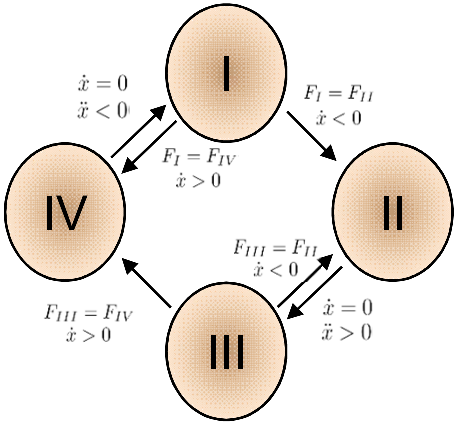
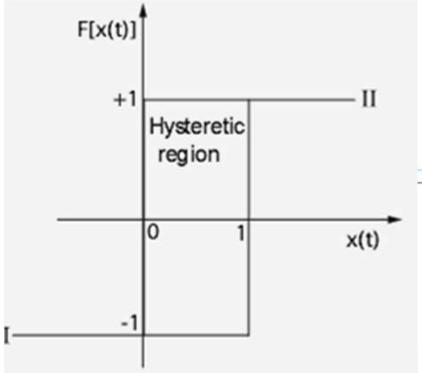
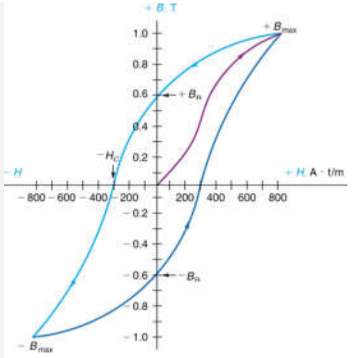
Devilish Eigenvalues: Hysteretic Systems and Mechanistic Turbulence

Tamás Kalmár-Nagy



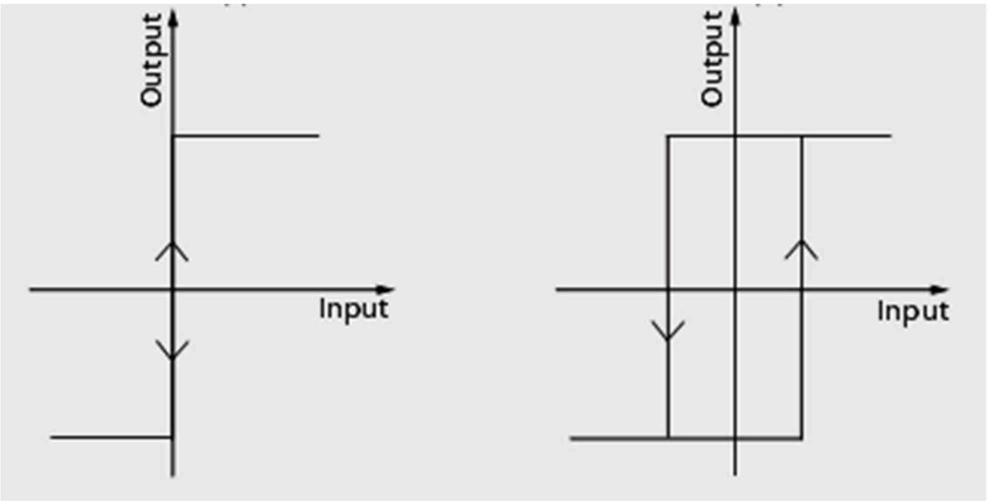


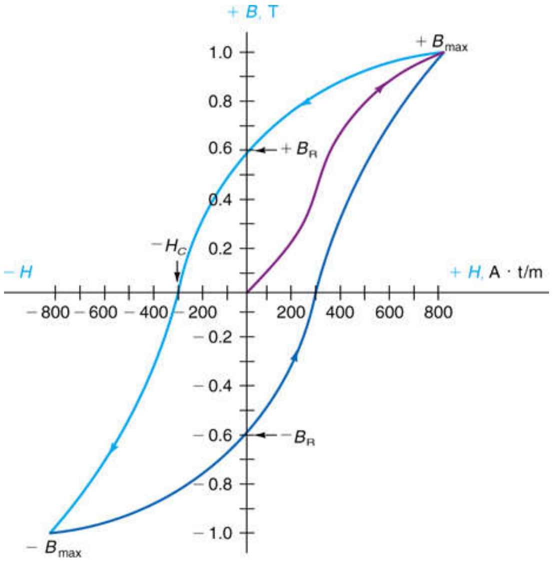


OUTLINE

- Hysteresis
- Measuring rate independence
- Preisach model
- Transition graph, eigenvalue distribution
- Mechanistic turbulence

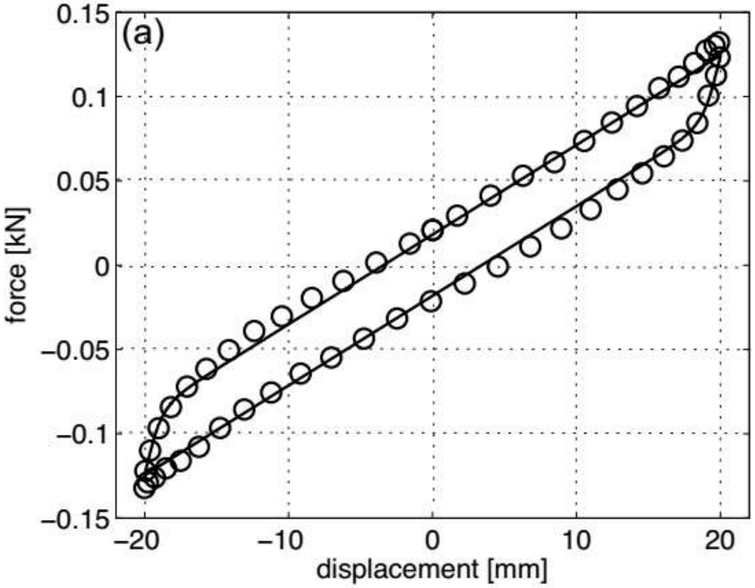
HYSTERESIS





-magnetic hysteresis
-material hysteresis

Class of piecewise smooth dynamical systems:
relay, backlash, friction, impacts



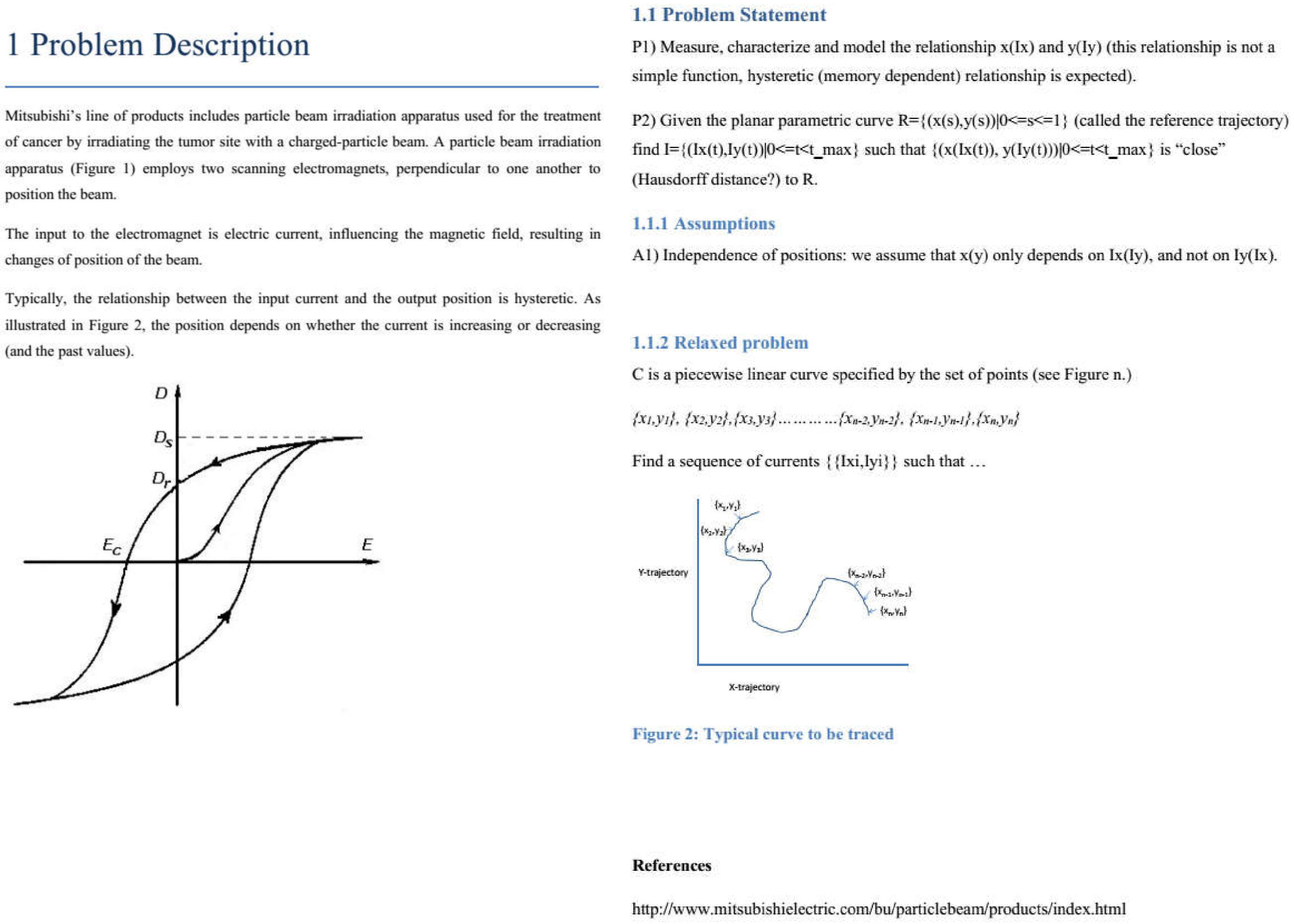
WHY DO WE CARE?

Engineering problems:

- Voltage regulators
- DC motors
- Servomechanisms
- Hysteretic materials
- Structures

Fame and fortune for solving cool problems

ENGINEERING PROBLEM STATEMENT



1 Problem Description

Mitsubishi's line of products includes particle beam irradiation apparatus used for the treatment of cancer by irradiating the tumor site with a charged-particle beam. A particle beam irradiation apparatus (Figure 1) employs two scanning electromagnets, perpendicular to one another to position the beam.

The input to the electromagnet is electric current, influencing the magnetic field, resulting in changes of position of the beam.

Typically, the relationship between the input current and the output position is hysteretic. As illustrated in Figure 2, the position depends on whether the current is increasing or decreasing (and the past values).

1.1 Problem Statement

P1) Measure, characterize and model the relationship x(Ix) and y(Iy) (this relationship is not a simple function, hysteretic (memory dependent) relationship is expected).

P2) Given the planar parametric curve R={(x(s),y(s))|0<=s<=1} (called the reference trajectory) find I={(Ix(t),Iy(t))|0<=t<t_max} such that {(x(Ix(t)), y(Iy(t)))|0<=t<t_max} is "close" (Hausdorff distance?) to R.

1.1.1 Assumptions

A1) Independence of positions: we assume that x(y) only depends on Ix(Iy), and not on Iy(Ix).

1.1.2 Relaxed problem

C is a piecewise linear curve specified by the set of points (see Figure n.)

