v}})\dot{\varphi}(t), \quad |\delta_{\mathrm{v}}| \leq \varepsilon_{\mathrm{v}}$

  ε_{v}: sensory uncertainty radius for the angular velocity

- $\ddot{\varphi}_{\mathrm{s}}(t) = (1+\delta_{\mathrm{a}})\ddot{\varphi}(t), \quad |\delta_{\mathrm{a}}| \leq \varepsilon_{\mathrm{a}}$

  ε_{a}: sensory uncertainty radius for the angular acceleration

- $u(t) = (1+\delta_{\mathrm{u}})u(t), \quad |\delta_{\mathrm{u}}| \leq \varepsilon_{\mathrm{u}}$

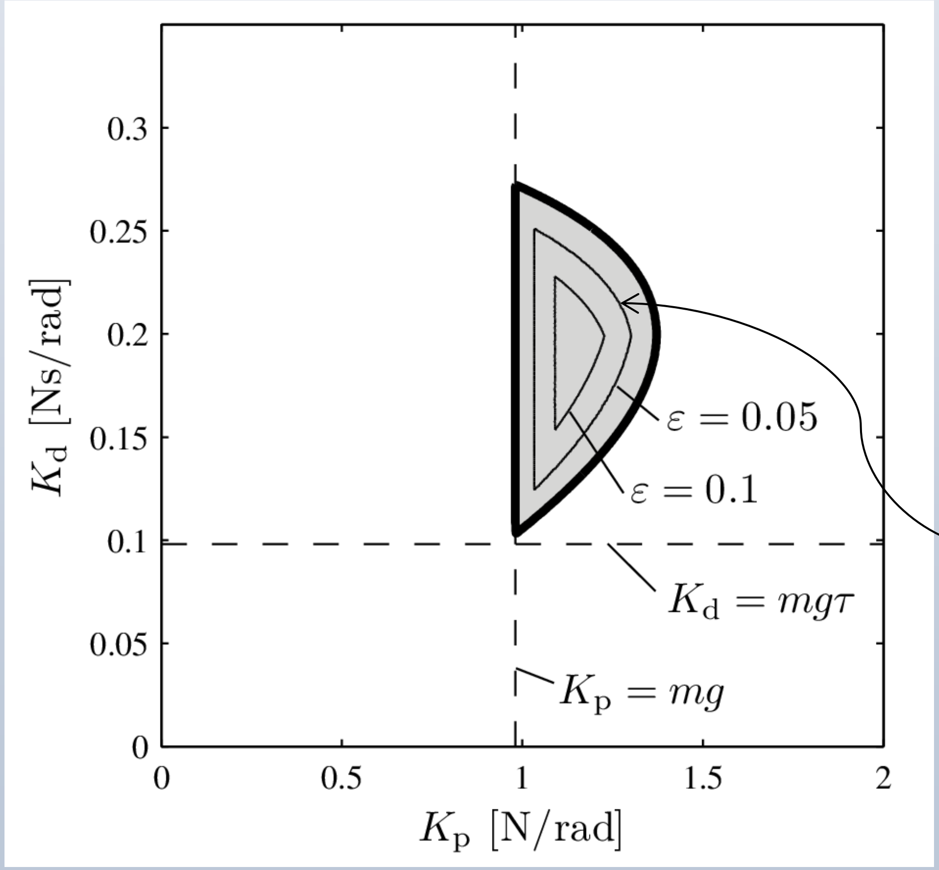
  ε_{u}: sensory uncertainty radius for the efferent copies

$\varepsilon \approx 7\text{–}13\%$ (Arieli 1996; Otmakhov 1993; Shadlen and Newsome 1998)

Modelling sensory uncertainties - PD

$$\ddot{\varphi}(t) - a\varphi(t) = -k_p \varphi_s(t-\tau) - k_d \dot{\varphi}_s(t-\tau)$$

$$\ddot{\varphi}(t) - a\varphi(t) = -k_p(1+\delta_\mathrm{p})\varphi(t-\tau) - k_d(1+\delta_\mathrm{v})\dot{\varphi}(t-\tau)$$



$$|\delta_\mathrm{p}| \le \varepsilon_\mathrm{p}, \quad |\delta_\mathrm{v}| \le \varepsilon_\mathrm{v}$$

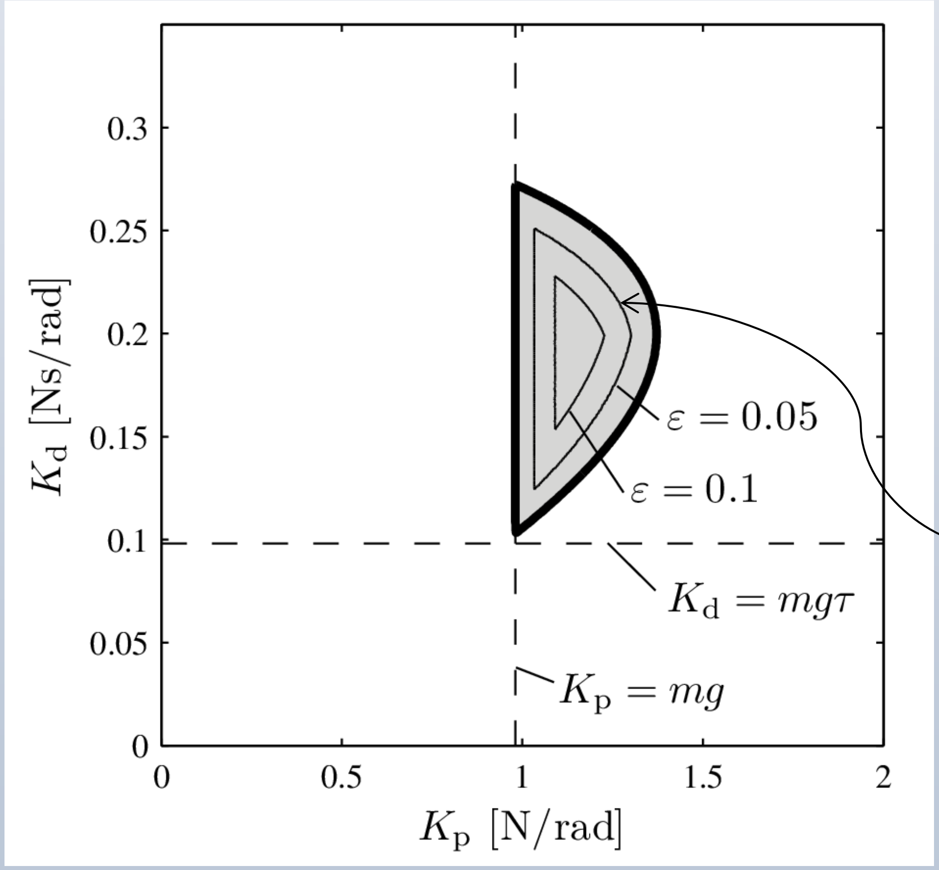
Parameters:
$l = 1\mathrm{m},\ \tau = 100\mathrm{ms}$
$\varepsilon_\mathrm{p} = \varepsilon_\mathrm{v} = \varepsilon$

Boundary of **robust stability** with respect to sensory input uncertainties of $\pm 5\%$

Modelling sensory uncertainties - PD

$$\ddot{\varphi}(t) - a\varphi(t) = -k_p\varphi_s(t-\tau) - k_d\dot{\varphi}_s(t-\tau)$$

$$\ddot{\varphi}(t) - a\varphi(t) = -k_p(1+\delta_\mathrm{p})\varphi(t-\tau) - k_d(1+\delta_\mathrm{v})\dot{\varphi}(t-\tau)$$



$$|\delta_\mathrm{p}| \le \varepsilon_\mathrm{p}, \qquad |\delta_\mathrm{v}| \le \varepsilon_\mathrm{v}$$

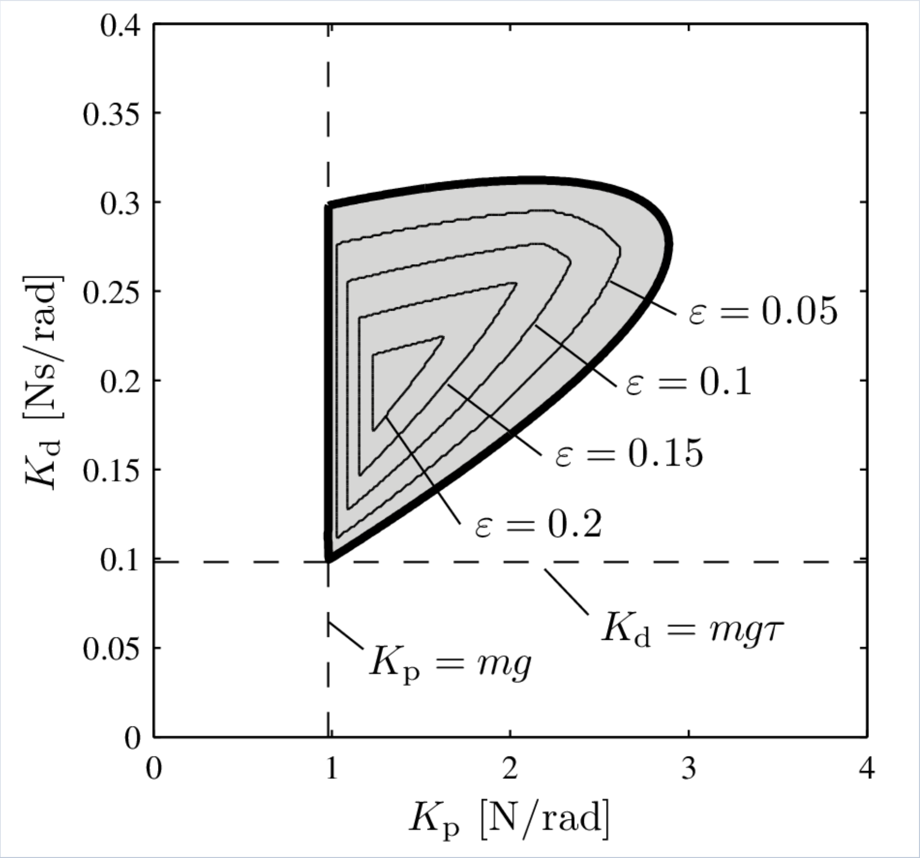
Parameters:
$l = 1\mathrm{m},\ \tau = 100\mathrm{ms}$
$\varepsilon_\mathrm{p} = \varepsilon_\mathrm{v} = \varepsilon$

Boundary of **robust stability** with respect to sensory input uncertainties of $\pm 5\%$

If $\tau = 230\mathrm{ms}$ and $\varepsilon = 0.05$, then $\boxed{l_{\mathrm{crit,PD}} = 306\ \mathrm{cm}}$

Modelling sensory uncertainties - PDA

$$\ddot{\varphi}(t) - a\varphi(t) = -k_p\varphi_s(t-\tau) - k_d\dot{\varphi}_s(t-\tau) - k_a\ddot{\varphi}_s(t-\tau)$$



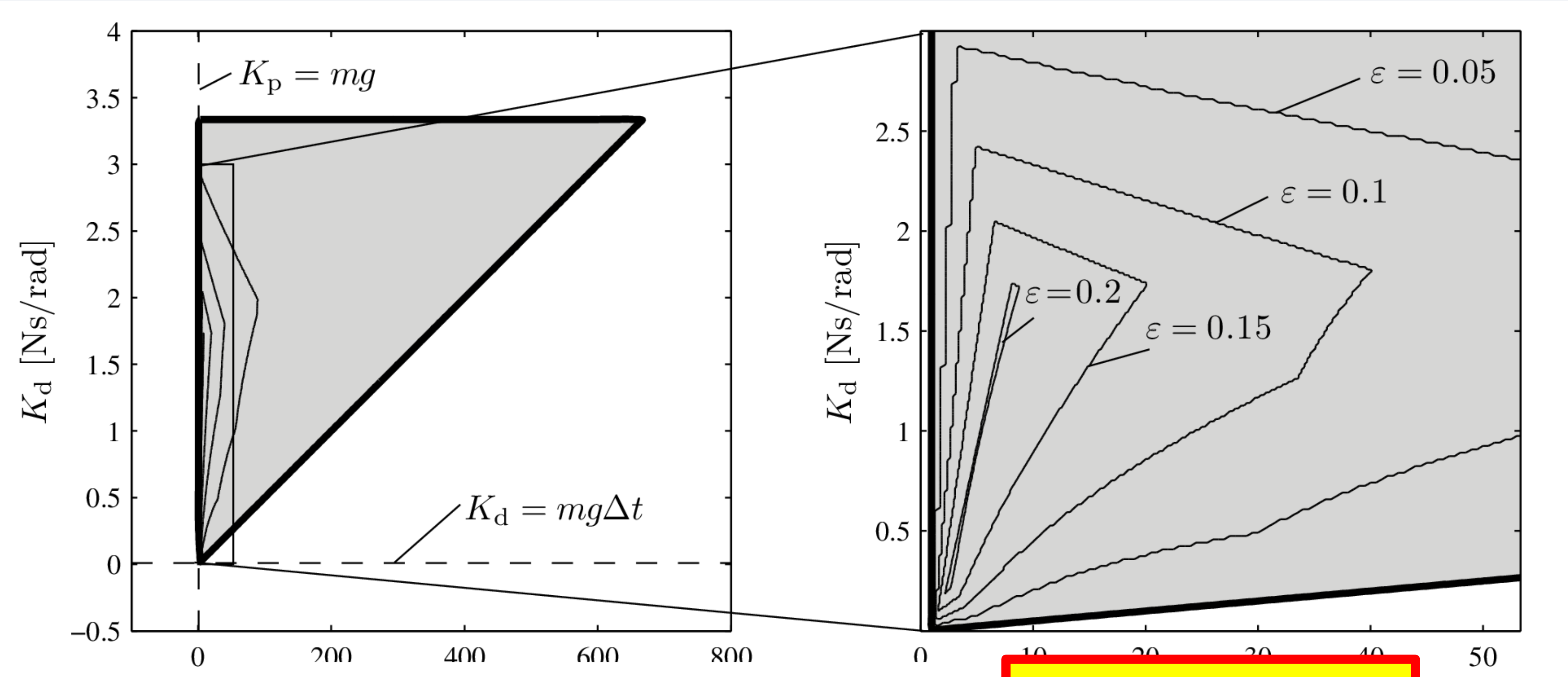
Parameters:
$l = 1\mathrm{m}, \ \tau = 100\mathrm{ms}$
$\varepsilon_\mathrm{p} = \varepsilon_\mathrm{v} = \varepsilon_\mathrm{a} = \varepsilon$