$\{x_1,y_1\}, \{x_2,y_2\}, \{x_3,y_3\}\ldots\ldots\ldots\{x_{n-2},y_{n-2}\}, \{x_{n-1},y_{n-1}\}, \{x_n,y_n\}$

Find a sequence of currents {{Ixi,Iyi}} such that …

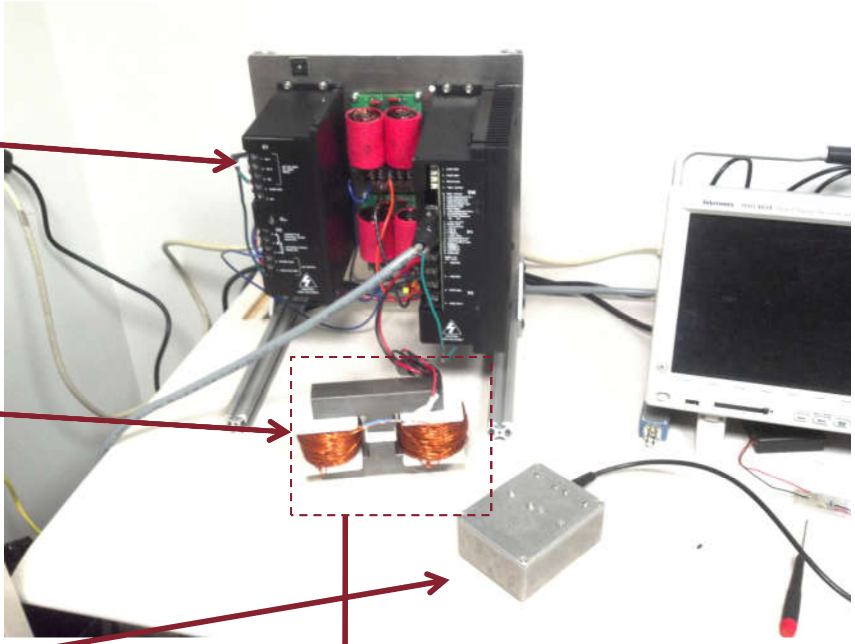
Figure 2: Typical curve to be traced

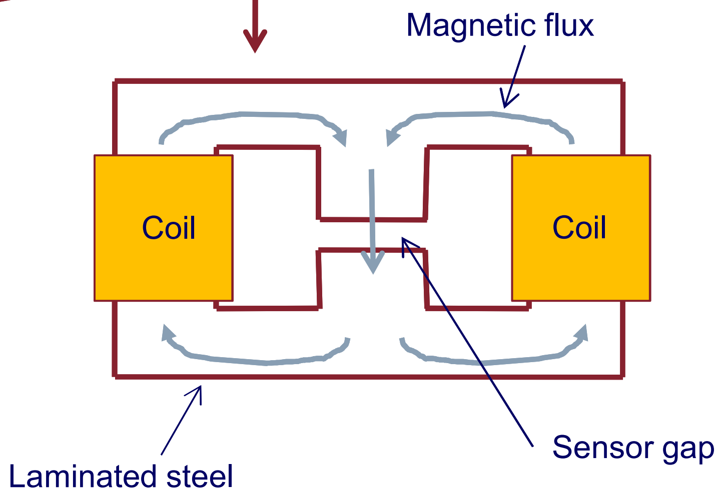
References

http://www.mitsubishielectric.com/bu/particlebeam/products/index.html

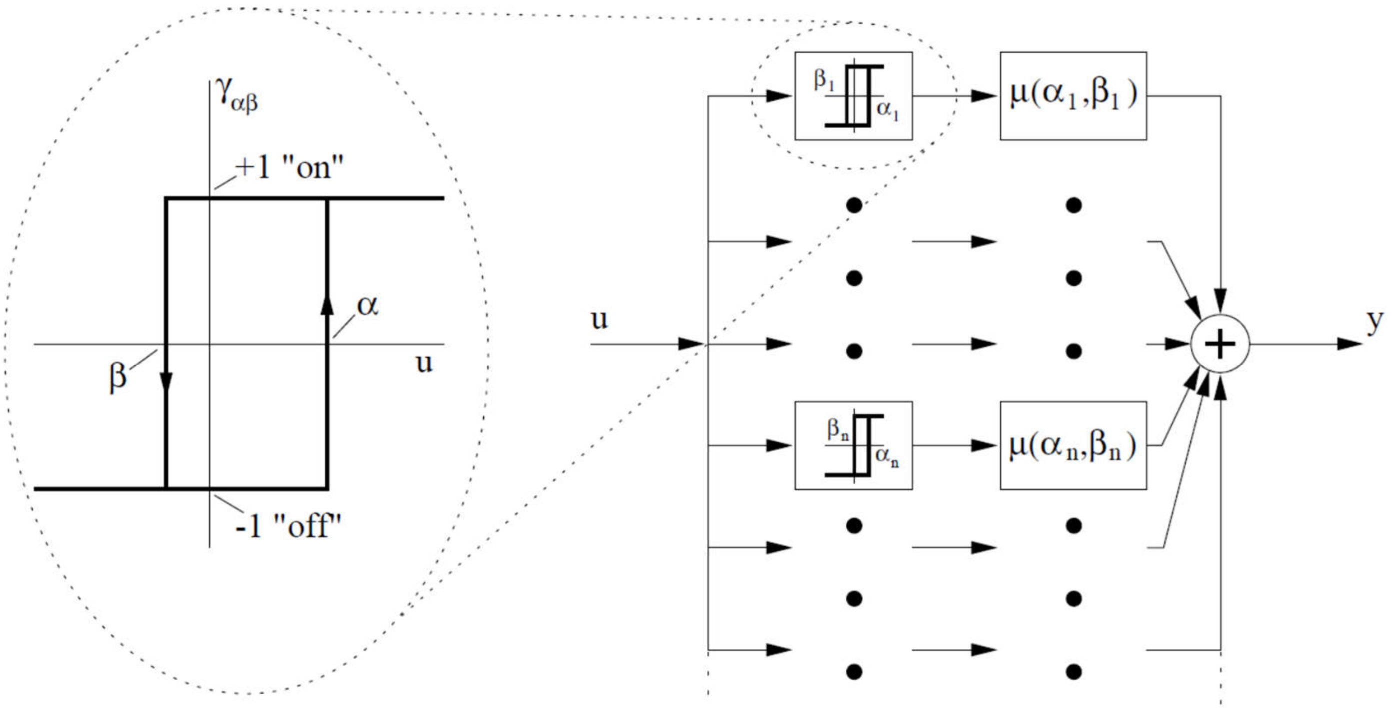
EXPERIMENTAL SYSTEM

- Current supply
  - Brushed DC motor driver
  - +/- 50A @ 170 V
  - PWM @ 14.5 kHz (inductive load filters out PWM signal)
- Magnet assembly
  - E-core magnet
  - Laminated M15 electrical steel
  - Coils are ~250 turns each of 16AWG magnet wire
  - R = 1.2 ohms and L = 30 mH
- B-field sensor
  - Hoeben hall effect sensor
  - Signal condition board with low pass filter and amplifier
- Data acquisition
  - National Instrument USB-6341
  - 16 analog input, 2 analog output
  - 16-bit resolution, 500 kS/S





PREISACH MODEL



Basic hysteron with thresholds α and β

Weighted superposition of hysterons

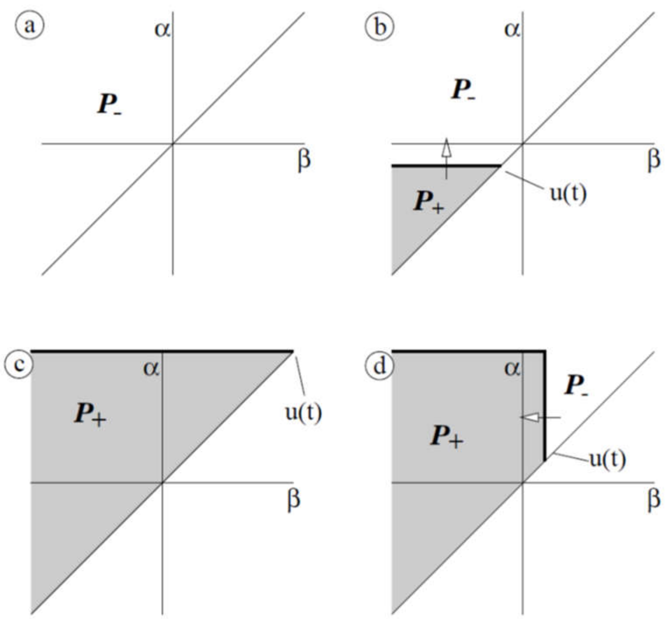
- Model of hysteresis as a weighted superposition of basic hysteretic elements called hysterons

$$y = \iint_{\alpha\geq\beta} \mu(\alpha,\beta)\gamma(\alpha,\beta,x(t))d\alpha d\beta$$

- Hysterons switch to +1 or “on” state when the input increase above a threshold a, and switch to -1 or “off” state when the input decrease below α lower threshold β.

PREISACH PLANE

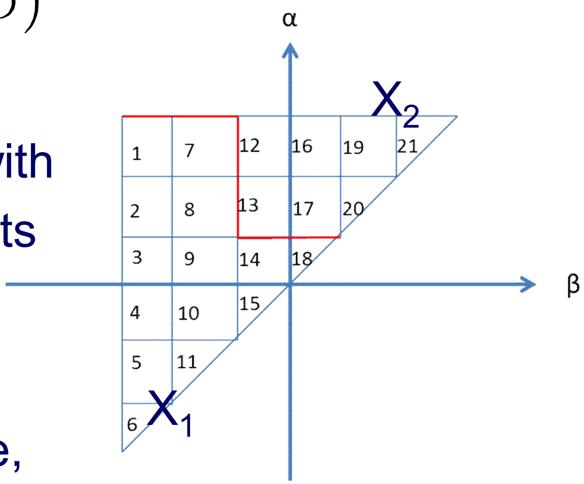
- Hysteron can be illustrated on a abstract plane called the Preisach plane with axes α and β
- By the definition of the hysteron, $\alpha \geq \beta$, so the Preisach plane is really a half-plane
- Hysterons can be in only two states, +1 or -1
- Curves, called memory curves, divide the Preisach plane into regions P_+ and P_- where all hysterons are in the +1 state or -1 state, respectively
- When the input current to the magnet is at its most negative value all hysterons are in the -1 state.
- Raising the current to value I_1causes all hysterons having $\alpha <= I_1$ to switch to the +1 state
- Reducing the current to value I_2 causes all hysterons having $\beta >= I_2$ to switch into the -1 state.



(a) is the initial state of the model when all hysterons are in the -1 state. (b) and (c) show memory curves for increasing current and (d) shows a memory curve when the current is decreasing from its maximum

SYSTEM IDENTIFICATION

- Need to determine the Preisach density function $\mu(\alpha, \beta)$ from experimental data.
- We discretize the Preisach plane into N x N hysterons with each hysteron having a density, or weight, x_i. The weights can be put into a weight vector X. The goal is to find an approximation X' of X.
- Starting from a known state, e.g. all hysterons in -1 state, any sequence of currents I_1, I_2, … I_n results in a memory curve. The output y is given by



$$y(t) \approx \sum_{m \in X_1} x_m + \sum_{m \in X_2} (-1) x_m + c + \varepsilon$$

Where c is contribution from the hysteron outside of the region of our inputs, and ε is noise.

- Define a row vector $A_1 = [d_1, d_2, \ldots d_{n-1}]$ where $d_i = +1$ if x_i is in X_1 and $d_i = -1$ if x_i is in X_2. Then the output can be written as $$y(t_1) = A_1 X' + c + \varepsilon$$ $$Y = AX' + \varepsilon$$
- Many measurements can be combined into matrix equation

SYSTEM IDENTIFICATION (2)

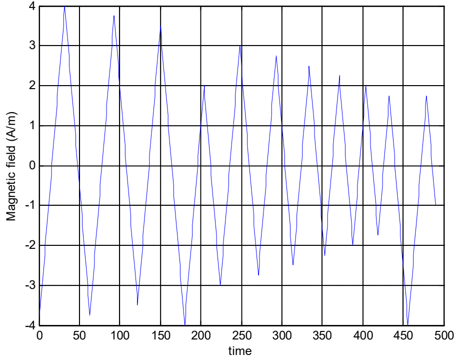
- We can determine X by minimizing the function $f(X)=\frac{1}{2}\left\|AX-Y\right\|^2$

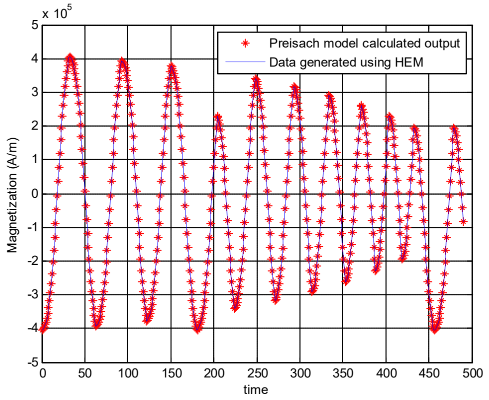
subject to the constraint g(X) = X $\geq$ 0.

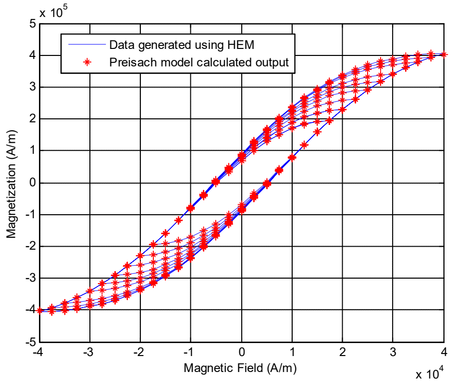
- We have fast solvers for this constrained quadratic optimization problem.
- Each row of A has length $N^2/2$. Ideally A is full rank ($N^2/2$ columns) or greater making the solution straightforward.
- For insufficient data, or excessive noise, the solution can be found using singular value decomposition.