If $\tau = 230\mathrm{ms}$ and $\varepsilon = 0.05$, then $\boxed{l_{\mathrm{crit,PDA}} = 156\ \mathrm{cm}}$

Modelling sensory uncertainties - MP

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\left(\mathbf{K}\mathrm{e}^{\widetilde{\mathbf{A}}\tilde{\tau}}\mathbf{x}_{\mathrm{s}}(t) + \mathbf{K}\int_{t}^{t+\tilde{\tau}} \mathrm{e}^{\widetilde{\mathbf{A}}(t+\tilde{\tau}-\vartheta)}\widetilde{\mathbf{B}}\mathbf{u}_{\mathrm{s}}(\vartheta - \tilde{\tau})\mathrm{d}\vartheta\right)$$

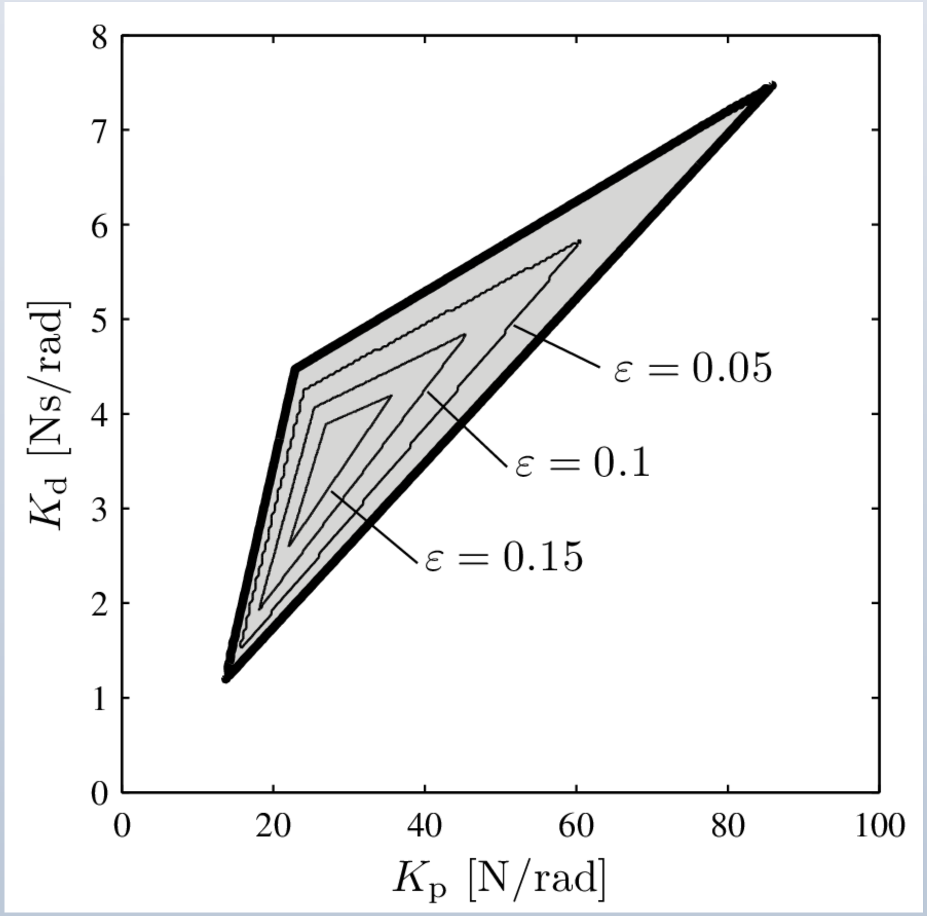
Parameters: $l = 1\mathrm{m},\ \tau = 100\mathrm{ms},\ \varepsilon_{\mathrm{p}} = \varepsilon_{\mathrm{v}} = \varepsilon_{\mathrm{u}} = \varepsilon$



If $\tau = 230\mathrm{ms}$ and $\varepsilon = 0.05$, then $l_{\mathrm{crit,MP}} = 67$ cm

Modelling sensory uncertainties - AAW

$$\ddot{\varphi}(t) - a\varphi(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < t_\mathrm{w} \quad \text{(wait)} \\ -k_\mathrm{p}\varphi(t-\tau) - k_\mathrm{d}\dot{\varphi}(t-\tau) & \text{if} \quad t_\mathrm{w} \le t < t_\mathrm{w} + t_\mathrm{a} = T \text{ (act)} \end{cases}$$



$$|\delta_\mathrm{p}| \le \varepsilon_\mathrm{p}, \qquad |\delta_\mathrm{v}| \le \varepsilon_\mathrm{v}$$

Parameters:
$l = 1\mathrm{m}, \ \tau = 100\mathrm{ms}$
$\varepsilon_\mathrm{p} = \varepsilon_\mathrm{v} = \varepsilon$

If $\tau = 230\mathrm{ms}$ and $\varepsilon = 0.05$, then $l_{\mathrm{crit,AAW}} = 146 \text{ cm}$

Comparison of different control concepts

- proportional-derivative (PD)
- proportional-derivative-acceleration (PDA)
- model predictive (MP) controllers
- act-and-wait (AAW)

$\boxed{l_{\text{crit,PD}} = 156 \text{ cm}}$

$\boxed{l_{\text{crit,PDA}} = 78 \text{ cm}}$

$\boxed{l_{\text{crit,MP}} = 0 \text{ cm}}$

$\boxed{l_{\text{crit,AAW}} = 0 \text{ cm}}$

But, in case of 5% sensory uncertainties:

(Insperger, Milton, Biol Cybern, 2014)

$\boxed{l_{\text{crit,PD}} = 306 \text{ cm}}$

$\boxed{l_{\text{crit,PDA}} = 156 \text{ cm}}$

$\boxed{l_{\text{crit,MP}} = 67 \text{ cm}}$

$\boxed{l_{\text{crit,AAW}} = 146 \text{ cm}}$

Intermittent control concepts

$$\ddot{\varphi}(t) - a\varphi(t) = -Q(t)$$

Clock-driven (time-dependent, parametrically forced)

or

Event-driven (sensory dead zone, state-dependent dead zones)

Intermittent control concepts

$$\ddot{\varphi}(t) - a\varphi(t) = -Q(t)$$

Clock-driven (time-dependent, parametrically forced)

- Intermittent predictive controller

(Gawthrop, Wang, Loram, Gollee, Lakie)

$$Q(t) = f\big(t, \varphi(t_i), \dot{\varphi}(t_i)\big), \qquad t \in [t_i, t_{i+1}]$$

- Act-and-wait controller (Insperger, Stépán)

$$Q(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < t_\mathrm{w} \quad \text{(wait)} \\ k_\mathrm{p}\varphi(t-\tau) + k_\mathrm{d}\dot{\varphi}(t-\tau) & \text{if} \quad t_\mathrm{w} \le t < t_\mathrm{w} + t_\mathrm{a} = T \ \text{(act)} \end{cases}$$

- Semi-discretization (Insperger, Stépán)

sampling + zero-order hold (~ digital effect)

Semi-discretization of delayed systems

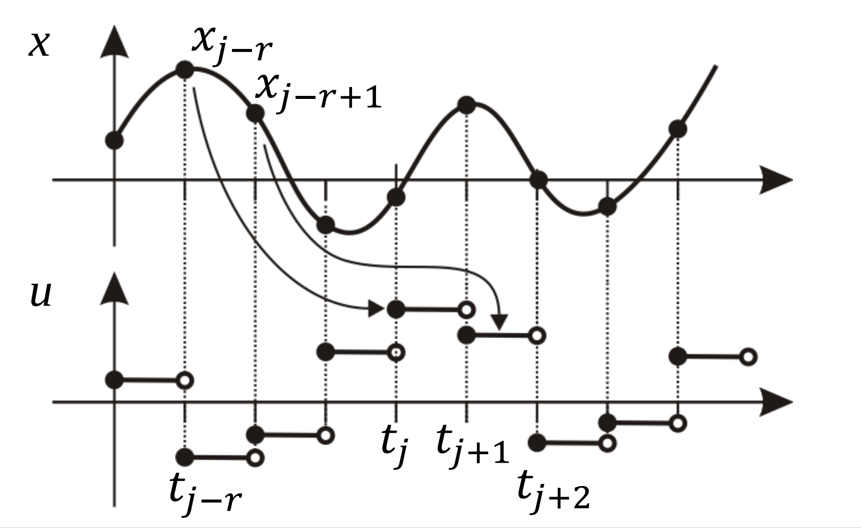
original equation: $(\mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m)$

$$\left.\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{u}(t) &= \mathbf{K}\mathbf{x}(t-\tau)\end{aligned}\right\} \quad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{x}(t-\tau)$$

approximate (semi-discrete) equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t_j), \qquad t \in [t_j, t_{j+1})$$

$$\mathbf{u}(t_j) = \mathbf{K}\mathbf{x}(t_j - r\Delta t), \qquad t_j = j\Delta t$$



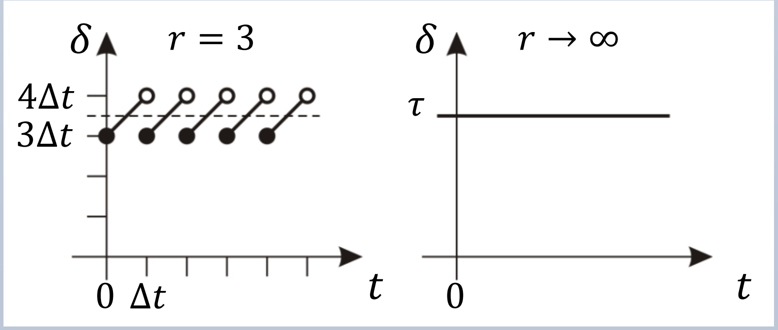
solution over a discretization step:

$$\mathbf{x}(t_{j+1}) = \underbrace{\mathrm{e}^{\mathbf{A}\Delta t}}_{\mathbf{P}}\, \mathbf{x}(t_j) + \underbrace{\int_0^{\Delta t} \mathrm{e}^{\mathbf{A}(\Delta t - s)}\,\mathrm{d}s\, \mathbf{B}}_{\mathbf{R}}\, \mathbf{u}(t_j)$$

finite dimensional discrete map:

$$\begin{pmatrix} \mathbf{x}(t_{j+1}) \\ \mathbf{u}(t_{j+r}) \\ \mathbf{u}(t_{j+r-1}) \\ \vdots \\ \mathbf{u}(t_{j+2}) \\ \mathbf{u}(t_{j+1}) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{P} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R} \\ \mathbf{K} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{I} & \mathbf{0} \end{pmatrix}}_{\boldsymbol{\Phi} \in \mathbb{R}^{(n+rm)\times(n+rm)}} \begin{pmatrix} \mathbf{x}(t_j) \\ \mathbf{u}(t_{j+r-1}) \\ \mathbf{u}(t_{j+r-2}) \\ \vdots \\ \mathbf{u}(t_{j+1}) \\ \mathbf{u}(t_j) \end{pmatrix}$$

~ time-varying delay $\mathbf{K}\mathbf{x}(t-\delta(t))$



(Insperger, Stépán, 2002)