RESULTS OF SYSTEM ID

- To generate artificial data for algorithm development, we used the Homogenized Energy Model (HEM).
- Preisach is a phenomenological model. HEM is an extension of Preisach that includes the microscopic physics of the system.
- Using generated data our algorithm gives excellent results





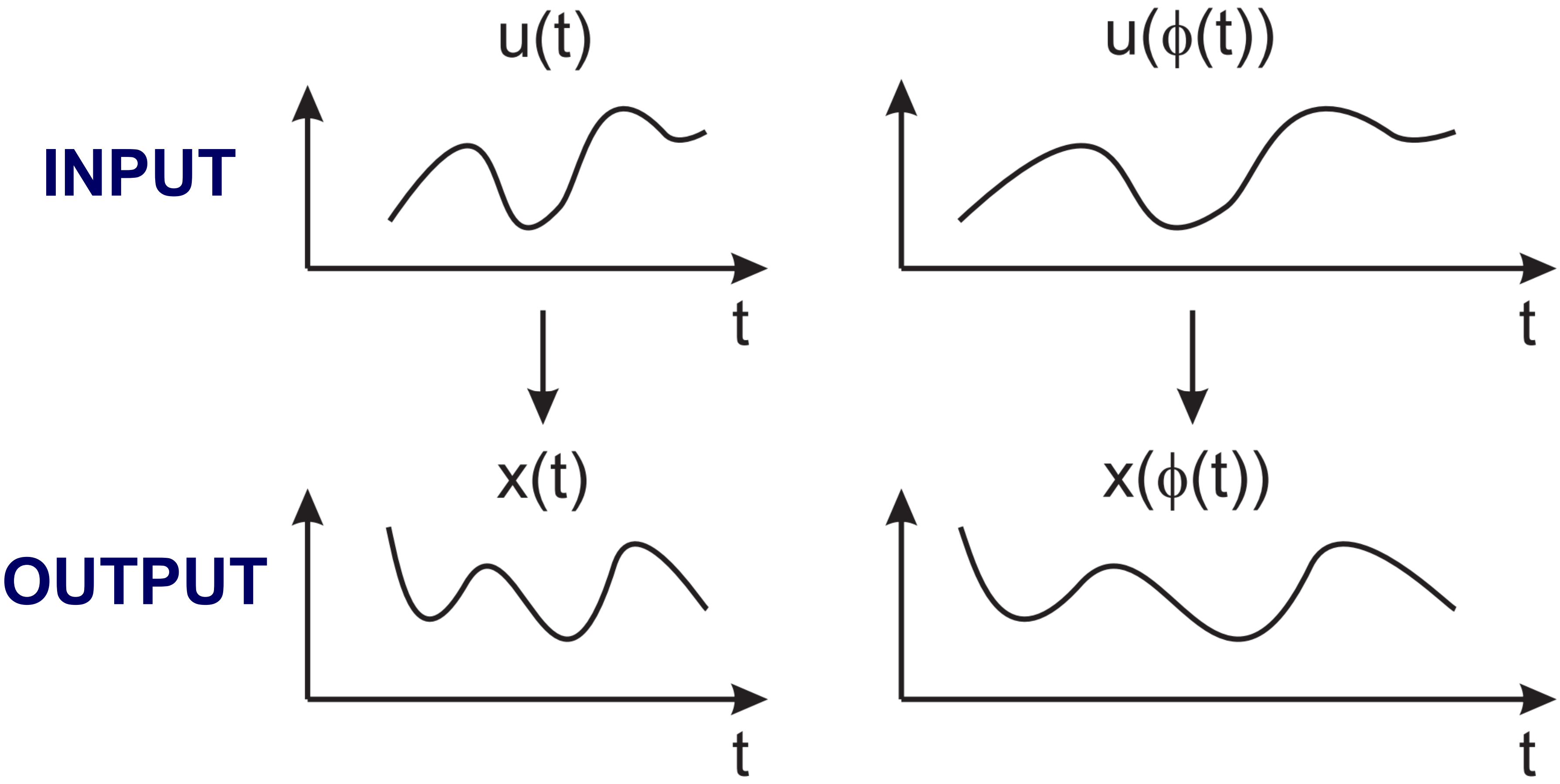


MEASURING RATE INDEPENDENCE
GOAL OF RESEARCH

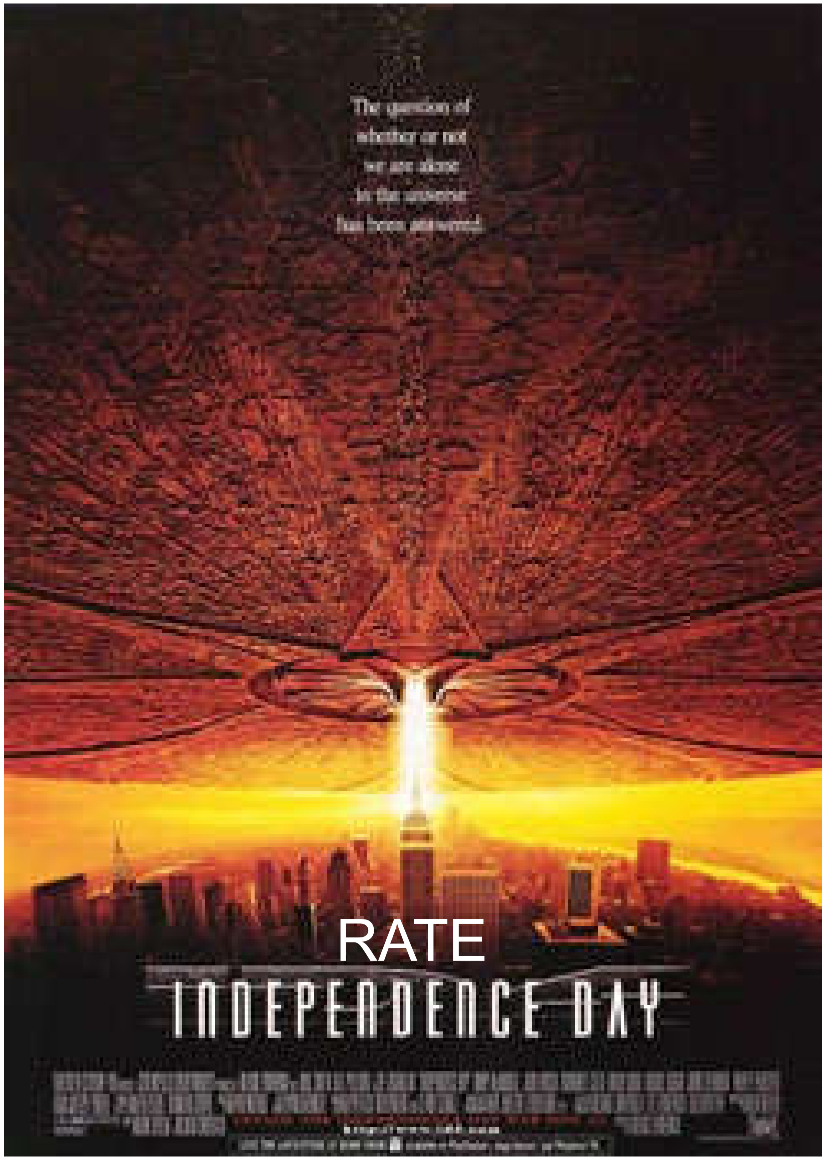
- The goal is to provide a practical metric ε to assess the rate-dependence of material hysteresis

- $\varepsilon \ll 1$ "almost" rate-independence useful for perturbation analysis

RATE INDEPENDENCE



10/17 Rate Independence Day



HYSTERESIS OPERATOR

A hysteresis operator maps input functions *u*(*t*) into output functions *x*(*t*)

$$x(t) = \mathcal{H}[u](t)$$

The map is rate independent if

$$\mathcal{H}[u(\varphi(t))] = x(\varphi(t))$$

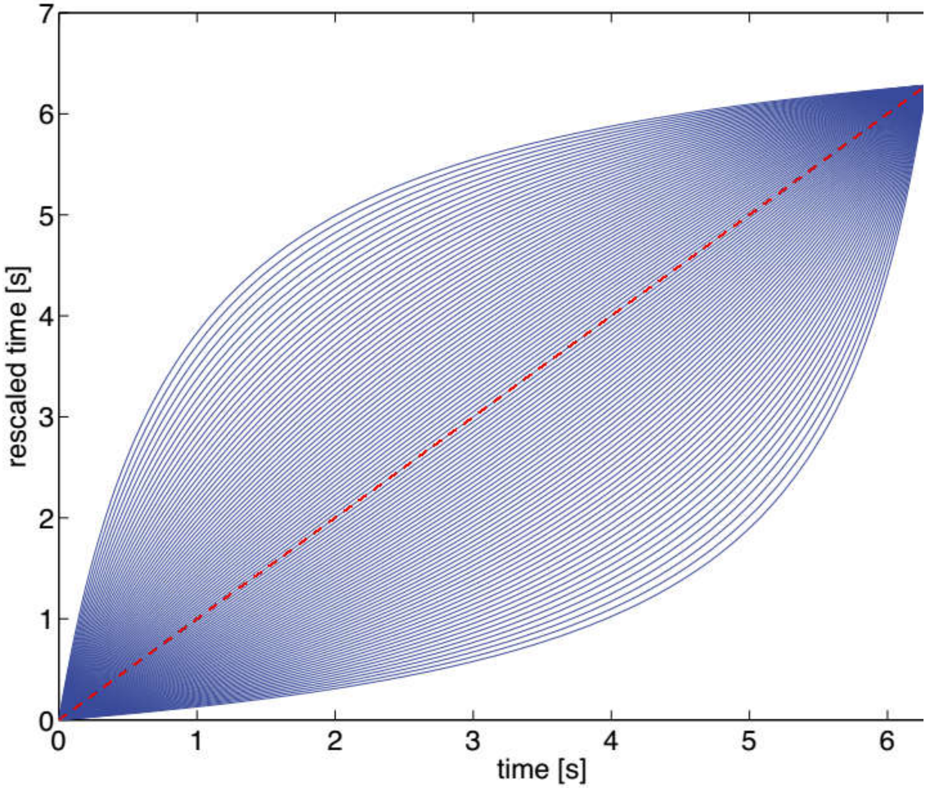
for all continuously increasing functions $\varphi : [0, T] \to [0, T]$
(homeomorphisms)

Our measure of rate dependence

$$\varepsilon = \left\| \mathcal{H}[u(\varphi(t))] - x(\varphi(t)) \right\|$$

SIMPLE HOMEOMORPHISMS

$$t_{new} = t_{i+1} - \frac{t_{i+1} - t_i}{1 + \frac{1-a}{a}\frac{t-t_i}{t_{i+1}-t}}, \qquad a \in [0, 1]$$



Simple homeomorphisms for $a \in [0.1, 0.9]$

BENCHMARK PROBLEM – FIRST ORDER ODE

$$\dot{x}(t) = f(x(t), u(t))$$

solution $\alpha_t[u(t); x_0]$ with $u(t)$

after time transformation

$$\dot{x}(t) = f(x(t), u(\phi(t)))$$

solution $\beta_t[u(\phi(t)); x_0]$

rate independence

$$\beta_t[u(\phi(t)); x_0] = \alpha_{\phi(t)}[u(t); x_0]$$

for all continuously increasing functions $\varphi : [0, T] \to [0, T]$

$$\varepsilon = \left\| \beta_t[u(\phi(t)); x_0] - \alpha_{\phi(t)}[u(t); x_0] \right\|$$

LINEARLY VISCOUS DAMPER

A linearly viscous damper subject to a ramp input

$$\dot{x} = t$$

$$\alpha_t = x_0 + t^2/2$$

time-transformed equation

$$\dot{x} = \phi(t)$$

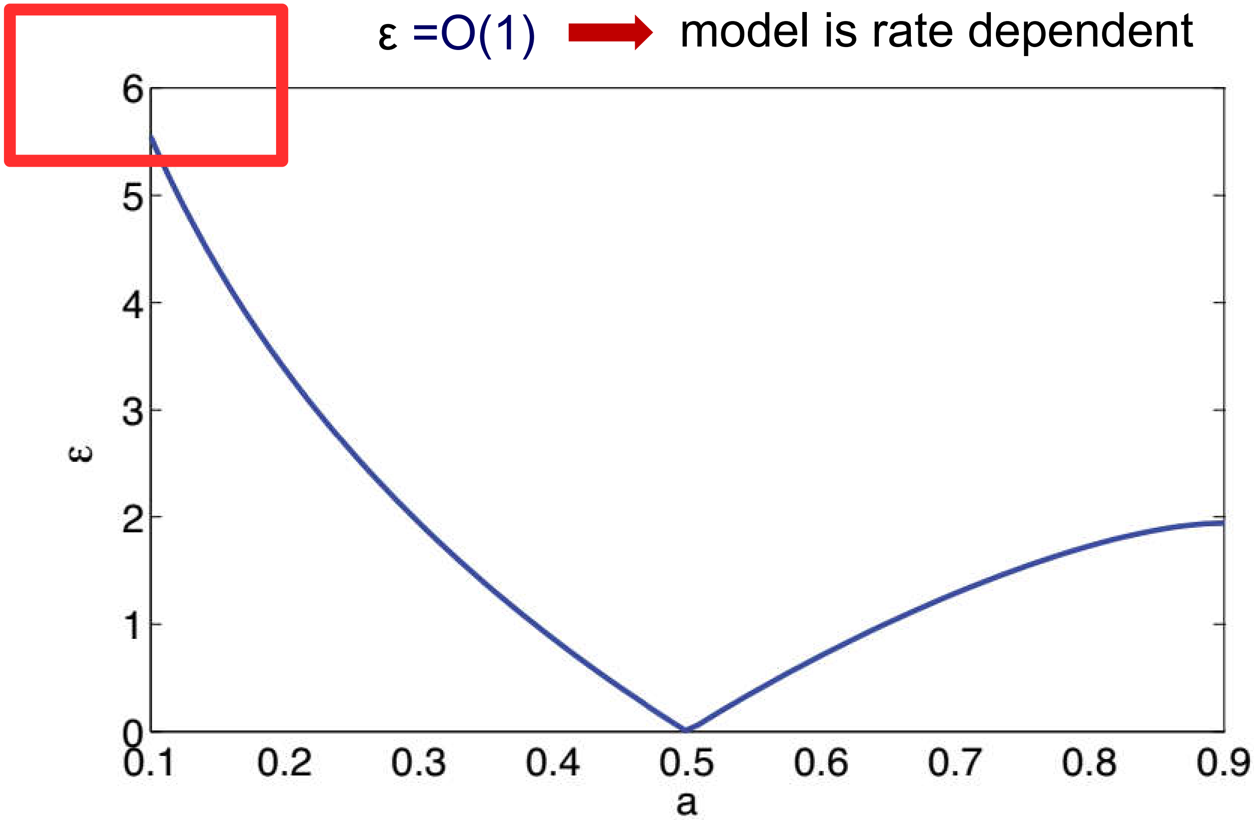
$$\beta_t = x_0 + \int_0^t \phi(\tau)\, d\tau$$

$$\alpha_{\phi(t)} = x_0 + \tfrac{1}{2}\phi(t)^2$$

LINEARLY VISCOUS DAMPER

$$\varepsilon = \left\| x_0 + \int_0^t \phi(\tau)\,d\tau - \left(x_0 + \tfrac{1}{2}\phi(t)^2\right) \right\| = \left\| \int_0^t \phi(\tau)\,d\tau - \tfrac{1}{2}\phi(t)^2 \right\|$$

ε =O(1) → model is rate dependent



Rate-dependence measure ε for the purely viscous damper as a function of *a*.