Intermittent control concepts

$$\ddot{\varphi}(t) - a\varphi(t) = -Q(t)$$

Event-driven (state-dependent, **nonlinear**)

- Sensory dead zone, discontinuous feedback (Eurich, Milton, Ohira)

$$Q(t) = \begin{cases} C & \text{if} \quad \varphi(t-\tau) \geq \varphi_{\mathrm{st}} \\ 0 & \text{if} \quad |\varphi(t-\tau)| < \varphi_{\mathrm{st}} \\ -C & \text{if} \quad \varphi(t-\tau) \leq -\varphi_{\mathrm{st}} \end{cases}$$

- Different sensory thresholds (Insperger, Milton, Stépán)

$$Q(t) = Q_{\mathrm{p}}(t) + Q_{\mathrm{d}}(t)$$

$$Q_{\mathrm{p}}(t) = \begin{cases} k_{\mathrm{p}}\varphi(t-\tau) & \text{if} \quad |\varphi(t-\tau)| \geq \varphi_{\mathrm{st}} \\ 0 & \text{if} \quad |\varphi(t-\tau)| < \varphi_{\mathrm{st}} \end{cases}$$

$$Q_{\mathrm{d}}(t) = \begin{cases} k_{\mathrm{d}}\dot{\varphi}(t-\tau) & \text{if} \quad |\dot{\varphi}(t-\tau)| \geq \omega_{\mathrm{st}} \\ 0 & \text{if} \quad |\dot{\varphi}(t-\tau)| < \omega_{\mathrm{st}} \end{cases}$$

Intermittent control concepts

$$\ddot{\varphi}(t) - a\varphi(t) = -Q(t)$$

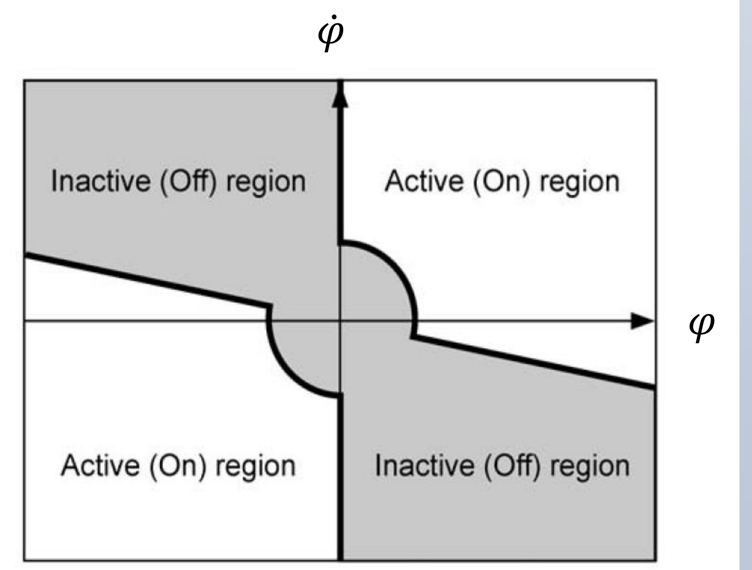
Event-driven (state-dependent, **nonlinear**)

- State-dependent threshold (Asai, Nomura, Suzuki, Casadio, Morasso, Bottaro)

$$Q(t) = \begin{cases} k_{\mathrm{p}}\varphi_\Delta + k_{\mathrm{d}}\dot{\varphi}_\Delta & \text{if} \quad \varphi_\Delta(\dot{\varphi}_\Delta - \alpha\varphi_\Delta) \geq 0 \text{ and } {\varphi_\Delta}^2 + {\dot{\varphi}_\Delta}^2 > r^2 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

$$\varphi_\Delta = \varphi(t - \tau)$$

$$\dot{\varphi}_\Delta = \dot{\varphi}(t - \tau)$$

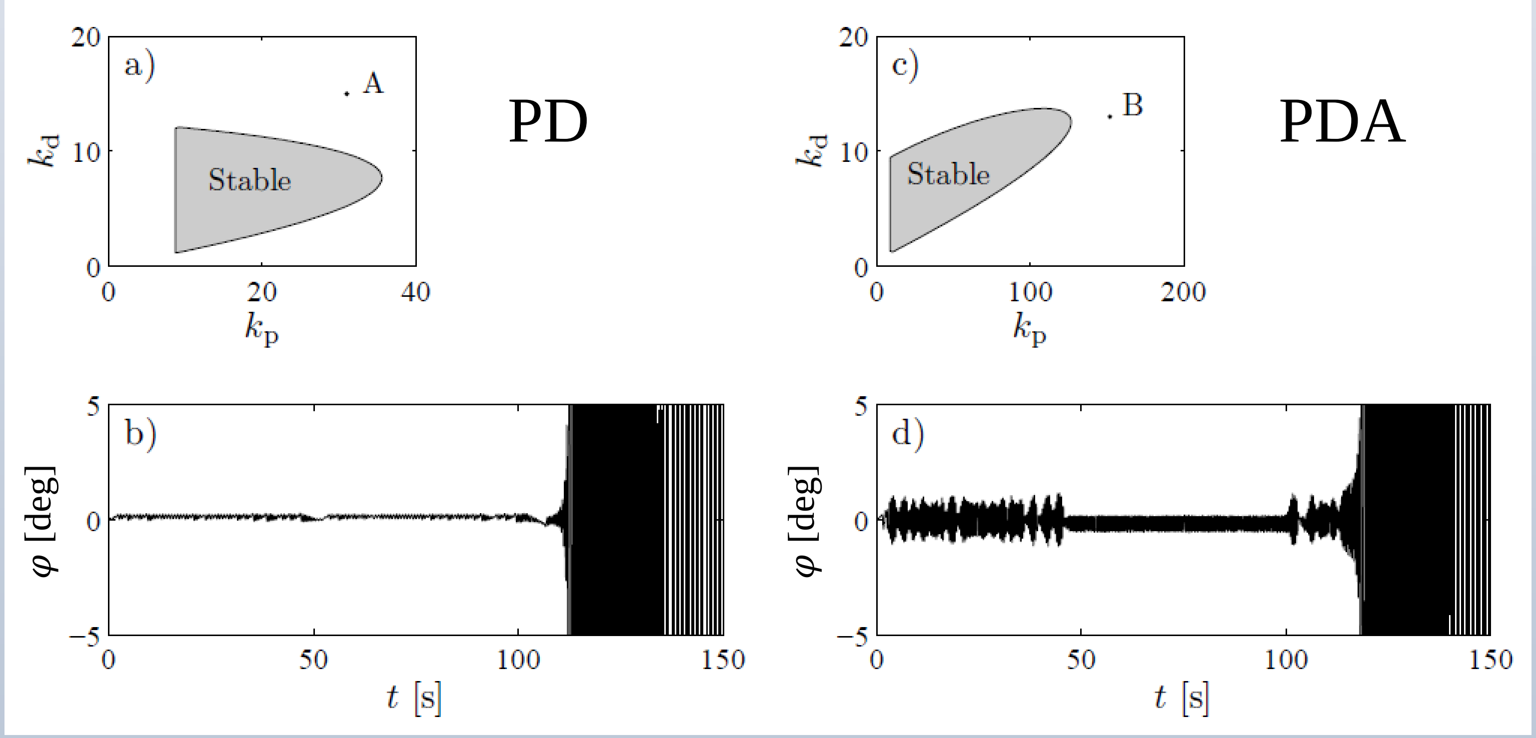


(Asai et al. PLOS ONE, 2009)

The effect of sensory dead zones

Modelling sensory dead zones: $\ddot{\varphi}(t) - a\,\varphi(t) = \begin{cases} 0 & \text{if} \quad |\varphi(t-\tau)| < \varphi_{st} \\ -\dfrac{6}{ml} Q(t) & \text{if} \quad |\varphi(t-\tau)| \geq \varphi_{st} \end{cases}$

Time domain simulations ~ transient chaos?



Bounded motions (chaos) rather then stability?

The effect of sensory dead zones

Modelling sensory dead zones: $\ddot{\varphi}(t) - a\,\varphi(t) = \begin{cases} 0 & \text{if} \quad |\varphi(t-\tau)| < \varphi_{\mathrm{st}} \\ -\dfrac{6}{ml} Q(t) & \text{if} \quad |\varphi(t-\tau)| \geq \varphi_{\mathrm{st}} \end{cases}$

Simplifying the model (Eurich, Milton, 1996):

$$\dot{x}(t) = \begin{cases} x(t) + C & \text{if} \quad x(t-\tau) < -1 \\ x(t) & \text{if} \quad -1 \leq x(t-\tau) \leq 1 \\ x(t) - C & \text{if} \quad x(t-\tau) > 1 \end{cases}$$

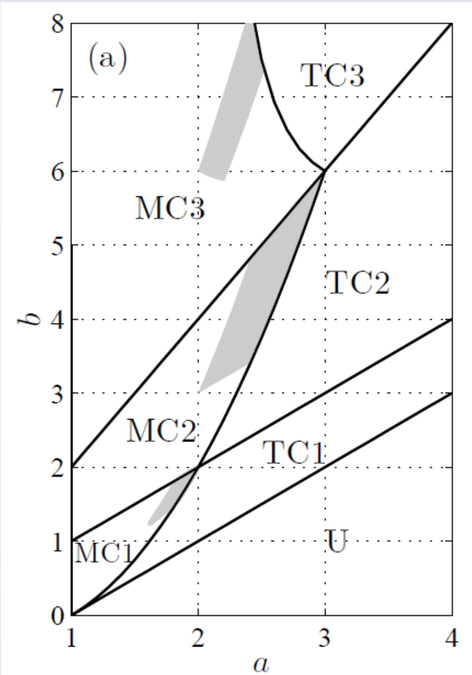
(1) an unstable upright position in the absence of feedback
(2) stabilizing time-delayed feedback
(3) a sensory dead zone

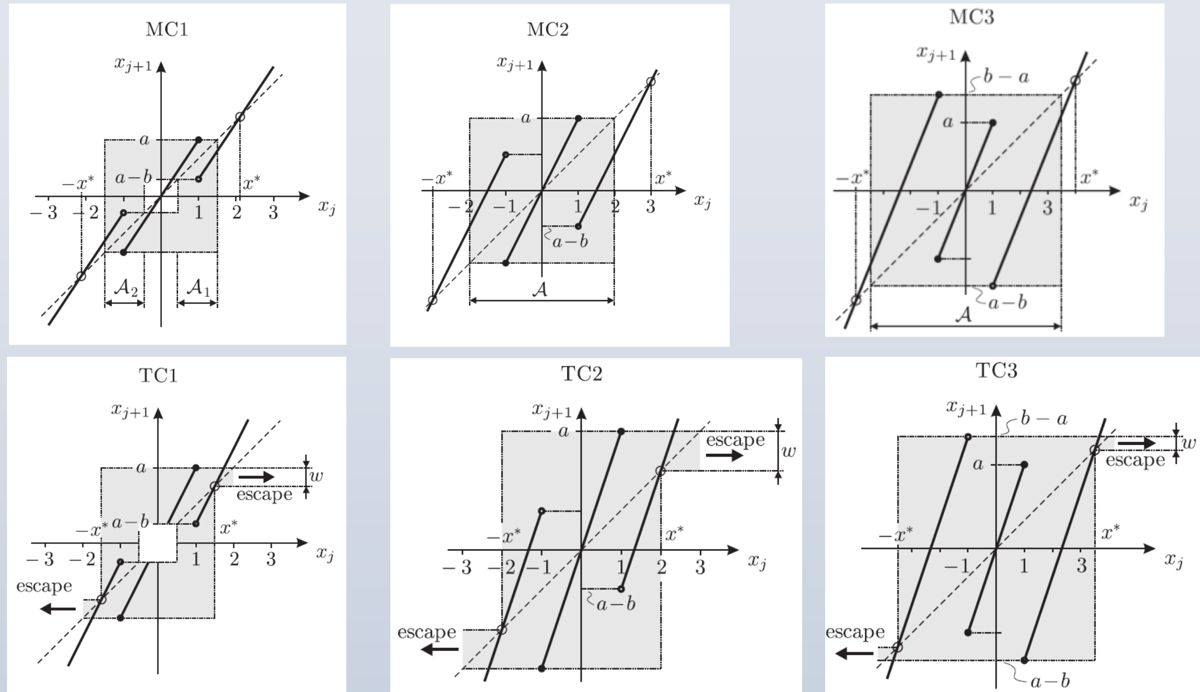
Even more simplified model (discrete-time model):

$$x(t_{j+1}) = \begin{cases} a\,x(t_j) + b & \text{if} \quad x(t_j) < -1 \\ a\,x(t_j) & \text{if} \quad -1 \leq x(t_j) \leq 1 \\ a\,x(t_j) - b & \text{if} \quad x(t_j) > 1 \end{cases}$$

(Insperger, Milton, Stépán, SIAM ADS, 2015)