THE EXTENDED BOUC-WEN MODEL OF PINCHED HYSTERESIS

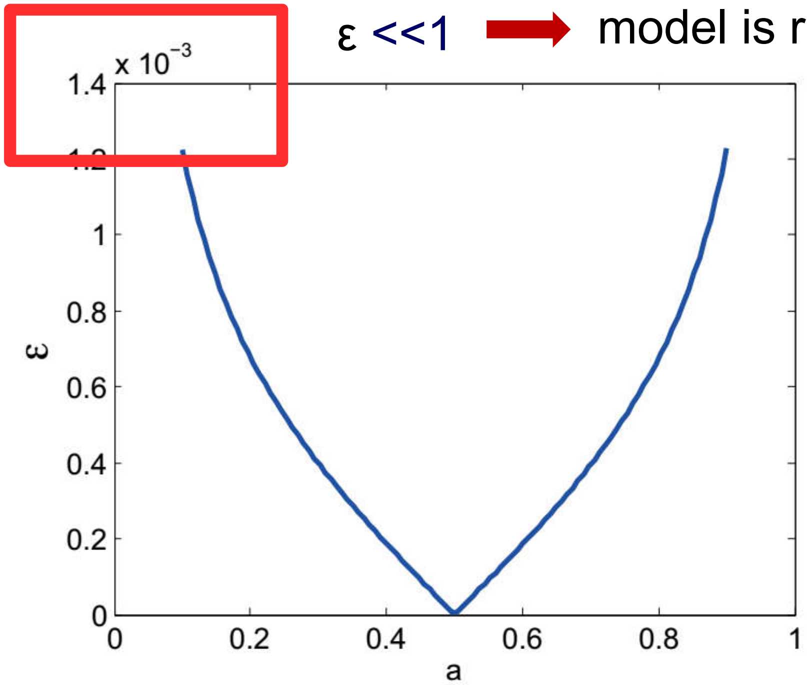
The original Bouc-Wen model is rate-independent (input displacement *x* and the output force *f*).

Pinched model:

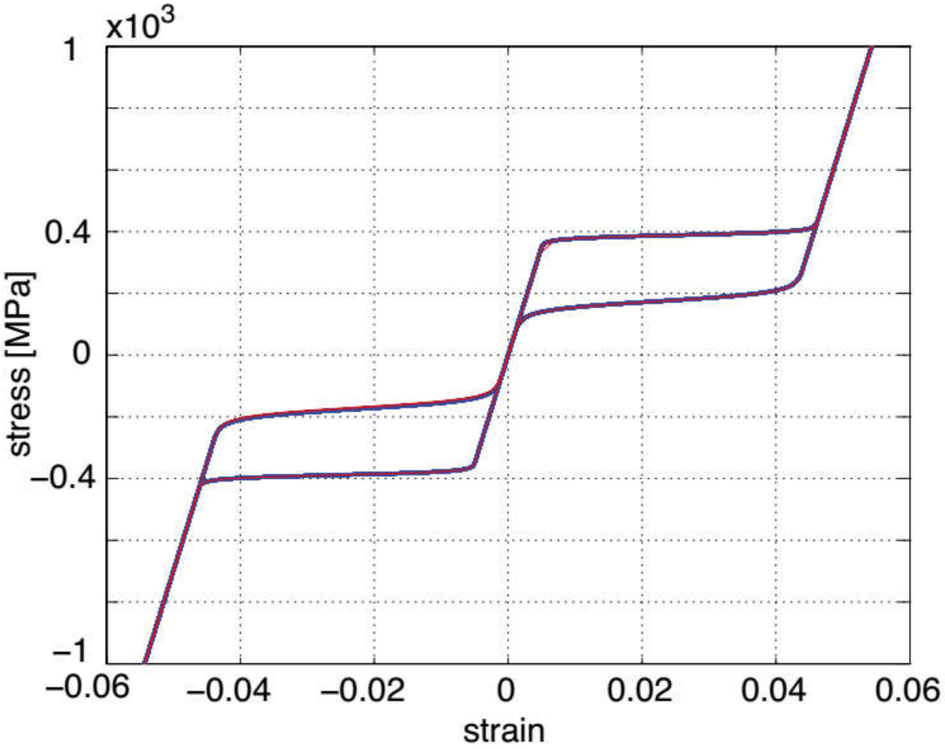
Pinching

$$f = k_e x + z, \qquad \dot{z} = [k_d h - (\gamma + \beta\,\mathrm{sgn}(\dot{x}z)\,|z|^n]\,\dot{x}$$

ε <<1 ➡ model is rate independent



SHAPE MEMORY ALLOYS

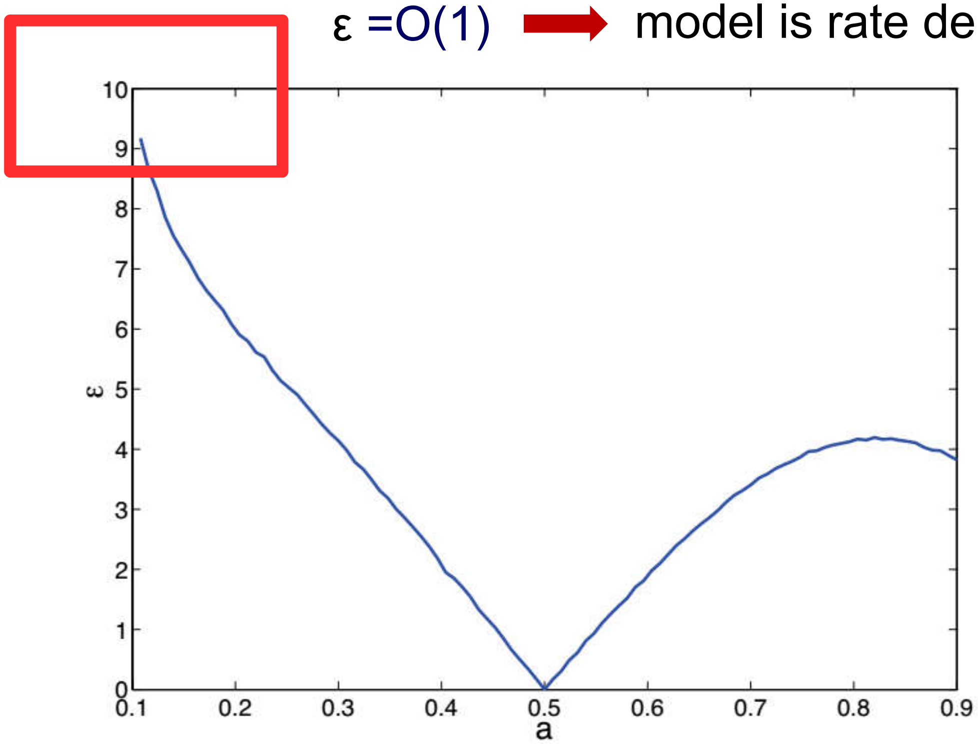


Stress-strain curves for a typical SMA material

BERNARDINI-PENCE MODEL

For fixed temperature, the model is rate-independent.
Thermomechanical coupling gives rate-dependent behavior

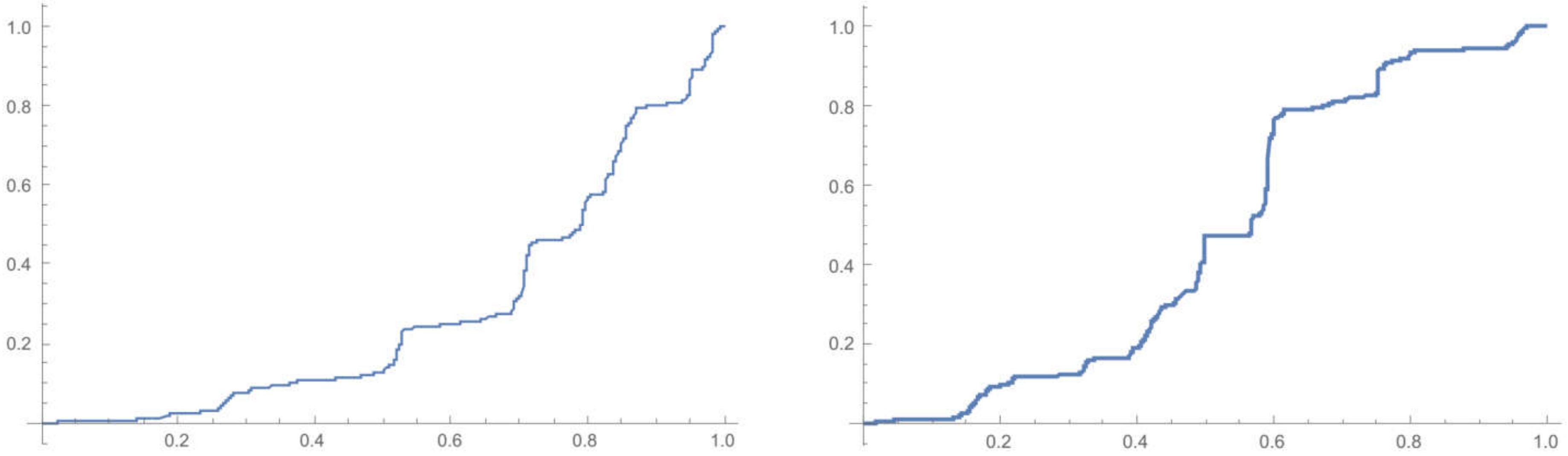
ε =O(1) ➡ model is rate dependent



Variation of the rate independence measure ε for a shape memory material in non-isothermal conditions

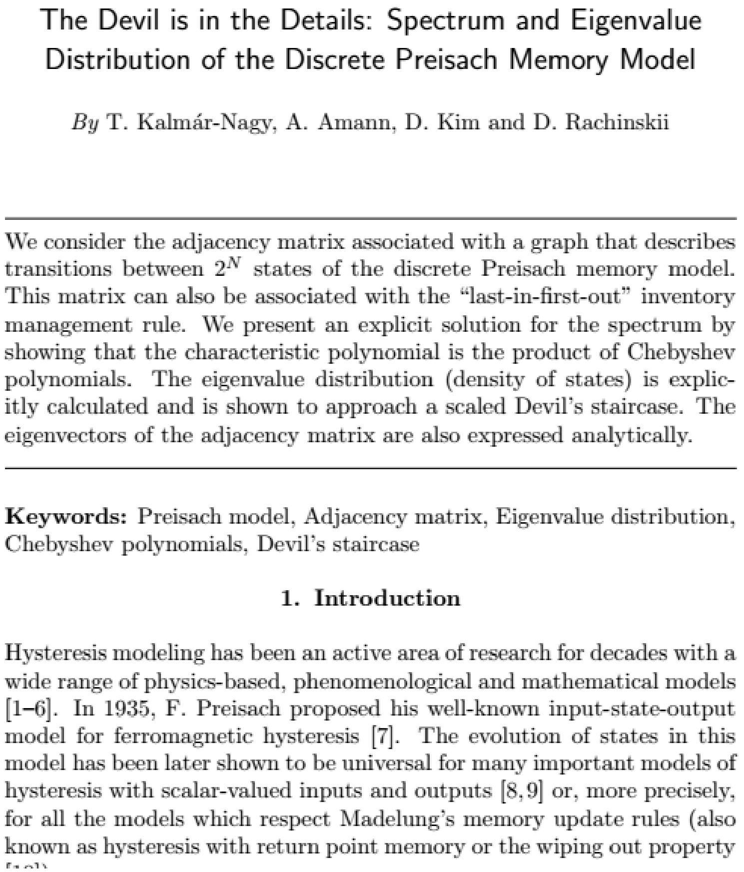
NEXT STEPS

- Experiments
- Average over random homeomorphisms



THE DEVIL'S IN THE DETAILS

Spectrum and Eigenvectors of the Discrete Preisach Memory Model



The Devil is in the Details: Spectrum and Eigenvalue Distribution of the Discrete Preisach Memory Model

By T. Kalmár-Nagy, A. Amann, D. Kim and D. Rachinskii

We consider the adjacency matrix associated with a graph that describes transitions between 2^N states of the discrete Preisach memory model. This matrix can also be associated with the "last-in-first-out" inventory management rule. We present an explicit solution for the spectrum by showing that the characteristic polynomial is the product of Chebyshev polynomials. The eigenvalue distribution (density of states) is explicitly calculated and is shown to approach a scaled Devil's staircase. The eigenvectors of the adjacency matrix are also expressed analytically.

Keywords: Preisach model, Adjacency matrix, Eigenvalue distribution, Chebyshev polynomials, Devil's staircase

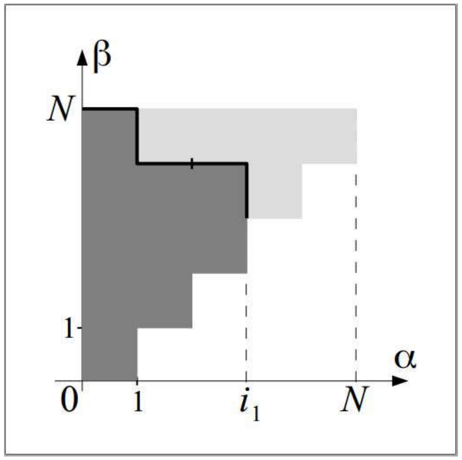
1. Introduction

Hysteresis modeling has been an active area of research for decades with a wide range of physics-based, phenomenological and mathematical models [1–6]. In 1935, F. Preisach proposed his well-known input-state-output model for ferromagnetic hysteresis [7]. The evolution of states in this model has been later shown to be universal for many important models of hysteresis with scalar-valued inputs and outputs [8,9] or, more precisely, for all the models which respect Madelung's memory update rules (also known as hysteresis with return point memory or the wiping out property

BME Math Modeling Seminar 2017/10/17, Budapest.

PREISACH MEMORY MODEL (PMM)

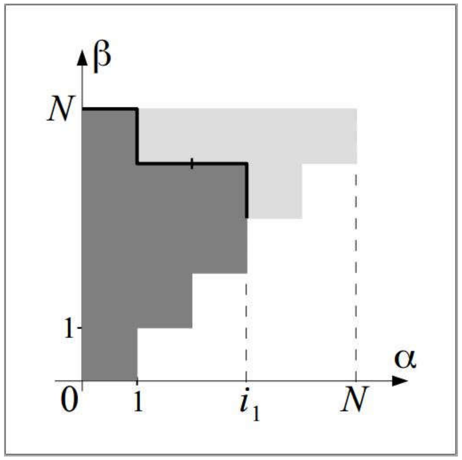
- An input-output model for magnetic **hysteresis** (dependence of a system's state on its history)
- Discrete form (DPPM) characterizes transitions in magnetic moments
- At each time step, the input changes ± 1



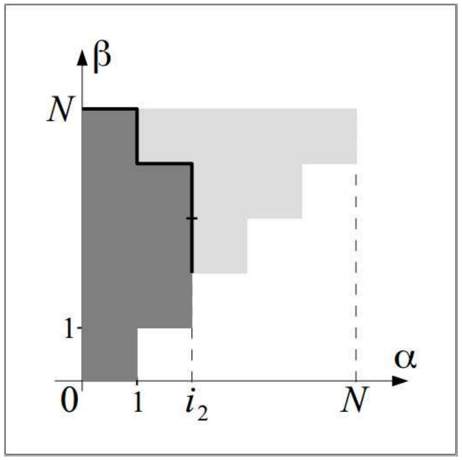
$(1, 0, 1, 1, 0)$

A black polyline of length N connects $(0, N)$ to (i, i), allowing us to encode the state of the DPPM in 1’s (horizontal line segment) and 0’s (vertical), where the squares are magnetic moments (up/down).

(unit step: -1)



$(1, 0, 1, 1, 0)$



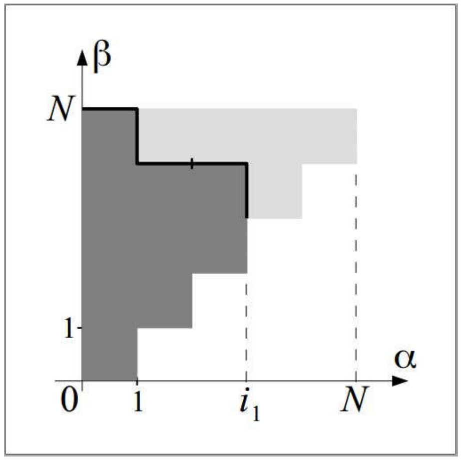
$(1, 0, 1, 0, 0)$

$(i_1 - 1, i_1 - 1) =$

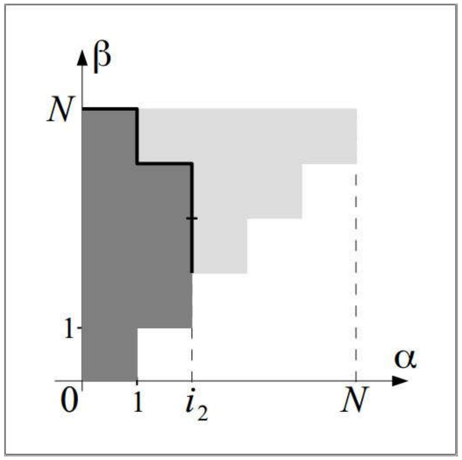
(i_2, i_2)

A black polyline of length N connects $(0, N)$ to (i, i), allowing us to encode the state of the DPPM in 1's (horizontal line segment) and 0's (vertical), where the squares are magnetic moments (up/down).

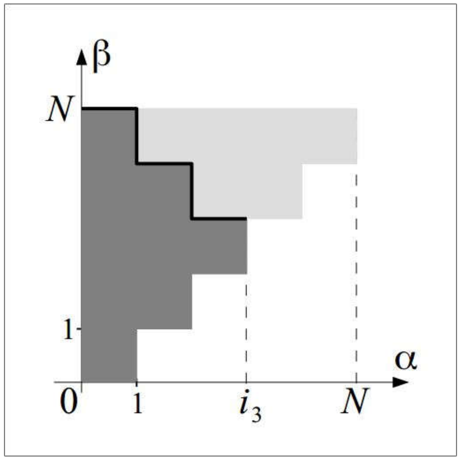
(unit step: +1)



$(1, 0, 1, 1, 0)$



$(1, 0, 1, 0, 0)$



$(1, 0, 1, 0, 1)$
(i_2+1, i_2+1)
$= (i_3, i_3)$

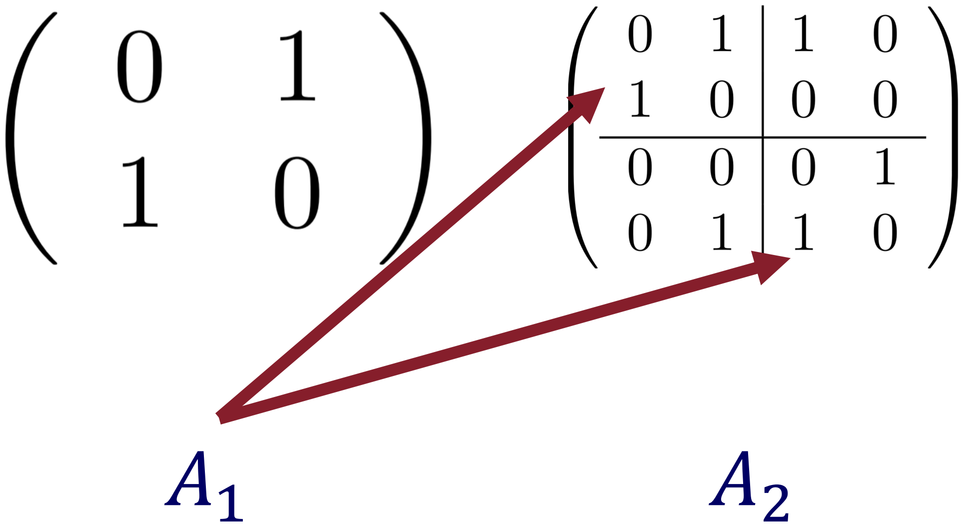
A black polyline of length N connects $(0, N)$ to (i, i), allowing us to encode the state of the DPPM in 1's (horizontal line segment) and 0's (vertical), where the squares are magnetic moments (up/down).

ADJACENCY MATRIX

- The **adjacency matrix** A_N can be associated with DPMM transitions
  - The index j of a column represents an input state (represented as a binary string of length N), and the index i of a row represents a potential output state
  - "1" denotes a horizontal step, "0" denotes a vertical step
  - Has a **self-similar** (block-hierarchical) structure
- A self-similar directed graph Γ is thus defined by this matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A_1



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

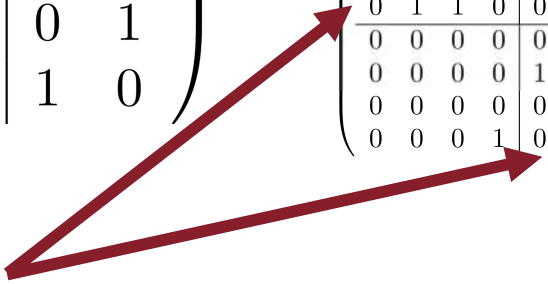
A_1 A_2

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A_1

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

A_2

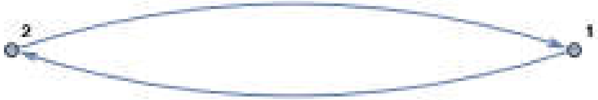


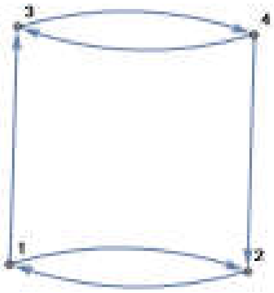
$$\left(\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right)$$

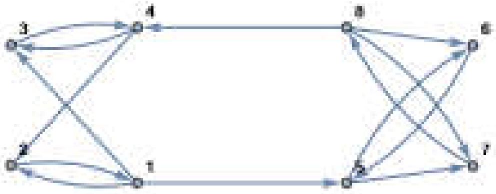
A_3

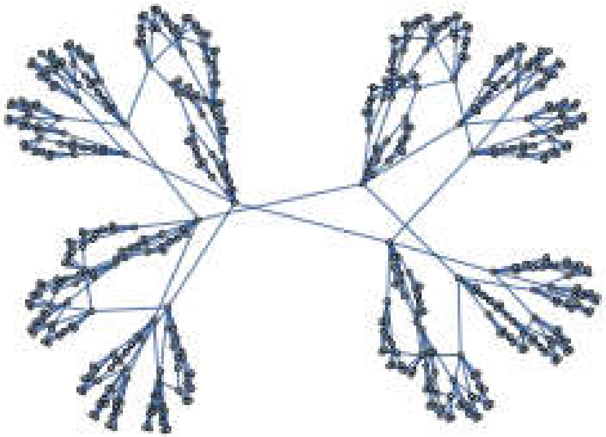
$$A_{k+1} = \left(\begin{array}{c|c} A_k & J_k \\ \hline J_k' & A_k \end{array} \right)$$

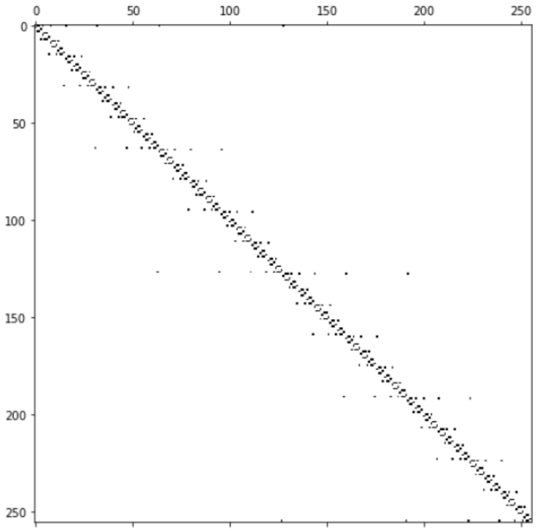
$$J_k = \operatorname{diag}\{1, 0, 0, \ldots, 0, 0\}, \qquad J_k' = \operatorname{diag}\{0, 0, \ldots, 0, 0, 1\}.$$