The effect of sensory dead zones





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(Insperger, Milton, Stépán, SIAM ADS, 2015)

- (Compact invariant set)
- Sensitivity to initial conditions
- Topological transitivity (mixing)

$$\Downarrow$$

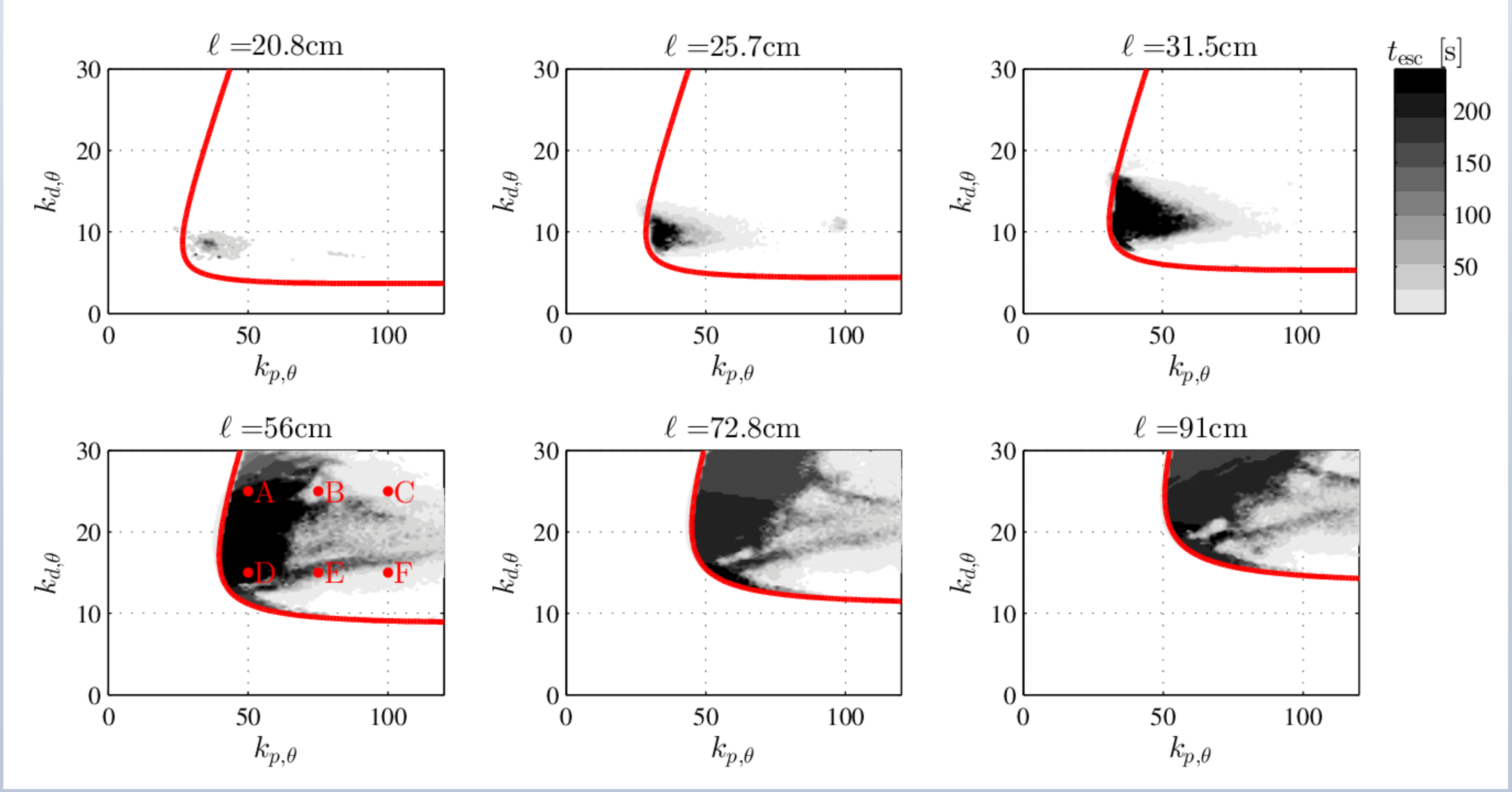
Both permanent and transient chaos are possible!!!

The effect of sensory dead zones

MP control – escape time diagram

Red: linear stability boundary (no dead zone)

Gray shading: escape time ($|\varphi| > 20$deg) in case of dead zone (2.86deg)



The effect of sensory dead zones

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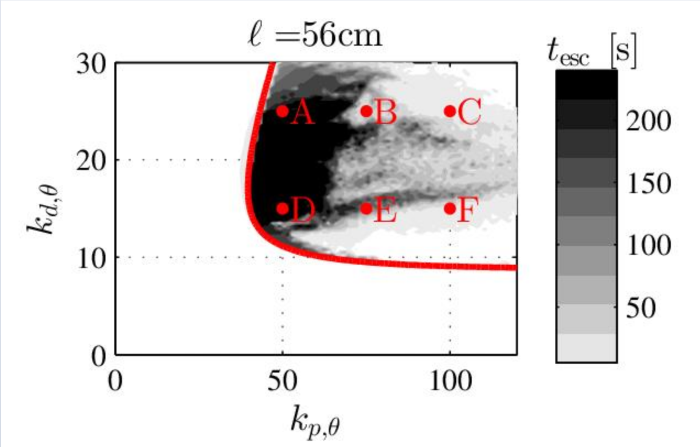