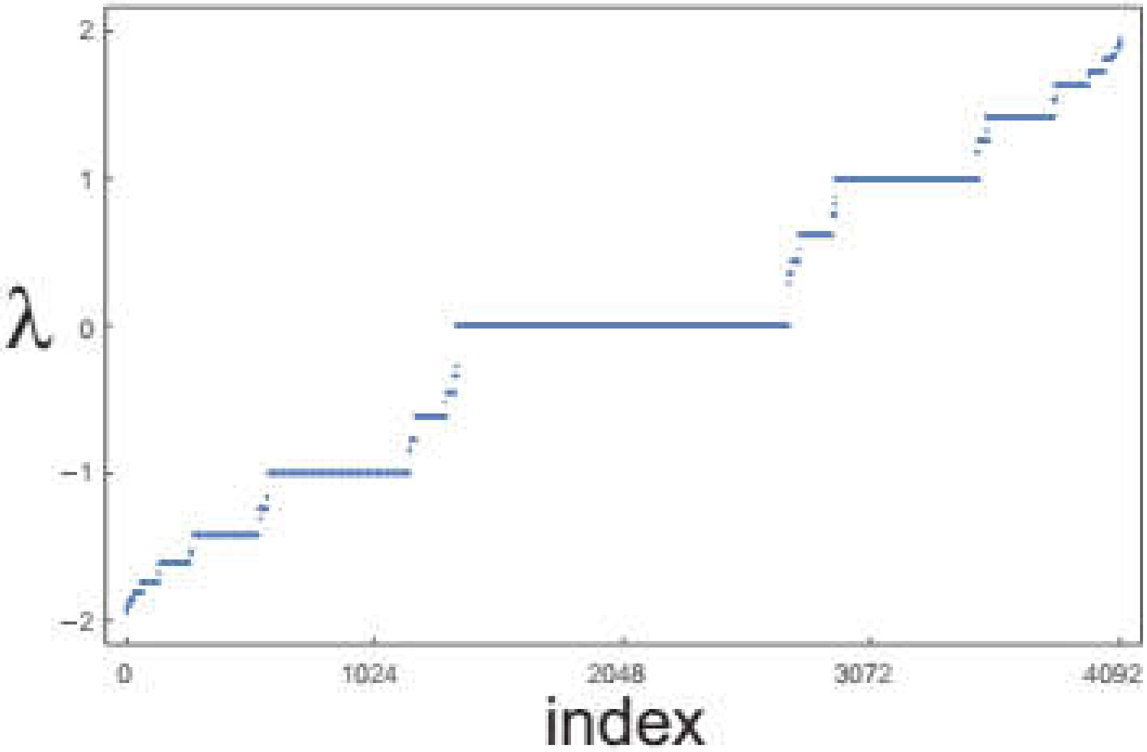




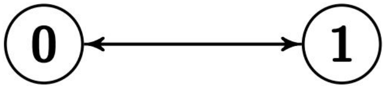




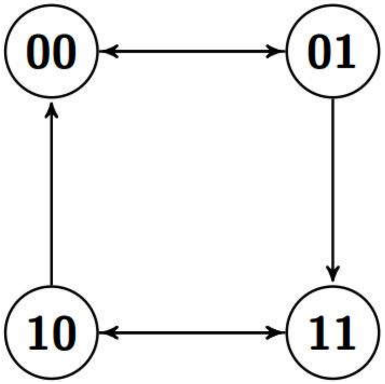
Adjacency matrix N=8



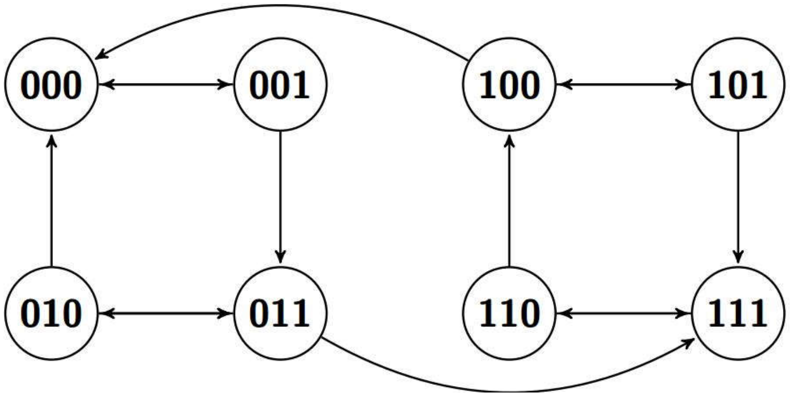
$Spectrum\ N = 12$



$N = 1$



$N = 2$



$N = 3$

SPECTRUM (THE MAIN RESULT)

- A product of **Chebyshev polynomials** U_k determines the **characteristic polynomial** χ_N of A_N (i.e., $\det(A_N - \lambda \mathrm{Id}_{2^N})$).

- Since the zeroes of Chebyshev polynomials are known, the eigenvalues and their multiplicities are relatively simple to compute

$$\chi_N(\lambda) = U_{N+1}(-\lambda/2) \prod_{i=0}^{N-1} \big(U_i(-\lambda/2)\big)^{2^{N-i-1}}$$

CHEBYSHEV POLYNOMIALS (OF THE SECOND KIND)

- A sequence of polynomials $U_N(\lambda)$ defined:
  - Recursively: $U_0(\lambda) = 1, U_1(\lambda) = 2\lambda,$
    $U_{N+1}(\lambda) = 2\lambda U_N(\lambda) - U_{N-1}(\lambda)$
    - This definition is suitable for computing determinants
  - Explicitly: $U_N(\cos(\theta)) = \frac{\sin((N+1)\theta)}{\sin(\theta)}$
    - This definition helps us to easily compute the eigenvalues

PROOF (MAIN IDEAS)

- $\chi_N = \det\left(A_N - \lambda \mathrm{Id}_{2^N}\right) = \det(T_N)$ is to be computed

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- Taking advantage of these recursions, we use induction to equate products of $U_j\left(-\frac{\lambda}{2}\right), j \in \{1, \ldots, N-1, N+1\}$ with $\det(P_k)$ and $\det(Q_k)$

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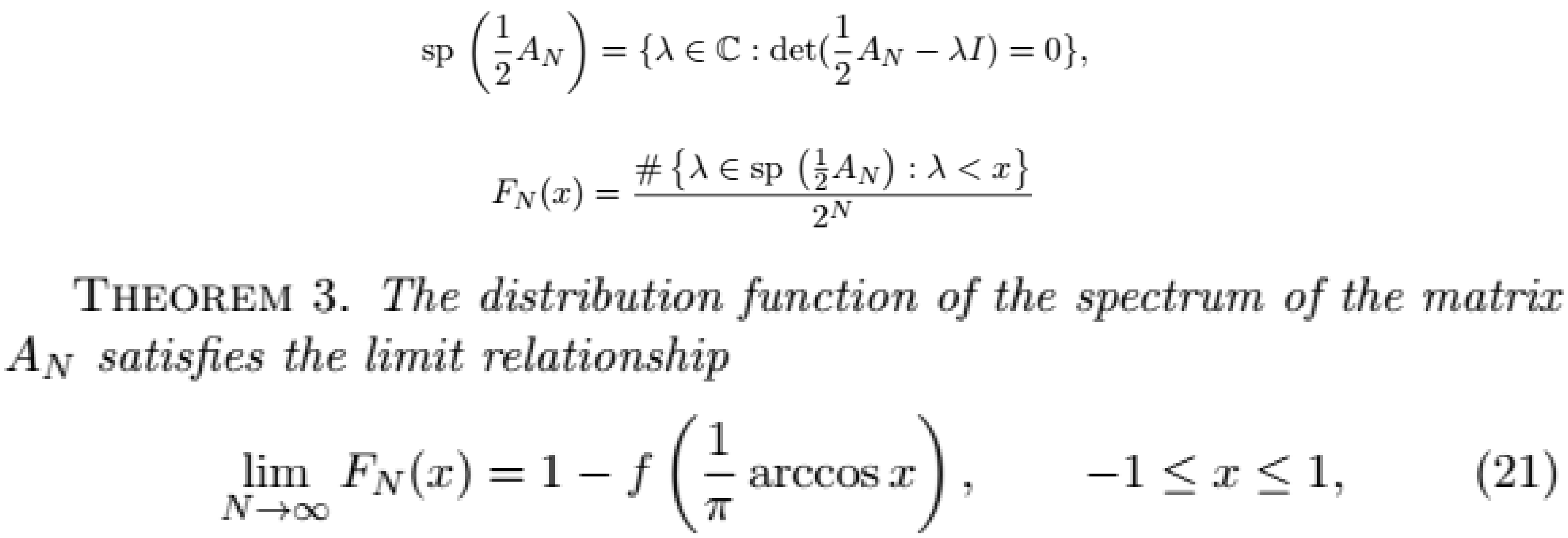
- Finally, we count the number of times each $U_j\left(-\frac{\lambda}{2}\right)$ appears a factor in the final expression of χ_k and verify our result. Q.E.D.

EIGENVALUE DISTRIBUTION

- Since the multiplicities of each U_k is given by the main result, we can sum the number of times each zero (eigenvalue) factors into the final characteristic polynomial, by employing the fact the zeroes of Chebyshev polynomials are given by $\lambda_i = \cos(\frac{i+1}{k+1}), i = 0, \ldots, k-1$
- The limit of the empirical spectral distribution (density of states) is a **devil's staircase** of the form $f(x) = \sum_{k\in\mathbb{N}} \frac{\lfloor kx \rfloor}{2^k}$



$$\mathrm{sp}\left(\frac{1}{2}A_N\right) = \{\lambda \in \mathbb{C} : \det(\frac{1}{2}A_N - \lambda I) = 0\},$$

$$F_N(x) = \frac{\#\{\lambda \in \mathrm{sp}\ (\frac{1}{2}A_N) : \lambda < x\}}{2^N}$$

THEOREM 3. *The distribution function of the spectrum of the matrix A_N satisfies the limit relationship*

$$\lim_{N\to\infty} F_N(x) = 1 - f\left(\frac{1}{\pi}\arccos x\right), \qquad -1 \le x \le 1, \qquad (21)$$

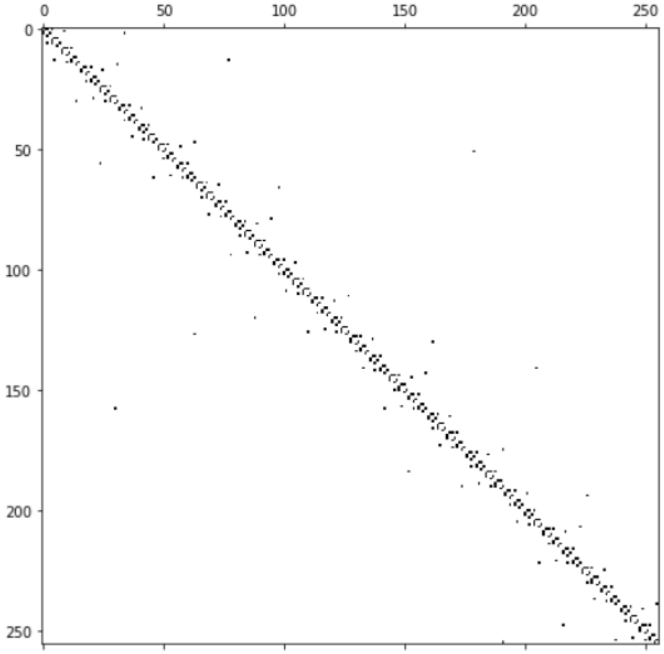
RANDOMIZING THE STRUCTURE (BALAZS HEIZER)

- The matrix:

  Coupling matrix has one 1 in the diagonal at a random position

- Studies:

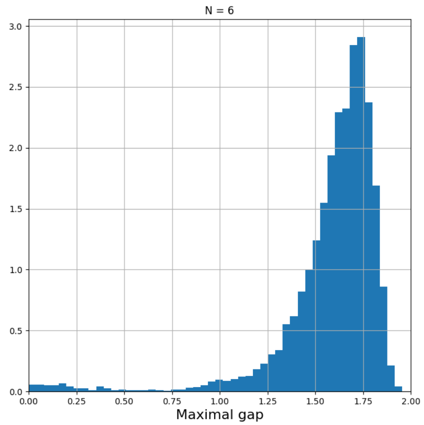
  maximal gap distribution as function of N



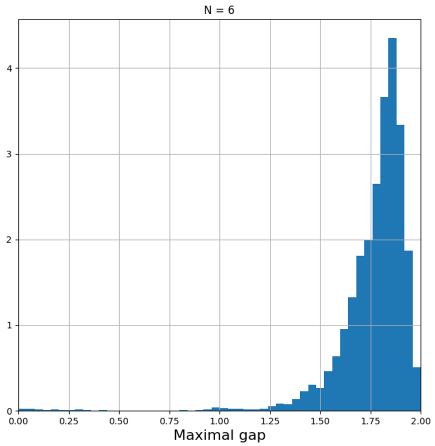
Adjacency matrix N=8

MULTIPLE I’S ON THE DIAGONAL

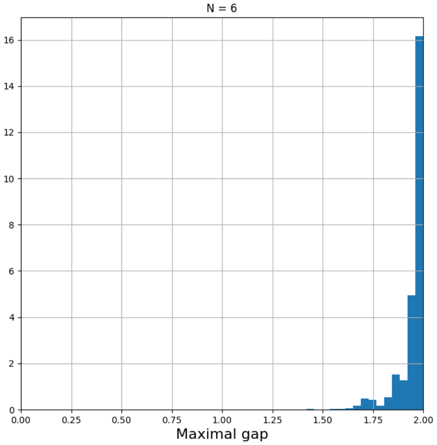
r=0.9

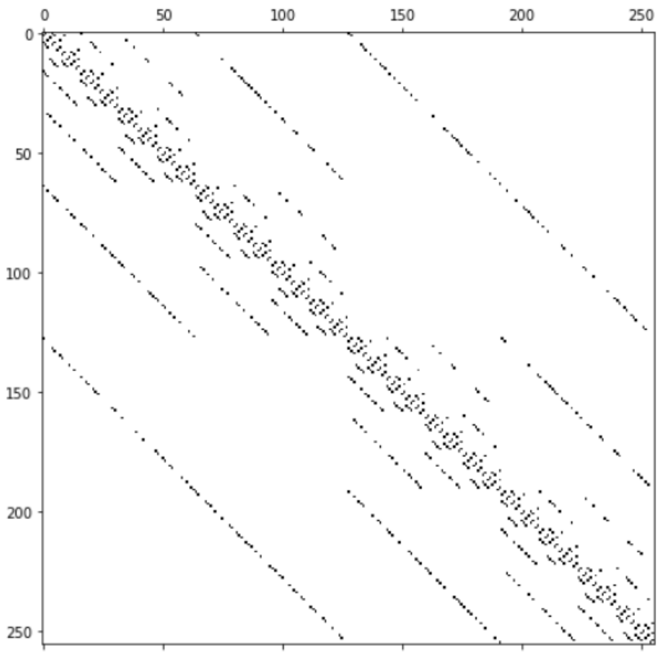


r=0.95



r=0.99





Adajcency matrix N=8
r=0.5

MECHANISTIC TURBULENCE MODEL

"Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."

Lewis Fry Richardson

Mechanistic model of turbulence: reproducing the Kolmogorov-cascade

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[***]

Abstract: The understanding of turbulent flow and the turbulent energy cascade is a main unresolved problem in physics. Although, the concept of the energy cascade and the laws of interaction among the turbulent eddies were already described by Lewis Fry Richardson and Kolmogorov in the early-mid 19's. To model the energy cascade of 3D homogeneous turbulence, we introduce the mechanistic model of turbulence which is a binary tree of masses connected by springs. The masses in the different levels model the vortices of different sizes. The structure is related to how the energy is transferred from the largest vortices (the mass in the top) to the smallest ones (the masses in the bottom). The masses in the bottom level are connected to the rigid ground with dampers to model the dissipation at the smallest scales. We analyzed the behavior of the model: a formula is presented for the analytical calculation of the eigenvalues and the eigenvalue distribution is analyzed for special mass and stiffness distributions. The discrete energy spectrum of the mechanistic model is defined as the total mechanical energy stored in each level. The spectrum is calculated for models having different mass and stiffness distributions and it is compared with the well-known Kolmogorov-spectrum. We found such parameter sets for which the energy spectrum of the mechanistic model shares the features of the Kolmogorov-spectrum. This finding demonstrates that the energy cascade of this simple model is similar to that of 3D isotropic turbulence.

Keywords: Turbulence, energy cascade, energy spectrum, mechanistic model, vibration.

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1. INTRODUCTION

The understanding of turbulent flow Pope [2000] is one of the most challenging fields in fluid mechanics. According to the notion of Richardson [1922], in a turbulent flow there are vortices of different sizes which interact with each other. Richardson introduced the concept of the turbulent energy cascade which refers to the energy transfer among the different scales of vortices in turbulent flow. The largest vortices (having length scale L_0) are induced externally. These large vortices are unstable and break up into smaller vortices. This is how the kinetic energy of the flow is transferred to smaller and smaller vortices. This kinetic energy is also continously dissipated to heat due to viscous friction. The smaller a vortex is, the larger is the effect of viscosity. The dissipation is characterized by the energy dissipation rate

$$\epsilon(t) = -\frac{dE(t)}{dt}, \tag{1}$$

where $E(t)$ is the total kinetic energy of the flow.

According to the first similarity hypothesis of Kolmogorov [1941], the dissipation becomes significant below the so-called Kolmogorov length scale η. At sufficiently high Reynolds-numbers η is related to the fluid viscosity ν and ϵ, i.e.

$$\eta = (\nu^3/\epsilon)^{1/4}. \tag{2}$$

The famous Kolmogorov-spectrum of turbulence shows the spectrum of the total kinetic energy $\hat{E}(\kappa)$ in the wavenumber domain Pope [2000], Ditlevsen [2010], i.e.

$$E(t) = \int \hat{E}(\kappa) d\kappa, \quad \kappa \sim 1/L. \tag{3}$$

For 3D homogeneous turbulence the kinetic energy spectrum is qualitatively depicted in Figure 1.

There are 3 notable regions in the Kolmogorov-spectrum:

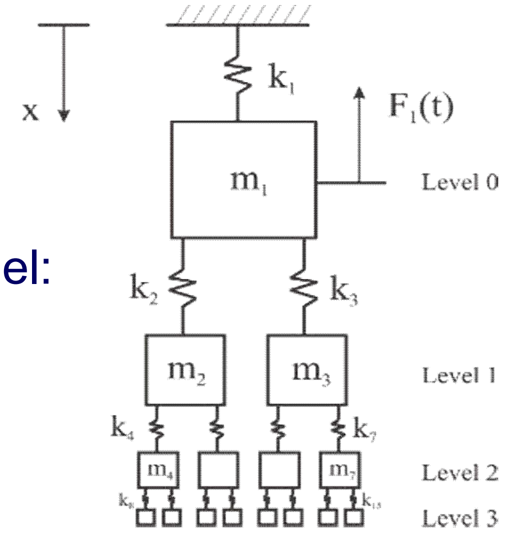
- The peak of the spectrum is located in the energy containing range which covers the smallest wavenumbers. The energy is injected into the system in this range.
- The inertial range covers the intermediate wavenumbers ($1/L_0 < \kappa < 1/\eta$) where the energy spectrum obeys the scaling law $\hat{E}(\kappa) \sim \kappa^{-5/3}$.
- The dissipation range is the part of the spectrum above the wavenumber $1/\eta$. In this range $\hat{E}(k)$ falls off due to the significant dissipation.

Kolmogorov's second similarity hypothesis states that at sufficiently high Reynolds number the energy spectrum

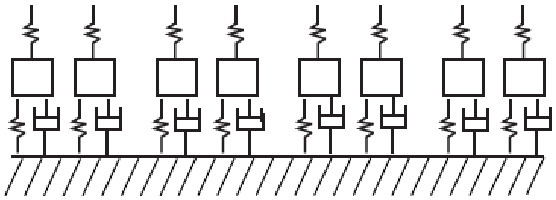
MECHANISTIC TURBULENCE MODEL

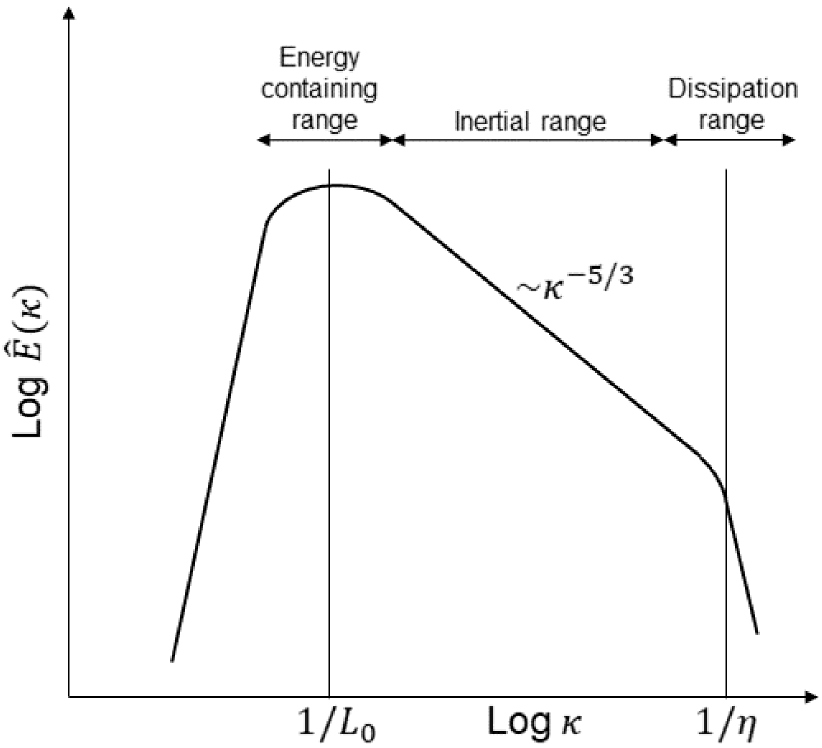
- Energy cascade: energy transfer among different scales
- Binary tree of masses connected by springs
- n level system consists of $N = 2^n - 1$ masses

The model:



Connection with the ground:

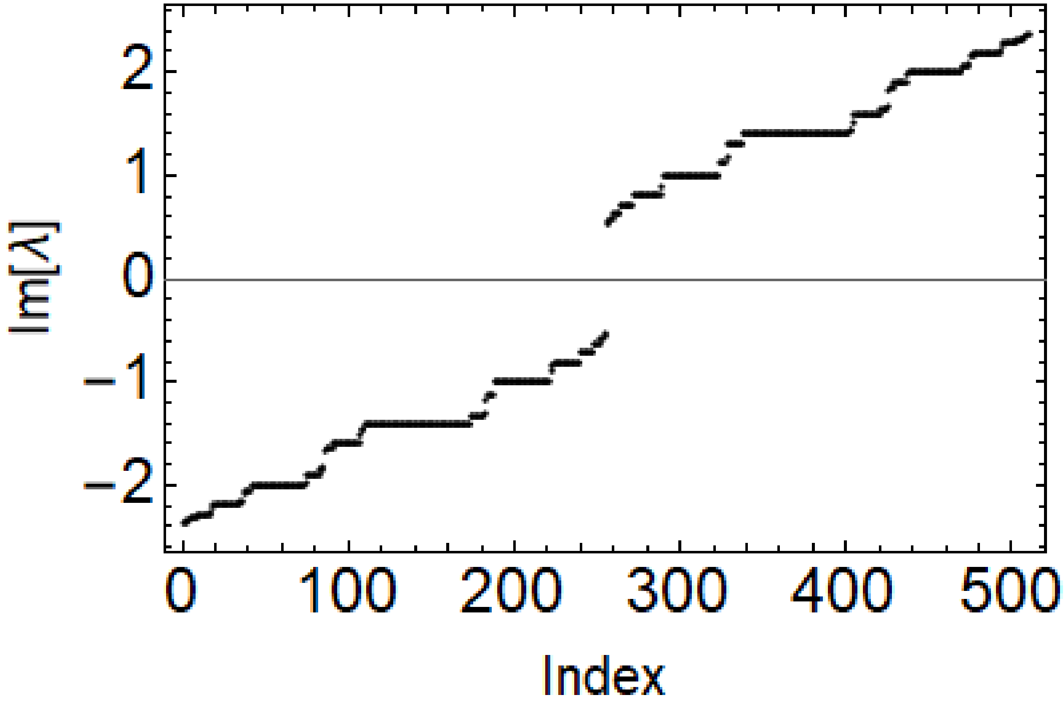




Kolmogorov-spectrum

EIGENVALUE DISTRIBUTION

- The system of differential equations $(*)$: $\mathcal{M}\ddot{x} + \mathcal{C}\dot{x} + \mathcal{K}x = 0$
- Total mechanical energy: $E(t) = \frac{1}{2}(\dot{x}^T\mathcal{M}\dot{x} + x^T\mathcal{K}x)$
- Devil's staircase type eigenvalue distribution

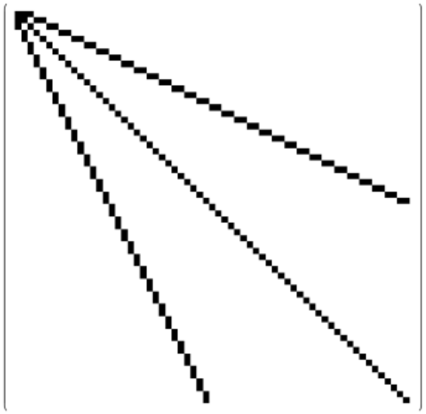


$n = 8$ levels, no damping, $m_i = k_i = 1$

$$m_i\ddot{x}_i = \begin{cases} k_{2i}(x_{2i} - x_i) + k_{2i+1}(x_{2i+1} - x_i) - \\ \quad -k_i x_i, \quad i = 1 \\ k_{2i}(x_{2i} - x_i) + k_{2i+1}(x_{2i+1} - x_i) - \\ \quad -k_i(x_i - x_{\lfloor i/2 \rfloor}), \quad i = 2, \dots, 2^{n-1} - 1 \\ -k_i(x_i - x_{\lfloor i/2 \rfloor}) - k_{i+2^{n-1}} x_i - \\ \quad -c_{i+2^{n-1}}\dot{x}_i, \quad i = 2^{n-1}, \dots, N. \end{cases} \quad (*)$$

CHARACTERISTIC EQUATION

$$m_i\ddot{x}_i = \begin{cases} k_{2i}(x_{2i}-x_i)+k_{2i+1}(x_{2i+1}-x_i)- \\ \quad -k_i x_i, \quad i=1 \\ k_{2i}(x_{2i}-x_i)+k_{2i+1}(x_{2i+1}-x_i)- \\ \quad -k_i(x_i - x_{\lfloor i/2 \rfloor}), \quad i = 2,...,2^{n-1}-1 \\ -k_i(x_i - x_{\lfloor i/2 \rfloor}) - k_{i+2^{n-1}} x_i - \\ \quad -c_{i+2^{n-1}}\dot{x}_i, \quad i = 2^{n-1},...,N. \end{cases} \qquad (*)$$



The characteristic equation of the system described by $(*)$ has the form

$$P_n(\lambda) = \frac{Q_n(\lambda)}{Q_{n-1}(\lambda)} P_{n-1}^2(\lambda), \; n \geq 1, \, P_0(\lambda) = Q_0(\lambda) = 1,$$

if $m_i = k_i = 1$, and $c_i = 0$ for all i. $Q_i(\lambda)$ are generalized Chebyshev-polynomials of the second kind which have the form

$$Q_i(\lambda) = \left(\sqrt{2}\right)^i U_i\left(\frac{\lambda^2+3}{2\sqrt{2}}\right) \text{-} \left(\sqrt{2}\right)^{i-1} U_{i-1}\left(\frac{\lambda^2+3}{2\sqrt{2}}\right), \; i \geq 1,$$

where U_i is the i. Chebyshev-polinomial of the second kind.