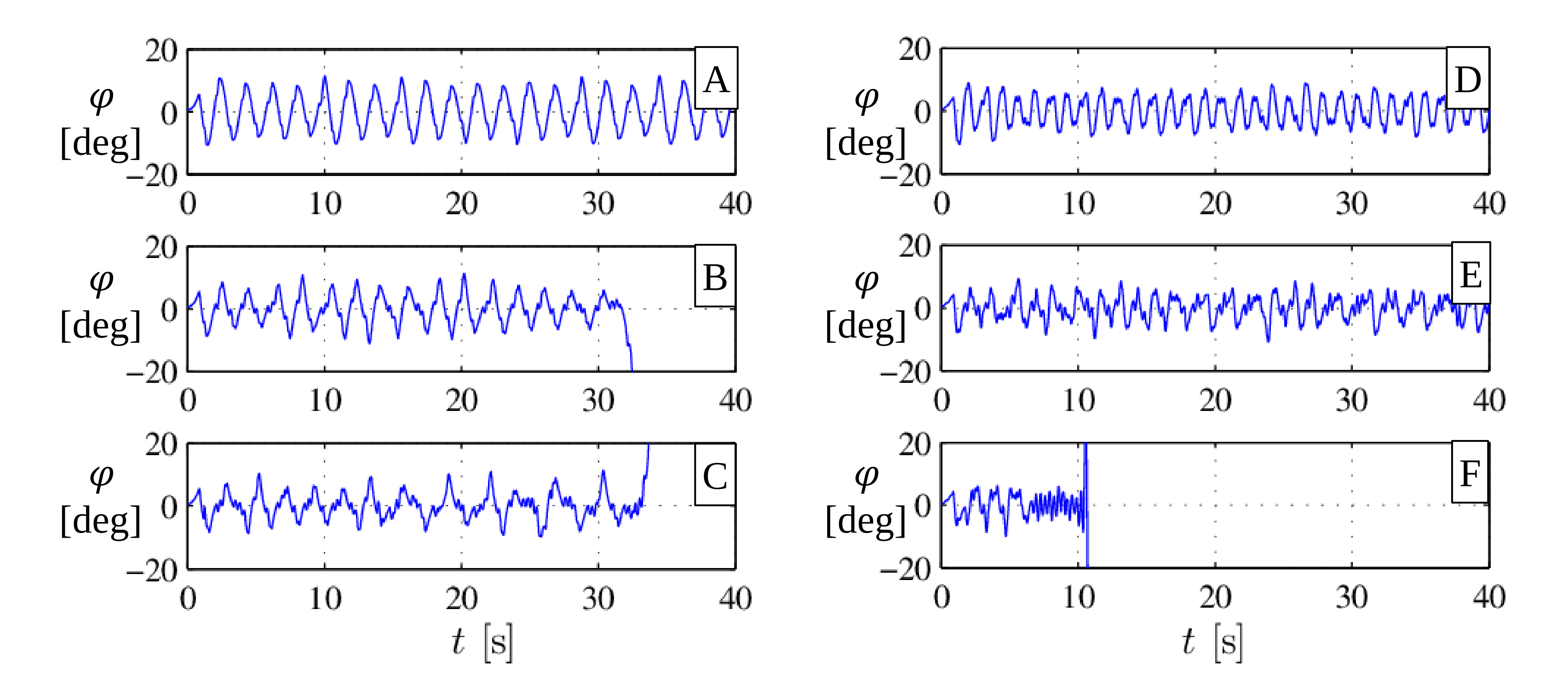

The effect of sensory dead zones

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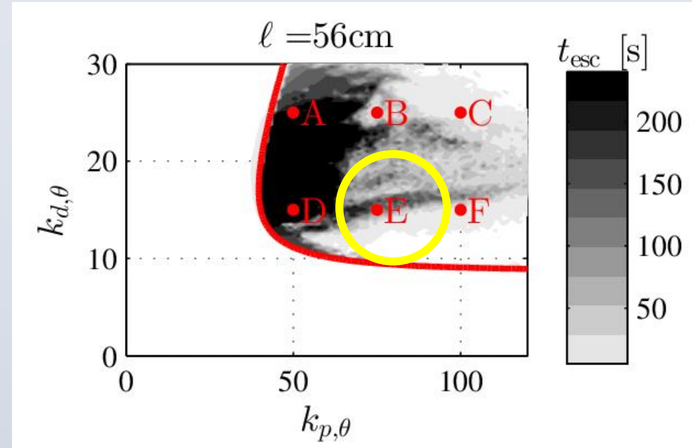
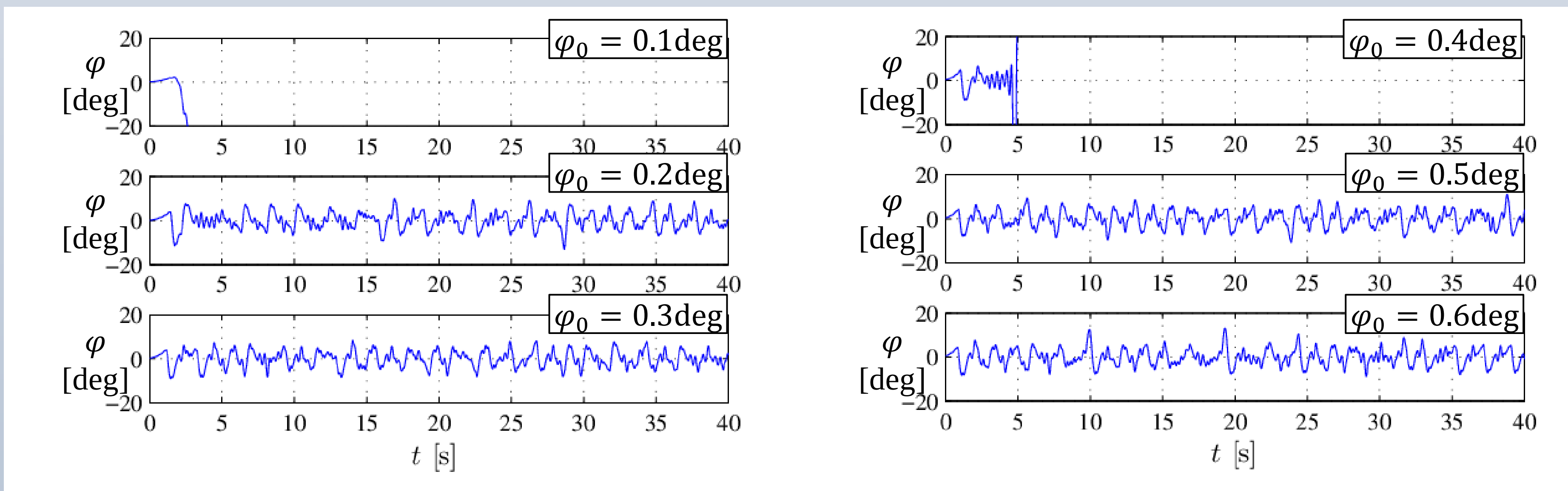


Different initial conditions at point E



Conclusions

- Feedback delay presents a strong limitation for human balancing abilities (among other factors such as uncertainties, dead zones, quantization…).
- $l_{\text{crit,PDA}} = \frac{1}{2} l_{\text{crit,PD}}$.
- For the MP and AAW controllers $l_{\text{crit}} = 0$, but they are sensitive to uncertainties.
- Transiently bounded motion instead of stability.

- Still don't know what control concept do we use during stick balancing. (Vote for MP or some nonlinear controller.)
- Whatever we do during stick balancing, we are doing it in a pretty good way!

Thank you!