ENERGY SPECTRUM

- The masses are smaller on lower levels ~ large and small scales

$$M(1) = m_1,\, M(2) = m_2 = m_3 \ldots$$

- Wavenumber of the lth level

$$\kappa_l = 1/M(l)$$

- Mean energy stored in level l during t $\in [t_1, t_2]$

$$E_l(\kappa) = \frac{1}{\Delta\tau}\int E_l(t)dt,\, \Delta\tau = t_2 - t_1$$

- The discrete energy spectrum $E(\kappa)$ consists of the $E_l(\kappa)$ values $(l = 1, \ldots, n)$

- Energy injection at the largest scales
  - Impulse excitation

$$\text{Initial condition: } \dot{x}_1 = \sqrt{2} \rightarrow E(0) = \frac{1}{2}M(1)\dot{x}_1^2 = 1$$

  - Harmonic excitation

$\cos(\omega t)$ forcing on $M(1)$

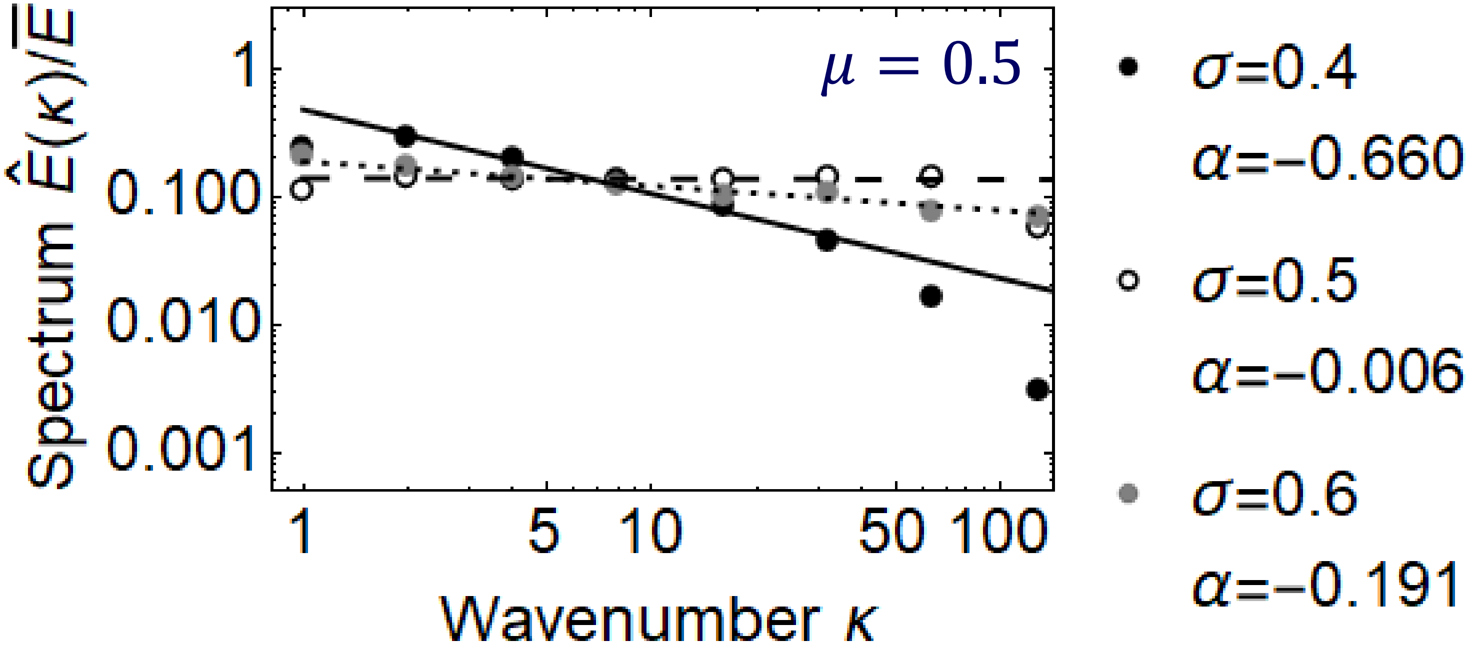
IMPULSE EXCITATION

- Mass- and spring distribution as the function of the level l

$$M_l = \mu^{l-1},\ \ 0 < \mu < 1,\ \ l = 1, \dots, n$$

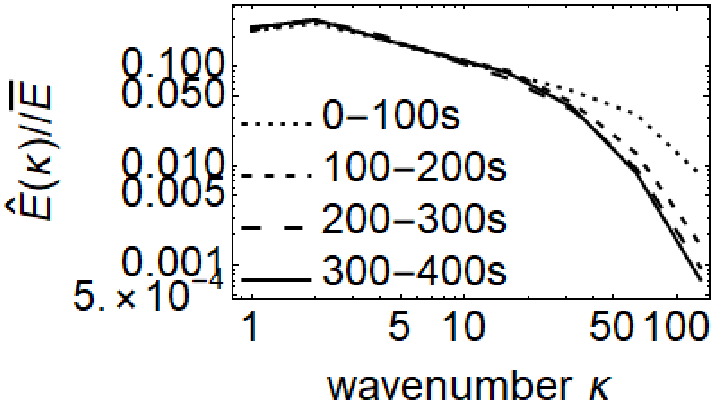
$$K_l = \sigma^{l-1}, \mu > 0,\ \ l = 1, \dots, n+1$$

- Equal damping coefficients ($c = c_i$)
- Dimensionless damping: $\xi = c/\sqrt{M_n K_{n+1}}$ (set to $\xi = 1$)

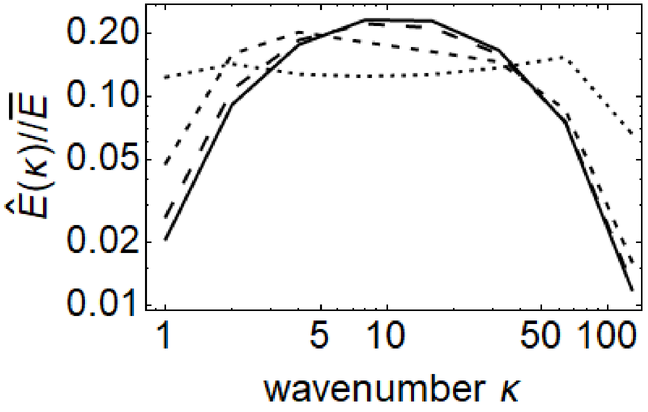


TIME EVOLUTION OF THE SPECTRUM

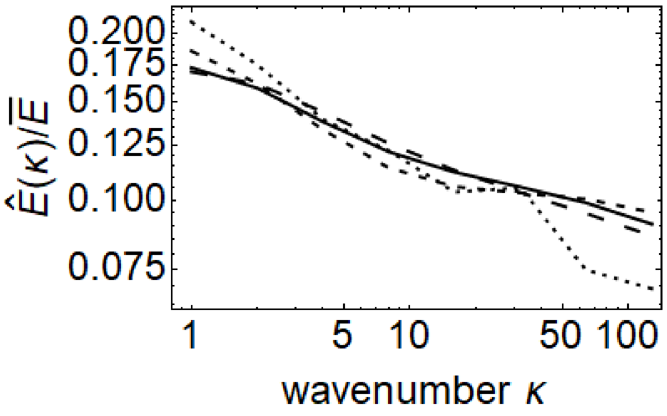
$\mu = 0.5$



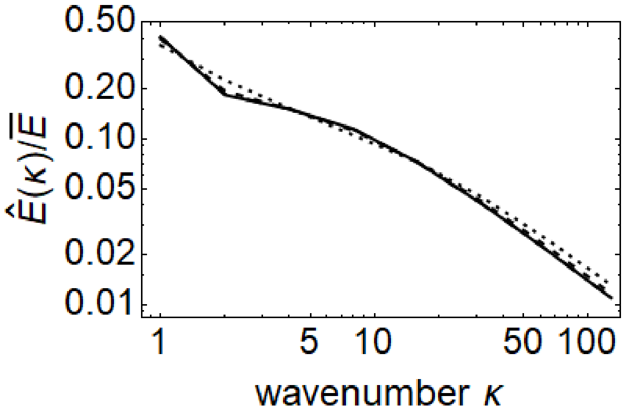
σ=0.4



σ=0.5



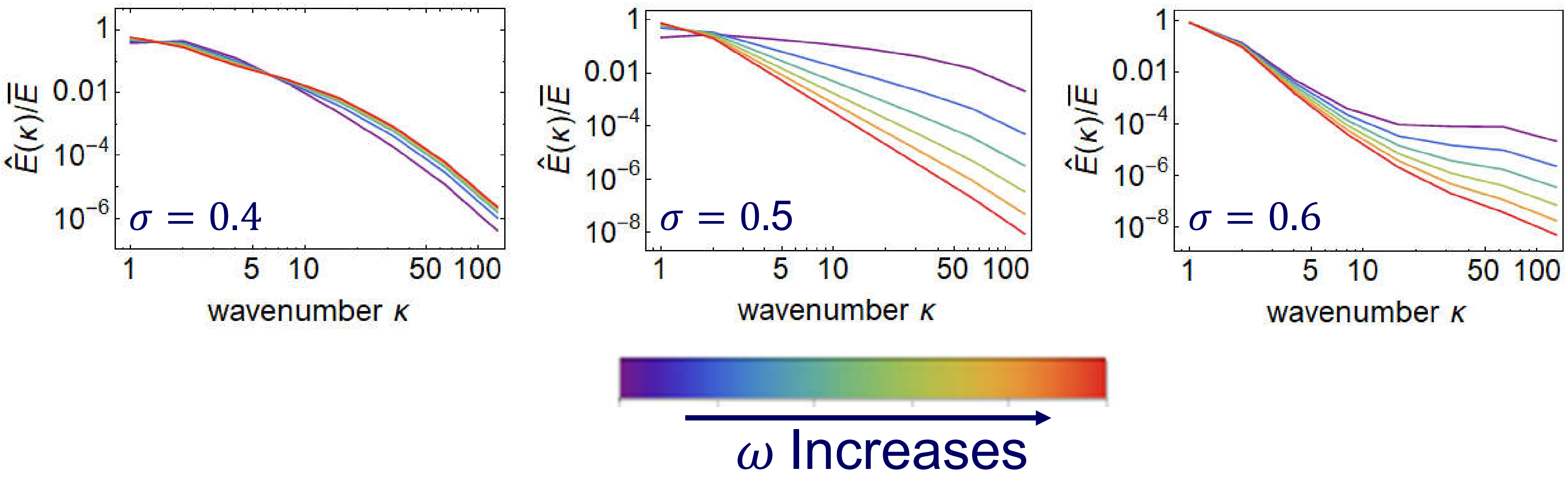
σ=0.6



σ=1.0

HARMONIC EXCITATION

- $F(t) = \cos(\omega t), \mu = 0.5,$ the parameter of the spectra is ω
- The characteristics of the spectra
  - quickly varies for $\min(\lambda) < \omega < \max(\lambda)$
  - are the same for $\omega > \max(\lambda)$ (this is shown in the figures)



We can tune parameters of the system to reproduce a Kolmogorov-like spectrum

THANK YOU!

QUESTIONS?
ANSWERS?

10/17
Rate Independence day